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BRANCHING PROCESSES IN RANDOM ENVIRONMENTS WITH THRESHOLDS

GIACOMO FRANCISCI * ** AND
ANAND N. VIDYASHANKAR,* *** *George Mason University*

Abstract

Motivated by applications to COVID dynamics, we describe a model of a branching process in a random environment $\{Z_n\}$ whose characteristics change when crossing upper and lower thresholds. This introduces a cyclical path behavior involving periods of increase and decrease leading to supercritical and subcritical regimes. Even though the process is not Markov, we identify subsequences at random time points $\{(\tau_j, \nu_j)\}$—specifically the values of the process at crossing times, viz. $\{(Z_{\tau_j}, Z_{\nu_j})\}$—along which the process retains the Markov structure. Under mild moment and regularity conditions, we establish that the subsequences possess a regenerative structure and prove that the limiting normal distributions of the growth rates of the process in supercritical and subcritical regimes decouple. For this reason, we establish limit theorems concerning the length of supercritical and subcritical regimes and the proportion of time the process spends in these regimes. As a byproduct of our analysis, we explicitly identify the limiting variances in terms of the functionals of the offspring distribution, threshold distribution, and environmental sequences.

Keywords: BPRE; COVID dynamics; ergodicity of Markov chains; estimators of growth rate; length of cycles; martingales; random sums; regenerative structure; size-dependent branching process; size-dependent branching process with a threshold; subcritical regime; supercritical regime

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Secondary 92D25; 92D30; 60G50

1. Introduction

Branching processes and their variants are used to model various biological, biochemical, and epidemic processes [1–4]. More recently, these methods have been used to model the spread of COVID cases in communities during the early stages of the pandemic [5, 6]. As time progressed, varying local containment efforts caused changes in the number of infected members in each community [7, 8], leading to periods of increase and decrease. In this paper, we describe a stochastic process model built on a branching process in random environments (BPRE) that explicitly takes into account periods of growth and decrease in the transmission rate of the virus.

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* Postal address: Department of Statistics, George Mason University, 4000 University Drive, Fairfax, VA 22030, USA.
** Email address: gfranci@gmu.edu
*** Email address: avidyash@gmu.edu

Specifically, we consider a branching process model initiated by a random number of ancestors (thought of as initiators of the pandemic within a community). During the first several generations, the process grows uncontrolled, allowing immigration into the system. This initial phase is modeled using a supercritical branching process with immigration in random environments, specifically independent and identically distributed (i.i.d.) environments. When consequences of rapid spread become significant, policymakers introduce restrictions to reduce the rate of growth, hopefully resulting in a reduced number of infected cases. The limitations are modeled using upper thresholds on the number of infected cases, and beyond the threshold the process changes its character to evolve as a subcritical branching process in random environments. During this period—owing to strict controls—immigration is also not allowed. In practical terms, this period typically involves a 'lockdown' and other social containment efforts, the intensity of which varies across communities.

The period of restrictions is not sustainable for various reasons, including political, social, and economic pressures leading to the easing of controls. Policymakers use multiple metrics to gradually reduce controls, leading to an 'opening of communities', resulting in increased human interaction. As a result, or because of changes undergone by the virus, the number of infected cases increases again. We use lower thresholds in the number of 'newly infected' to model the period of change and let the process evolve again as a supercritical BPRE in i.i.d. environments after it crosses the lower threshold. The process continues to evolve in this manner, alternating between periods of increase and decrease. In this paper, we provide a rigorous probabilistic analysis of this model.

Although we have taken the dynamics of COVID spread as a motivation for the proposed model, the aforementioned cyclic behavior is often observed in other biological systems, such as those modeled by predator–prey models or the susceptible–infected–recovered (SIR) model. In some biological populations, the cyclical behavior can be attributed to a decline in fecundity as the population size approaches some threshold [9]. Deterministic models such as ordinary differential equations, dynamical systems, and corresponding discrete-time models are used for analysis in the applications mentioned above [10–12]. While many of the models described above yield good qualitative descriptions, uncertainty estimates are typically unavailable. It is worth pointing out that the previously described branching process methods also produce reasonable point estimates for the mean growth during the early stages of the pandemic. However, these point estimates are unreliable during the later stages of the pandemic. In this paper, we address statistical estimation of the mean growth and characterize the variance of the estimates. We end the discussion with a plot, Figure 1, of the total number of confirmed COVID cases per week in Italy from 23 February 2020 to 20 July 2022. The plot also includes the number of cases estimated using the proposed model. Other examples with similar plots include the hare–lynx predator–prey dynamics and measles cases [12–14].

Before we provide a precise description of our model, we begin with a brief description of BPREs with immigration. Let $\Pi_n = (P_n, Q_n)$ be i.i.d. random variables taking values in $\mathcal{P} \times \mathcal{P}$, where $\mathcal{P}$ is the space of probability distributions on $\mathbb{N}_0$; that is, $P_n = \{P_{n,r}\}_{r=0}^{\infty}$ and $Q_n = \{Q_{n,r}\}_{r=0}^{\infty}$ for some non-negative integers $P_{n,r}$ and $Q_{n,r}$ such that $\sum_{r=0}^{\infty} P_{n,r} = 1$ and $\sum_{r=0}^{\infty} Q_{n,r} = 1$. The process $\Pi = \{\Pi_n\}_{n=0}^{\infty}$ is referred to as the environmental sequence. For each realization of Π, we associate a population process $\{Z_n\}_{n=0}^{\infty}$ defined recursively as follows: let Z_0 take values on the positive integers, and for $n \geq 0$, let

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i} + I_n,$$

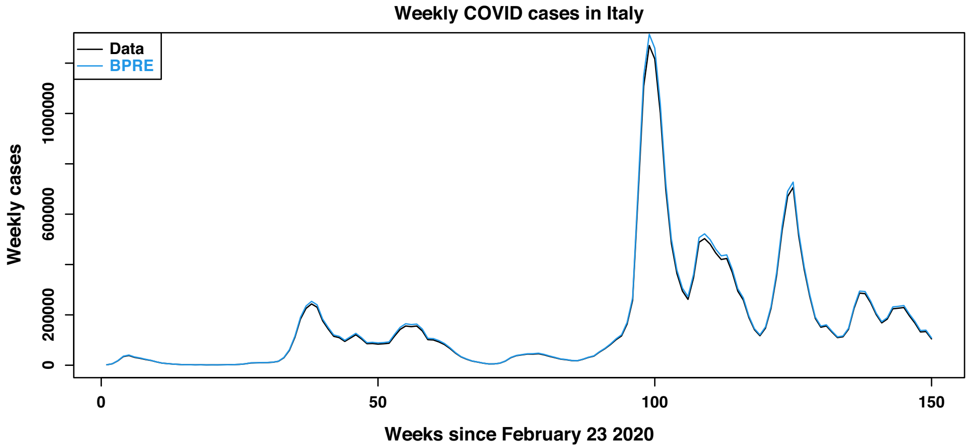

FIGURE 1. In black weekly COVID cases in Italy from February 23, 2020 to February 3, 2023. In blue a BPRE starting with the same initial value and offspring mean having the negative binomial distribution with predefined number of successful trials $r = 10$ and Gamma-distributed mean with shape parameter equal to the mean of the data and rate parameter 1.

where, given $\Pi_n = \big(P_n, Q_n\big)$, $\{\xi_{n,i}\}_{i=1}^{\infty}$ are i.i.d. with distribution P_n and I_n is an independent random variable with distribution Q_n. The random variable $Y_n = \log(\overline{P}_n)$, where $\overline{P}_n = \sum_{r=0}^{\infty} rP_{n,r}$, plays an important role in the classification of BPREs with immigration. It is well known that when $\mathbb{E}[Y_0] > 0$, the process diverges to infinity with probability one, and if $\mathbb{E}[Y_0] \leq 0$ and the immigration is degenerate at zero for all environments, then the process becomes extinct with probability one [15]. Furthermore, in the subcritical case, that is, $\mathbb{E}[Y_0] < 0$, one can identify three distinct regimes: (i) weakly subcritical, (ii) moderately subcritical, and (iii) strongly subcritical. The regime (i) corresponds to the case when there exists a $0 < \rho < 1$ such that $\mathbb{E}\big[Y_0 e^{\rho Y_0}\big] = 0$, while (ii) corresponds to the case when $\mathbb{E}[Y_0 e^{Y_0}] = 0$. Finally, (iii) corresponds to the case when $\mathbb{E}[Y_0 e^{Y_0}] < 0$ [16]. In this paper, when working with the subcritical regime, we will assume that the process is strongly subcritical, and we will refer to it as a subcritical process in the rest of the manuscript.

We now turn to a description of the model. Let $\Pi^U = \{\Pi_n^U\}_{n=0}^{\infty}$, where $\Pi_n^U = \big(P_n^U, Q_n^U\big)$, denote a collection of supercritical environmental sequences. Here, $P_n^U = \{P_{n,r}^U\}_{r=0}^{\infty}$ indicates the offspring distribution and $Q_n^U = \{Q_{n,r}^U\}_{r=0}^{\infty}$ represents the immigration distribution. Also, let $\Pi^L = \{\Pi_n^L\}_{n=0}^{\infty}$, where $\Pi_n^L = P_n^L = \{P_{n,r}^L\}_{r=0}^{\infty}$, denote a collection of subcritical environmental sequences. We now provide an evolutionary description of the process: at time zero the process starts with a random number of ancestors Z_0. Each of them lives one unit of time and reproduces according to the distribution P_0^U. Thus, the size of the first-generation population is

$$Z_1 = \sum_{i=1}^{Z_0} \xi_{0,i}^U + I_0^U,$$

where, given $\Pi_0^U = (P_0^U, Q_0^U)$, the $\xi_{0,i}^U$ are i.i.d. random variables with offspring distribution P_0^U and are independent of the immigration random variable I_0^U with distribution Q_0^U. The random variable $\xi_{0,i}^U$ is interpreted as the number of children produced by the ith parent in the 0th

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generation, and I_0^U is interpreted as the number of immigrants whose distribution is generated by the same environmental random variable Π_0^U.

Let U_1 denote the random variable representing the upper threshold. If $Z_1 < U_1$, each member of the first-generation population lives one unit of time and evolves, conditionally on the environment, as the ancestors independent of the population size at time one. That is,

$$Z_2 = \sum_{i=1}^{Z_1} \xi_{1,i}^U + I_1^U.$$

As before, given $\Pi_1^U = \left(P_1^U, Q_1^U\right)$, the $\xi_{1,i}^U$ are i.i.d. with distribution P_1^U and I_1^U has distribution Q_1^U. The random variables $\xi_{1,i}^U$ are independent of Z_1, $\xi_{0,i}^U$, and I_0^U, I_1^U. If $Z_1 \geq U_1$, then

$$Z_2 = \sum_{i=1}^{Z_1} \xi_{1,i}^L,$$

where, given $\Pi_1^L = P_1^L$, the $\xi_{1,i}^L$ are i.i.d. with distribution P_1^L. Thus, the size of the second-generation population is

$$Z_2 = \begin{cases} \sum_{i=1}^{Z_1} \xi_{1,i}^U + I_1^U & \text{if } Z_1 < U_1, \\ \sum_{i=1}^{Z_1} \xi_{1,i}^L & \text{if } Z_1 \geq U_1. \end{cases}$$

The process Z_3 is defined recursively as before. As an example, if $Z_1 < U_1$, $Z_2 < U_1$ or $Z_1 \geq U_1$, $Z_2 \leq L_1$, for a random lower threshold L_1, then the process will evolve like a supercritical BPRE with offspring distribution P_2^U and immigration distribution Q_2^U. Otherwise (that is, $Z_1 < U_1$ and $Z_2 \geq U_1$ or $Z_1 \geq U_1$ and $Z_2 > L_1$), the process will evolve like a subcritical BPRE with offspring distribution P_2^L. This dynamics continues with different thresholds (U_j, L_j), yielding the process $\{Z_n\}_{n=0}^{\infty}$, which we refer to as a branching process in random environments with thresholds (BPRET). The consecutive set of generations where the reproduction is governed by a supercritical BPRE is referred to as the *supercritical regime*, while the other is referred to as the *subcritical regime*. As we will see below, non-trivial immigration in the supercritical regime is required to obtain alternating periods of increase and decrease.

The model described above is related to size-dependent branching processes with a threshold as studied by Klebaner [9] and more recently by Athreya and Schuh [17]. Specifically, in that model the offspring distribution depends on a *fixed threshold* K and the size of the previous generation. As observed in these papers, these Markov processes either explode to infinity or are absorbed at zero. In our model the thresholds are *random and dynamic*, resulting in a non-Markov process; however, the offspring distribution does not depend on the size of the previous generation *as long as they belong to the same regime*. Indeed, when $U_j - 1 = L_j = K$ for all $j \geq 1$, the immigration distribution is degenerate at zero, and the environment is fixed, one obtains as a special case the density-dependent branching process (see for example [9, 17–20]. Additionally, while the model of Klebaner [9] uses Galton–Watson processes as a building block, our model uses branching processes in i.i.d. environments.

Continuing with our discussion on the literature, Athreya and Schuh [17] show that in the fixed-environment case, the special case of a size-dependent process with a single threshold becomes extinct with probability one. We show that this is also the case for the BPRE when there is no immigration; the details are in Theorem 2.1. Similar phenomena have been observed in slightly different contexts in Jagers and Zuyev [21, 22]. The incorporation of an immigration component ensures that the process is not absorbed at zero and hence may be useful for modeling stable populations at equilibrium as done in deterministic models. For additional discussion see Section 7.

For ease of further discussion, we introduce some notation. Let $Y_n^U := \log\left(\overline{P}_n^U\right)$ and $Y_n^L := \log\left(\overline{P}_n^L\right)$, where

$$\overline{P}_n^U = \sum_{r=0}^{\infty} rP_{n,r}^U \quad \text{and} \quad \overline{P}_n^L = \sum_{r=0}^{\infty} rP_{n,r}^L;$$

that is, $\overline{P}_n^U$ and $\overline{P}_n^L$ represent the offspring means conditional on the environments $\Pi_n^U = \left(P_n^U, Q_n^U\right)$ and $\Pi_n^L = P_n^L$, respectively. Also, let $\overline{Q}_n^U = \sum_{r=0}^{\infty} rQ_{n,r}^U$ denote the immigration mean conditional on the environment, and let

$$\overline{\overline{P}}_n^U = \sum_{r=0}^{\infty} \left(r - \overline{P}_n^U\right)^2 P_{n,r}^U \quad \text{and} \quad \overline{\overline{P}}_n^L = \sum_{r=0}^{\infty} \left(r - \overline{P}_n^L\right)^2 P_{n,r}^L$$

denote the conditional variance of the offspring distributions given the environment.

From the description, it is clear that the crossing times at the thresholds (U_j, L_j) of Z_n, namely τ_j and ν_j, will play a significant role in the analysis. It will turn out that $\{Z_{\tau_j}\}$ and $\{Z_{\nu_j}\}$ form time-homogeneous Markov chains with state spaces $S^L := \mathbb{N}_0 \cap [0, L_U]$ and $S^U := \mathbb{N} \cap [L_U + 1, \infty)$, respectively, where we take $L_j \leq L_U$ and $U_j \geq L_U + 1$ for all $j \geq 1$. Under additional conditions on the offspring distribution and the environment sequence, the processes $\{Z_{\tau_j}\}$ and $\{Z_{\nu_j}\}$ will be uniformly ergodic. These results are established in Section 3.

The amount of time the process spends in the supercritical and subcritical regimes, beyond its mathematical and scientific interest, will also arise in the study of the central limit theorem for the estimates of $M^U := \mathbb{E}\left[\overline{P}_n^U\right]$ and $M^L := \mathbb{E}\left[\overline{P}_n^L\right]$. Using the uniform ergodicity alluded to above, we will establish that the time averages of $\tau_j - \nu_{j-1}$ and $\nu_j - \tau_j$ converge to finite positive constants, μ^U and μ^L. Additionally, we establish a central limit theorem related to this convergence under a finite-second-moment hypothesis after an appropriate centering and scaling, that is,

$$\frac{1}{\sqrt{n}} \sum_{j=1}^{n} (\tau_j - \nu_{j-1}) \xrightarrow[n\to\infty]{d} N\left(\mu^U, \sigma^{2,U}\right),$$

and we characterize $\sigma^{2,U}$ in terms of the stationary distribution of the Markov chain. A similar result also holds for $\nu_j - \tau_j$. This, in turn, provides qualitative information regarding the proportion of time the process spends in these regimes. That is, if C_n^U is the amount of time the process spends in the supercritical regime up to time $n - 1$, we show that $n^{-1}C_n^U$ converges to $\mu^U\left(\mu^U + \mu^L\right)^{-1}$; a related central limit theorem is also established, and in the process we characterize the limiting variance. Interestingly, we show that the central limit theorem prevails even for the joint distribution of the length of time and the proportion of time the process spends in the supercritical and subcritical regimes. These results are described in Sections 4 and 5.

An interesting question concerns the rate of growth of the BPRET in the supercritical and subcritical regimes described by the corresponding expectations, namely M^U and M^L.

Specifically, we establish that the limiting joint distribution of the estimators is bivariate normal with a diagonal covariance matrix, yielding asymptotic independence of the mean estimators derived using data from supercritical and subcritical regimes. In the classical setting of a supercritical BPRE *without immigration*, this problem has received some attention (see for instance Dion and Esty [23]). The problem considered here is different in the following four ways: (i) the population size does not converge to infinity, (ii) the lengths of the regimes are random, (iii) in the supercritical regime the population size may be zero, and (iv) there is an additional immigration term. While (iii) and (iv) can be accounted for in the classical settings as well, their effect on the point estimates is minimized because of the exponential growth of the population size. Here, while the exponential growth is ruled out, perhaps as anticipated, the Markov property of the process at crossing times, namely $\{Z_{\tau_j}\}$ and $\{Z_{\nu_j}\}$, and their associated regeneration times plays a central role in the proof. It is important to note that it is possible for both regimes to occur between regeneration times. Hence, the proportion of time that the process spends in the supercritical and subcritical regimes also plays a vital role in the derivation of the asymptotic limit distribution. The limiting variance of the estimators depends additionally on μ^U and μ^L, beyond $V_1^U := \mathbb{V}\left[\overline{P}_0^U\right]$, $V_1^L := \mathbb{V}\left[\overline{P}_0^L\right]$, $V_2^U := \mathbb{E}\left[\overline{\overline{P}}_0^U\right]$, and $V_2^L := \mathbb{E}\left[\overline{\overline{P}}_0^L\right]$. In the special case of fixed environments, the limit behavior of the estimators takes a different form compared to the traditional results, as described for example in Heyde [24]. These results are in Section 6.

Finally, in Appendix B we provide some numerical experiments illustrating the behavior of the model. Specifically, we illustrate the effects of different distributions on the path behavior of the process and describe how they change when the thresholds increase. The experiments also suggests that if different regimes are not taken into account, the true growth rate of the virus may be underestimated. We now turn to Section 2, where we develop additional notation and provide precise statements of the main results.

2. Main results

The branching process in random environments with thresholds (BPRET) is a supercritical BPRE with immigration until it reaches an upper threshold, after which it transitions to a subcritical BPRE until it crosses a lower threshold. Beyond this time, the process reverts to a supercritical BPRE with immigration, and the above cycle continues. Specifically, let $\{(U_j, L_j)\}_{j=1}^{\infty}$ denote a collection of thresholds (assumed to be i.i.d.). Then the BPRET evolves like a supercritical BPRE with immigration until it reaches the upper threshold U_1, at which time it becomes a subcritical BPRE. The process remains subcritical until it crosses the threshold L_1; after that it evolves again as a supercritical BPRE with immigration, and so on. We now provide a precise description of the BPRET.

Let $\{(U_j, L_j)\}_{j=1}^{\infty}$ be i.i.d. random vectors with support $S_B^U \times S_B^L$, where $S_B^U := \mathbb{N} \cap [L_U + 1, \infty)$, $S_B^L := \mathbb{N} \cap [L_0, L_U]$, and $1 \le L_0 \le L_U$ are fixed integers. We denote by Π^U and Π^L the supercritical and subcritical environmental sequences; that is,

$$\Pi^U = \{\Pi_n^U\}_{n=0}^{\infty} = \left\{\left(P_n^U, Q_n^U\right)\right\}_{n=0}^{\infty} \quad \text{and} \quad \Pi^L = \{\Pi_n^L\}_{n=0}^{\infty} = \{P_n^L\}_{n=0}^{\infty}.$$

We use the notation $\mathbb{P}_{E^U}$ and $\mathbb{P}_{E^L}$ for probability statements with respect to the supercritical and subcritical environmental sequences. As in the introduction, given the environment, the $\xi_{n,i}^U$ are i.i.d. random variables with distribution P_n^U and are independent of the immigration random variable I_n^U. Similarly, conditionally on the environment, the $\xi_{n,i}^L$ are i.i.d. random variables with offspring distribution P_n^L. Finally, let Z_0 be an independent random variable with support included in $\mathbb{N} \cap [1, L_U]$. We emphasize that the thresholds are independent of the environmental sequences, offspring random variables, immigration random variables, and Z_0. For technical details regarding the construction of the probability space we refer the reader to Appendix A.1. We denote by $M^T := \mathbb{E}\Big[\overline{P}_0^T\Big]$, $T \in \{L, U\}$, and $N^U := \mathbb{E}\Big[\overline{Q}_0^U\Big]$ the annealed (averaged over the environment) offspring mean and the annealed immigration mean, respectively. Throughout the manuscript, we make the following assumptions on the environmental sequences.

Assumptions:

(H1) $\Pi^T = \{\Pi_n^T\}_{n=0}^{\infty}$ *are i.i.d. environments such that* $P_{0,0}^T < 1$ *and* $0 < \overline{P}_0^T < \infty$ $\mathbb{P}_{E^T}$*-almost surely (a.s.).*

(H2) $\mathbb{E}\Big[Y_0^L e^{Y_0^L}\Big] < 0$, $\mathbb{E}\big[Y_0^U\big] > 0$, $M^U < \infty$, *and* $\mathbb{E}\big[\log\big(1 - P_{0,0}^U\big)\big] > -\infty$.

(H3) $\mathbb{P}_{E^U}(Q_{0,0}^U < 1) > 0$ *and* $N^U < \infty$.

(H4) $\big\{(U_j, L_j)\big\}_{j=1}^{\infty}$ *are i.i.d. and have support* $S_B^U \times S_B^L$, *where* $1 \le L_0 \le M^L L_U$ *and* $\mathbb{E}[U_1] < \infty$.

The above assumptions rule out degenerate behavior of the process and are commonly used in the literature on BPRE (see Assumption R and Theorem 2.2 of Kersting and Vatutin [16]). Assumption (**H2**) states that Π_n^U is a supercritical environment and Π_n^L is a (strongly) subcritical environment. Additionally, by Jensen's inequality it follows that $M^L < 1$ and $1 < M^U < \infty$. Assumption (**H3**) states that immigration is positive with positive probability and has finite expectation N^U, while (**H4**) states that the upper thresholds U_j have finite expectation.

We are now ready to give a precise definition of the BPRET. Let $\nu_0 := 0$. Starting from Z_0, the BPRET $\{Z_n\}_{n=0}^{\infty}$ is defined recursively over $j \ge 0$ as follows:

1j. For $n \ge \nu_j$ and until $Z_n < U_{j+1}$,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}^U + I_n^U. \tag{1}$$

Next, let $\tau_{j+1} := \inf\{n \ge \nu_j : Z_n \ge U_{j+1}\}$.

2j. For $n \ge \tau_{j+1}$ and until $Z_n > L_{j+1}$,

$$Z_{n+1} = \sum_{i=1}^{Z_n} \xi_{n,i}^L. \tag{2}$$

Next, let $\nu_{j+1} := \inf\{n \ge \tau_{j+1} : Z_n \le L_{j+1}\}$.

It is clear from the definition that ν_j and τ_j are stopping times with respect to the σ-algebra $\mathcal{F}_n$ generated by $\{Z_j\}_{j=0}^{n}$ and the thresholds $\{(U_j, L_j)\}_{j=1}^{\infty}$. Thus, Z_{ν_j}, Z_{τ_j}, $\xi_{\nu_j,i}^U$, $\xi_{\tau_{j+1},i}^L$, and $I_{\nu_j}^U$ are well-defined random variables.

It is also clear from the above definition that the intervals $[\nu_{j-1}, \tau_j)$ and $[\tau_j, \nu_j)$ represent supercritical and subcritical intervals, respectively. We show below that the process $\{Z_n\}_{n=0}^{\infty}$ exits and enters the above intervals infinitely often. Let $\Delta_j^U := \tau_j - \nu_{j-1}$ and $\Delta_j^L := \nu_j - \tau_j$ denote the lengths of these intervals. Since a supercritical BPRE with immigration diverges

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with probability one (see Theorem 2.2 of Kersting and Vatutin [16]), it follows that τ_{j+1} is finite whenever ν_j is finite:

$$\mathbb{P}\Big(\Delta_{j+1}^U = \infty | \nu_j < \infty\Big) = \mathbb{P}\big(\cap_{l=1}^{\infty} \{Z_{\nu_j+l} < U_{j+1}\} | \nu_j < \infty\big) = 0. \tag{3}$$

We emphasize that Assumption (**H3**) is required, since otherwise, if $I_0^U \equiv 0$, the process may fail to cross the upper threshold and thus may become extinct (see Theorem 2.1 below). On the other hand, since a strongly subcritical BPRE becomes extinct with probability one, $\Delta_{j+1}^L < \infty$ whenever $\tau_{j+1} < \infty$; that is,

$$\mathbb{P}\Big(\Delta_{j+1}^L < \infty | \tau_{j+1} < \infty\Big) = 1. \tag{4}$$

Using $\nu_0 = 0$ and induction over j, we see that Δ_{j+1}^U, Δ_{j+1}^L, τ_{j+1}, and ν_{j+1} are finite a.s. We emphasize that (4) holds whenever Π^L is a subcritical or critical (but not strongly critical) environmental sequence (see Definition 2.3 in Kersting and Vatutin [16]). That is, it remains valid if the assumption $\mathbb{E}\Big[Y_0^L e^{Y_0^L}\Big] < 0$ in (**H2**) is weakened to $\mathbb{E}\big[Y_0^L\big] \leq 0$ and $\mathbb{P}_{E^L}\big(Y_0^L \neq 0\big) > 0$, which leads to the following assumption:

(**H2′**) $\mathbb{E}\big[Y_0^L\big] \leq 0$, $\mathbb{P}_{E^L}\big(Y_0^L \neq 0\big) > 0$, and $\mathbb{E}\big[Y_0^U\big] > 0$.

The next theorem shows that if immigration is zero, the process becomes extinct a.s.

Theorem 2.1. *Assume (**H1**), (**H2′**), and* $Q_{0,0}^U \equiv 1$ *a.s. Let* $\mathrm{T} := \inf\{n \geq 1 : Z_n = 0\}$. *Then* $\mathbb{P}(\mathrm{T} < \infty) = 1$.

Theorem 1 of Athreya and Schuh [16] follows from the above theorem by taking $L_U = K$, $L_j \equiv K$, $U_j \equiv K + 1$, where K is a finite positive integer, and assuming that the environments are fixed in both regimes.

2.1. Path properties of BPRET

We now turn to transience and recurrence of the BPRET $\{Z_n\}_{n=0}^{\infty}$. Notice that even though $\{Z_n\}_{n=0}^{\infty}$ is not Markov, the concepts of recurrence and transience can be studied using the definition given below (due to [25, 26]).

Definition 2.1. A non-negative stochastic process $\{X_n\}_{n=0}^{\infty}$ satisfying $\mathbb{P}(\limsup_{n\to\infty} X_n = \infty) = 1$ is said to be *recurrent* if there exists an $r < \infty$ such that $\mathbb{P}(\liminf_{n\to\infty} X_n \leq r) = 1$, and *transient* if $\mathbb{P}(\lim_{n\to\infty} X_n = \infty) = 1$.

Our next result is concerned with the path behavior of $\{Z_n\}_{n=0}^{\infty}$ and the stopped sequences $\big\{Z_{\nu_j}\big\}_{j=0}^{\infty}$ and $\big\{Z_{\tau_j}\big\}_{j=1}^{\infty}$.

Theorem 2.2. *Assume (**H1**)–(**H4**). Then*

(i) *the process* $\{Z_n\}_{n=0}^{\infty}$ *is recurrent;*

(ii) $\big\{Z_{\nu_j}\big\}_{j=0}^{\infty}$ *and* $\big\{Z_{\tau_j}\big\}_{j=1}^{\infty}$ *are time-homogeneous Markov chains.*

We now turn to the ergodicity properties of $\big\{Z_{\nu_j}\big\}_{j=0}^{\infty}$ and $\big\{Z_{\tau_j}\big\}_{j=1}^{\infty}$. These rely on conditions on the offspring distribution that ensure that the Markov chains $\big\{Z_{\nu_j}\big\}_{j=0}^{\infty}$ and $\big\{Z_{\tau_j}\big\}_{j=1}^{\infty}$ are irreducible and aperiodic. While several sufficient conditions are possible, we provide below some possible conditions:

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(H5) $\mathbb{P}_{E^L}\big(\cap_{r=0}^{1}\{P_{0,r}^L>0\}\big)>0$.

(H6) $\mathbb{P}_{E^U}\big(\cap_{r=0}^{\infty}\{P_{0,r}^U>0\}\cap\{Q_{0,0}^U>0\}\big)>0$ and $\mathbb{P}_{E^U}\big(Q_{0,s}^U>0\big)>0$ for some $s\in\{1,\ldots,L_U\}$.

(H7) $\mathbb{P}_{E^U}\big(\{P_{0,0}^U>0\}\cap\cap_{r=L_U+1}^{\infty}\{Q_{0,r}^U>0\}\big)>0$.

The condition (**H5**) requires that on a set of positive $\mathbb{P}_{E^L}$ probability, an individual can produce zero and one offspring, while (**H6**) requires that on a set of positive $\mathbb{P}_{E^U}$ probability, $P_{0,r}^U>0$ for all $r\in\mathbb{N}_0$ and $Q_{0,0}^U>0$. Also, on a set of positive $\mathbb{P}_{E^U}$ probability, $Q_{0,s}^U>0$ for some $s\in\{1,\ldots,L_U\}$. Finally, (**H7**) states that on a set of positive $\mathbb{P}_{E^U}$ probability, $P_{0,0}^U>0$ and $Q_{0,r}^U>0$ for all $r\geq L_U+1$. These are weak conditions on the environment sequences and are part of the standard BPRE literature. We recall that S^L is the set of non-negative integers not larger than L_U, and S^U is the set of integers larger than L_U.

Theorem 2.3. *Assume (**H1**)–(**H4**). (i) If (**H5**) also holds, then $\{Z_{\nu_j}\}_{j=0}^{\infty}$ is a uniformly ergodic Markov chain with state space S^L. (ii) If (**H6**) (or (**H7**)) holds, then $\{Z_{\tau_j}\}_{j=1}^{\infty}$ is a uniformly ergodic Markov chain with state space S^U.*

When the assumptions (**H1**)–(**H6**) (or (**H7**)) hold, we denote by $\pi^L=\{\pi_i^L\}_{i\in S^L}$ and $\pi^U=\{\pi_i^U\}_{i\in S^U}$ the stationary distributions of the ergodic Markov chains $\{Z_{\nu_j}\}_{j=0}^{\infty}$ and $\{Z_{\tau_j}\}_{j=1}^{\infty}$, respectively. While π^L has moments of all orders, we show in Proposition A.1 below that π^U has a finite first moment. These distributions will play a significant role in the study of the lengths of the supercritical and subcritical regimes, which we now undertake.

2.2. Lengths of supercritical and subcritical regimes

We now turn to the law of large numbers and central limit theorem for the differences Δ_j^U and Δ_j^L. We denote by $\mathbb{P}_{\pi^L}(\cdot)$, $\mathbb{E}_{\pi^L}[\cdot]$, $\mathbb{V}_{\pi^L}[\cdot]$, and $\mathbb{C}_{\pi^L}[\cdot,\cdot]$ the probability, expectation, variance, and covariance conditionally on $Z_{\nu_0}\sim\pi^L$. Similarly, when π^L is replaced by π^U in the above quantities, we understand that they are conditioned on $Z_{\tau_1}\sim\pi^U$. We define $\mu^U:=\mathbb{E}_{\pi^L}\big[\Delta_1^U\big]$, $\mu^L:=\mathbb{E}_{\pi^U}\big[\Delta_1^L\big]$,

$$\sigma^{2,U}:=\mathbb{V}_{\pi^L}\big[\Delta_1^U\big]+2\sum_{j=1}^{\infty}\mathbb{C}_{\pi^L}\big[\Delta_1^U,\Delta_{j+1}^U\big],\qquad\text{and}\tag{5}$$

$$\sigma^{2,L}:=\mathbb{V}_{\pi^U}\big[\Delta_1^L\big]+2\sum_{j=1}^{\infty}\mathbb{C}_{\pi^U}\big[\Delta_1^L,\Delta_{j+1}^L\big].\tag{6}$$

In the supercritical regime, we impose the additional assumption (**H8**) below, to avoid needing to qualify our statements with the phrase 'on the set of non-extinction'. Assumption (**H9**) below ensures that the immigration distribution stochastically dominates the upper threshold.

(H8) $P_{0,0}^U=0$ $\mathbb{P}_{E^U}$-a.s.

(H9) $\mathbb{E}\Big[\frac{U_1}{\mathbb{P}(I_0^U\geq U_1|U_1)}\Big]<\infty$.

Let $S_n^U:=\sum_{j=1}^{n}\Delta_j^U$ and $S_n^L:=\sum_{j=1}^{n}\Delta_j^L$. We now state the main result of this subsection.

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Theorem 2.4. *Assume (H1)–(H4). (i) If (H5) and (H8) hold, then*

$$\lim_{n\to\infty} \frac{1}{n} S_n^U = \mu^U \text{ a.s.,} \quad \text{and} \quad \frac{1}{\sqrt{n}}\big(S_n^U - n\mu^U\big) \xrightarrow[n\to\infty]{d} N\big(0, \sigma^{2,U}\big).$$

(ii) If (H6) (or (H7)) and (H9) hold, then

$$\lim_{n\to\infty} \frac{1}{n} S_n^L = \mu^L \text{ a.s.,} \quad \text{and} \quad \frac{1}{\sqrt{n}}\big(S_n^L - n\mu^L\big) \xrightarrow[n\to\infty]{d} N(0, \sigma^{2,L}).$$

2.3. Proportion of time spent in supercritical and subcritical regimes

We now consider the proportion of time the process spends in the subcritical and supercritical regimes. To this end, for $n \geq 0$, let $\chi_n^U := \mathbf{I}_{\cup_{j=1}^{\infty}[\nu_{j-1}, \tau_j)}(n)$ be the indicator function assuming value 1 if at time n the process is in the supercritical regime and 0 otherwise. Similarly, let $\chi_n^L := 1 - \chi_n^U = \mathbf{I}_{\cup_{j=1}^{\infty}[\tau_j, \nu_j)}(n)$ take value 1 if at time n the process is in the subcritical regime and 0 otherwise. Furthermore, let $C_n^U := \sum_{j=1}^n \chi_{j-1}^U$ and $C_n^L := \sum_{j=1}^n \chi_{j-1}^L = n - C_n^U$ be the total time that the process spends in the supercritical and the subcritical regime, respectively, up to time $n-1$. Let

$$\theta_n^U := \frac{C_n^U}{n} \quad \text{and} \quad \theta_n^L := \frac{C_n^L}{n}$$

denote the proportion of time the process spends in the supercritical and the subcritical regime. Our main result in this section is concerned with the central limit theorem for θ_n^U and θ_n^L. To this end, let

$$\theta^U := \frac{\mu^U}{\mu^U + \mu^L} \quad \text{and} \quad \theta^L := \frac{\mu^L}{\mu^U + \mu^L}.$$

Theorem 2.5. *Assume (H1)–(H6) (or (H7)) and (H8)–(H9). Then, for $T \in \{L, U\}$, θ_n^T converges a.s. to θ^T. Furthermore,*

$$\sqrt{n}(\theta_n^T - \theta^T) \xrightarrow[n\to\infty]{d} N(0, \eta^{2,T}),$$

where $\eta^{2,T}$ is defined in (19).

We now use these results to describe the growth rate of the process in the supercritical and subcritical regime, as defined by their expectations (that is, M^U and M^L).

2.4. Offspring mean estimation

We begin by noticing that $Z_{\tau_j} \geq L_U + 1$ and $Z_{\tau_j+1}, \ldots, Z_{\nu_j - 1} \geq L_0$ are positive for all $j \in \mathbb{N}$. However, there may be instances where $Z_{\nu_j}, \ldots, Z_{\tau_j - 1}$ could be zero. To avoid division by zero in (7) below, we let $\tilde{\chi}_n^U := \chi_n^U \mathbf{I}_{\{Z_n \geq 1\}}$, $\tilde{C}_n^U := \sum_{j=1}^n \tilde{\chi}_{j-1}^U$, and use the convention that $0/0 = 0 \cdot \infty = 0$. The generalized method-of-moments estimators of M^U and M^L are given by

$$M_n^U := \frac{1}{\tilde{C}_n^U} \sum_{j=1}^n \frac{Z_j - I_{j-1}^U}{Z_{j-1}} \tilde{\chi}_{j-1}^U \quad \text{and} \quad M_n^L := \frac{1}{C_n^L} \sum_{j=1}^n \frac{Z_j}{Z_{j-1}} \chi_{j-1}^L, \tag{7}$$

where the last term is non-trivial whenever $C_n^L \geq 1$, that is, $n \geq \tau_1 + 1$. Our assumptions will involve first- and second-moment assumptions on the centered offspring means $\left(\overline{P}_n^T - M^T\right)$

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and the centered offspring random variables $\left(\xi_{n,i}^T - \overline{P}_n^T\right)$. To this end, we define the quantities $\Lambda_{n,1}^{T,s} := \left|\overline{P}_n^T - M^T\right|^s$ and $\Lambda_{n,2}^{T,s} := \mathbb{E}\left[|\xi_{n,1} - \overline{P}_n^T|^s|\Pi_n^T\right]$. Next, let $\boldsymbol{M}_n := \left(M_n^U, M_n^L\right)^\top$, $\boldsymbol{M} := \left(M^U, M^L\right)^\top$, and let Σ be the 2×2 diagonal matrix with elements

$$\frac{1}{\tilde{\theta}^U}\left(V_1^U + \frac{\tilde{A}^U V_2^U}{\tilde{\mu}^U}\right) \quad \text{and} \quad \frac{1}{\theta^L}\left(V_1^L + \frac{A^L V_2^L}{\mu^L}\right),$$

where $\tilde{\mu}^U := \mathbb{E}_{\pi^L}\left[\sum_{k=1}^{\tau_1} \tilde{\chi}_{k-1}^U\right]$ is the average length of supercritical regime, not taking into account the times at which the process is zero;

$$\tilde{\theta}^U := \frac{\tilde{\mu}^U}{\mu^U + \mu^L}$$

is the average proportion of time the process spends in the supercritical regime and is positive;

$$\tilde{A}^U := \mathbb{E}_{\pi^L}\left[\sum_{k=1}^{\tau_1} \frac{\tilde{\chi}_{k-1}^U}{Z_{k-1}}\right]$$

is the average sum of $\frac{1}{Z_n}$ over a supercritical regime, discarding the times at which Z_n is zero; and

$$A^L := \mathbb{E}_{\pi^U}\left[\sum_{k=\tau_1+1}^{\nu_1} \frac{\chi_{k-1}^L}{Z_{k-1}}\right]$$

is the average sum of $\frac{1}{Z_n}$ over a subcritical regime. Obviously, $0 \le \tilde{\mu}^U \le \mu^U$. Finally, we recall that $V_1^T = \mathbb{V}\left[\overline{P}_0^T\right]$ is the variance of the random offspring mean $\overline{P}_0^T$ and $V_2^T = \mathbb{E}\left[\overline{\overline{P}}_0^T\right]$ is the expectation of the random offspring variance $\overline{\overline{P}}_0^T$.

Theorem 2.6. *Assume (H1)–(H6) (or (H7)). (i) If* $\mu^T < \infty$ *and if* $\mathbb{E}\left[\Lambda_{0,i}^{T,s}\right] < \infty$ *for some* $s > 1$*, where* $i = 1, 2$ *and* $T \in \{L, U\}$*, then* $\boldsymbol{M}_n$ *is a strongly consistent estimator of* $\boldsymbol{M}$*. (ii) If additionally for some* $\delta > 0$ $\mathbb{E}\left[\Lambda_{0,i}^{T,2+\delta}\right] < \infty$ *for* $i = 1, 2$ *and* $T \in \{L, U\}$*, then*

$$\sqrt{n}(\boldsymbol{M}_n - \boldsymbol{M}) \xrightarrow[n\to\infty]{d} N(\boldsymbol{0}, \Sigma).$$

Remark 2.1. In the fixed-environment case, $\overline{P}_0^T = M^T$ and $\overline{\overline{P}}_0^T = V_2^T$ are deterministic constants. Therefore, $V_1^T = 0$, and Σ is the 2×2 diagonal matrix with elements

$$\frac{\tilde{A}^U V_2^U}{\tilde{\theta}^U \tilde{\mu}^U} \quad \text{and} \quad \frac{A^L V_2^L}{\theta^L \mu^L}.$$

3. Path properties of BPRET

In this section we provide the proofs of Theorems 2.1, 2.2, and 2.3, along with the required probability estimates. The proofs rely on the fact that both the environmental sequence and the thresholds are i.i.d. It follows that probability statements like $\mathbb{P}(Z_{\tau_{j+1}} = k|Z_{\nu_j} = i, \nu_j < \infty)$

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and $\mathbb{P}(Z_{\nu_{j+1}} = i | Z_{\tau_{j+1}} = k, \tau_{j+1} < \infty)$ do not depend on the index j. This idea is made precise in Lemma A.1 in Appendix A.2 and will lead to time-homogeneity of $\{Z_{\nu_j}\}_{j=0}^{\infty}$ and $\{Z_{\tau_j}\}_{j=1}^{\infty}$. As expected, this property does not depend on the process being strongly subcritical: Assumptions (**H1**) and (**H2′**) are more than enough. We denote by $\mathbb{P}_{\delta_i^L}(\cdot)$, $\mathbb{E}_{\delta_i^L}[\cdot]$, $\mathbb{V}_{\delta_i^L}[\cdot]$, and $\mathbb{C}_{\delta_i^L}[\cdot, \cdot]$ the probability, expectation, variance, and covariance conditionally on $Z_{\nu_0} \sim \delta_i^L$, where δ_θ^L is the restriction of the Dirac delta to S^L. Similarly, when δ_i^L is replaced by δ_i^U in the above quantities, we understand that they are conditioned on $Z_{\tau_1} \sim \delta_i^U$, where δ_θ^U is the restriction of the Dirac delta to S^U.

3.1. Extinction when immigration is zero

In this subsection, we provide the proof of Theorem 2.1, which is an adaptation of Theorem 1 of Athreya and Schuh [17] for BPRE. Recall that for this theorem there is no immigration in the supercritical regime, and hence the extinction time T is finite with probability one.

Proof of Theorem 2.1. For simplicity, set $\tau_0 := -1$. We partition the sample space as

$$\Omega = \left(\cup_{j=0}^{\infty} \{\tau_{j+1} = \infty, \tau_j < \infty\}\right) \cup \left(\cap_{j=1}^{\infty} \{\tau_j < \infty\}\right)$$

and show that (i) $\{\tau_{j+1} = \infty, \tau_j < \infty\} \subset \{\mathrm{T} < \infty\}$ for all $j \in \mathbb{N}_0$ and (ii) $\mathbb{P}\left(\cap_{j=1}^{\infty} \{\tau_j < \infty\}\right) = 0$. First, we notice that if $\tau_j < \infty$, then $\nu_j < \infty$ by Theorem 2.1 of Kersting and Vatutin [16]. Thus,

$$\{\tau_{j+1} = \infty, \tau_j < \infty\} = \{Z_n < U_{j+1} \ \forall n \geq \nu_j, \tau_j < \infty\},$$

where $\{Z_n\}_{n=\nu_j}^{\infty}$ is a supercritical BPRE until U_{j+1} is reached. Since $Z_n < U_{j+1}$ for all $n \geq \nu_j$, (2.6) of Kersting and Vatutin [16] yields that $\lim_{n\to\infty} Z_n = 0$ a.s. and $\{\tau_{j+1} = \infty, \tau_j < \infty\} \subset \{\mathrm{T} < \infty\}$. Turning to (ii), since the events $\{\tau_j < \infty\}$ are nonincreasing, $\mathbb{P}\left(\cap_{j=1}^{\infty} \{\tau_j < \infty\}\right) = \lim_{j\to\infty} \mathbb{P}(\tau_{j+1} < \infty)$ and $\mathbb{P}(\tau_{j+1} < \infty) = \mathbb{P}(\tau_{j+1} < \infty | \tau_j < \infty)\mathbb{P}(\tau_j < \infty)$. Since $\tau_j = \infty$, if $Z_{\nu_{j-1}} = 0$, it follows that

$$\begin{aligned}
\mathbb{P}(\tau_{j+1} < \infty | \tau_j < \infty) &\leq \mathbb{P}\left(\tau_{j+1} < \infty | \tau_j < \infty, Z_{\nu_{j-1}} \in [1, L_U]\right) \\
&\leq \max_{i=1,\ldots,L_U} \mathbb{P}\left(\tau_{j+1} < \infty | \tau_j < \infty, Z_{\nu_{j-1}} = i\right) \\
&\leq 1 - \min_{i=1,\ldots,L_U} \mathbb{P}\left(Z_{\tau_j+1} = 0 | \tau_j < \infty, Z_{\nu_{j-1}} = i\right).
\end{aligned}$$

Lemma A.1 yields that for all $k \in S_B^U$,

$$\mathbb{P}\left(Z_{\tau_j} = k | \tau_j < \infty, Z_{\nu_{j-1}} = i\right) = \mathbb{P}_{\delta_i^L}\left(Z_{\tau_1} = k | \tau_1 < \infty\right). \tag{8}$$

Also, for all $j \geq 1$,

$$\begin{aligned}
\mathbb{P}\left(Z_{\tau_j+1} = 0 | \tau_j < \infty, Z_{\tau_j} = k\right) &= \mathbb{P}\left(\cap_{i=1}^{k} \left\{\xi_{\tau_j,i}^L = 0\right\} | \tau_j < \infty\right) \\
&= \sum_{n=1}^{\infty} \mathbb{P}\left(\cap_{i=1}^{k} \left\{\xi_{n,i}^L = 0\right\} | \tau_j = n\right)\mathbb{P}(\tau_j = n) \\
&= \mathbb{P}\left(\xi_{0,1}^L = 0\right)^k.
\end{aligned}$$

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Multiplying by $\mathbb{P}\big(Z_{\tau_j+1}=0|\tau_j<\infty, Z_{\tau_j}=k\big)$ and $\mathbb{P}\big(Z_{\tau_1+1}=0|\tau_1<\infty, Z_{\tau_1}=k\big)$ and summing over $k\geq L_U+1$ in (8), we obtain that

$$\mathbb{P}\big(Z_{\tau_j+1}=0|\tau_j<\infty, Z_{\nu_{j-1}}=i\big)=\mathbb{P}_{\delta_i^L}\big(Z_{\tau_1+1}=0|\tau_1<\infty\big)$$
$$=\sum_{k=L_U+1}^{\infty}\mathbb{P}_{\delta_i^L}\big(Z_{\tau_1}=k|\tau_1<\infty\big)\mathbb{P}\big(\xi_{0,1}^L=0\big)^k.$$

Set $\underline{p}:=\min_{i=1,\ldots,L_U}p_i$, where $p_i:=\mathbb{P}_{\delta_i^L}\big(Z_{\tau_1+1}=0|\tau_1<\infty\big)$. Since $\mathbb{P}\big(\xi_{0,1}^L=0\big)>0$ and $\mathbb{P}_{\delta_i^L}\big(Z_{\tau_1}=k|\tau_1<\infty\big)>0$ for some k, we have that $p_i>0$ and $\underline{p}>0$. Hence, $\mathbb{P}(\tau_{j+1}<\infty)\leq(1-\underline{p})\mathbb{P}(\tau_j<\infty)$. Iterating the above argument, it follows that $\mathbb{P}(\tau_{j+1}<\infty)\leq\big(1-\underline{p}\big)^j\mathbb{P}(\tau_1<\infty)$, yielding $\lim_{j\to\infty}\mathbb{P}(\tau_{j+1}<\infty)=0$. □

3.2. Markov property at crossing times

Proof of Theorem 2.2. We begin by proving (i). We first notice that since $\{U_j\}_{j=1}^{\infty}$ are i.i.d. random variables with unbounded support S_B^U, $\limsup_{j\to\infty}U_j=\infty$ with probability one. Next, observe that along the subsequence $\{\tau_j\}_{j=1}^{\infty}$, $Z_{\tau_j}\geq U_j$. Hence, $\limsup_{n\to\infty}Z_n=\infty$. On the other hand, along the subsequence $\{\nu_j\}_{j=1}^{\infty}$, we have $Z_{\nu_j}\leq L_j$. Thus, $0\leq\liminf_{j\to\infty}Z_j\leq L_U<\infty$. It follows that $\{Z_n\}_{n=0}^{\infty}$ is recurrent in the sense of Definition 2.1. Turning to (ii), we first notice that, since $Z_0\leq L_U$, $Z_{\nu_j}\leq L_j\leq L_U$, and $Z_{\tau_j}\geq U_j\geq L_U+1$ for all $j\geq 1$, the state spaces S^L of $\big\{Z_{\nu_j}\big\}_{j=0}^{\infty}$ and S^U of $\big\{Z_{\tau_j}\big\}_{j=1}^{\infty}$ are included in S^L and S^U, respectively. We now establish the Markov property of $\big\{Z_{\nu_j}\big\}_{j=0}^{\infty}$. For all $j\geq 0$, $k\in S^L$, and $i_0, i_1,\ldots,i_j\in S^L$, we consider the probability $\mathbb{P}(Z_{\nu_{j+1}}=k|Z_{\nu_0}=i_0,\ldots,Z_{\nu_j}=i_j)$. By the law of total expectation, this is equal to

$$\mathbb{E}\big[\mathbb{P}\big(Z_{\nu_{j+1}}=k|Z_{\nu_0}=i_0,\ldots,Z_{\nu_j}=i_j, L_{j+1}, U_{j+1},\pi^L,\pi^U\big)\big].$$

Now, setting $A_{\nu_j,s}(u):=\{Z_{\nu_j+s}\geq u, Z_{\nu_j+s-1}<u,\ldots,Z_{\nu_j+1}<u\}$, $B_{\nu_j,s,t}(l):=\{Z_{\nu_j+t}=k, Z_{\nu_j+t-1}>l,\ldots,Z_{\nu_j+s+1}>l\}$, we have that

$$\mathbb{P}\big(Z_{\nu_{j+1}}=k|Z_{\nu_0}=i_0,\ldots,Z_{\nu_j}=i_j, L_{j+1}, U_{j+1},\pi^L,\pi^U\big)$$
$$=\sum_{s=1}^{\infty}\sum_{t=s+1}^{\infty}\mathbb{P}\big(A_{\nu_j,s}(U_{j+1})|Z_{\nu_j}=i_j, U_{j+1},\pi^U\big)\mathbb{P}\big(B_{\nu_j,s,t}(L_{j+1})|A_{\nu_j,s}, L_{j+1},\pi^L\big)$$
$$=\mathbb{P}\big(Z_{\nu_{j+1}}=k|Z_{\nu_j}=i_j, L_{j+1}, U_{j+1},\pi^L,\pi^U\big),$$

where in the second line we have used that $\{Z_n\}_{n=\nu_j}^{\infty}$ is a supercritical BPRE with immigration until it crosses the threshold U_{j+1} at time $\tau_{j+1}=\nu_j+s$, and similarly $\{Z_n\}_{n=\tau_{j+1}}^{\infty}$ is a subcritical BPRE until it crosses the threshold L_{j+1} at time $\nu_{j+1}=\nu_j+t$. By taking the expectation on both sides, we obtain that

$$\mathbb{P}\big(Z_{\nu_{j+1}}=k|Z_{\nu_0}=i_0,\ldots,Z_{\nu_j}=i_j\big)=\mathbb{P}\big(Z_{\nu_{j+1}}=k|Z_{\nu_j}=i_j\big).$$

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Turning to the time-homogeneity property, we obtain from Lemma A.1(iii) that

$$\begin{aligned}\mathbb{P}\big(Z_{\nu_{j+1}}=k|Z_{\nu_j}=i_j\big) &= \sum_{l=L_U+1}^{\infty} \mathbb{P}\big(Z_{\nu_{j+1}}=k|Z_{\tau_{j+1}}=l\big)\mathbb{P}\big(Z_{\tau_{j+1}}=l|Z_{\nu_j}=i_j\big)\\ &= \sum_{l=L_U+1}^{\infty} \mathbb{P}\big(Z_{\nu_1}=k|Z_{\tau_1}=l\big)\mathbb{P}\big(Z_{\tau_1}=l|Z_{\nu_0}=i_j\big)\\ &= \mathbb{P}\big(Z_{\nu_1}=k|Z_{\nu_0}=i_j\big).\end{aligned}$$

The proof for $\{Z_{\tau_j}\}_{j=1}^{\infty}$ is similar. □

3.3. Uniform ergodicity of $\{Z_{\nu_j}\}_{j=0}^{\infty}$ and $\{Z_{\tau_j}\}_{j=1}^{\infty}$

In this subsection we prove Theorem 2.3. The proof relies on the following lemma. We denote by $p_{ik}^L(j)=\mathbb{P}_{\delta_i^L}(Z_{\nu_j}=k)$, $i, k\in S^L$, and $p_{ik}^U(j)=\mathbb{P}_{\delta_i^U}(Z_{\tau_{j+1}}=k)$, $i, k\in S^U$, the j-step transition probability of the (time-homogeneous) Markov chains $\{Z_{\nu_j}\}_{j=0}^{\infty}$ and $\{Z_{\tau_j}\}_{j=1}^{\infty}$. For $j=1$, we also write $p_{ik}^L=p_{ik}^L(1)$ and $p_{ik}^U=p_{ik}^U(1)$. Finally, let $p_i^L(j)=\{p_{ik}^L(j)\}_{k\in S^L}$ and $p_i^U(j)=\{p_{ik}^U(j)\}_{k\in S^U}$ be the j-step transition probability of the Markov chains $\{Z_{\nu_j}\}_{j=0}^{\infty}$ and $\{Z_{\tau_j}\}_{j=1}^{\infty}$ from state $i\in S^L$ (resp. $i\in S^U$).

Lemma 3.1. *Assume (**H1**)–(**H4**). Then (i) if (**H5**) also holds, then $p_{ik}^L\geq \underline{p}^L>0$ for all $i, k\in S^L$. Also, (ii) if 2 (or 3) holds, then $p_{ik}^U\geq \underline{p}_k^U>0$ for all $i, k\in S^U$.*

Proof of Lemma 3.1. The idea of proof is to establish a lower bound on p_{ik}^L and p_{ik}^U using (9) and (10) below, respectively. We begin by proving (i). Using Assumption (**H5**), let A be a measurable subset of $\mathcal{P}$ satisfying $\mathbb{P}_{E^L}\big(\Pi_0^L\in A\big)>0$ and $P_{0,r}^L>0$ for $r=0$, 1 and $\Pi_0^L\in A$. By the law of total expectation,

$$p_{ik}^L=\mathbb{E}_{\delta_i^L}\big[\mathbb{P}\big(Z_{\nu_1}=k|Z_{\tau_1}, L_1, \pi^L\big)\big]. \tag{9}$$

Since $\mathbb{P}\big(Z_{\nu_1}=k|Z_{\tau_1}, L_1, \pi^L\big)=0$ on the event $\{L_1<k\}$, it follows that

$$\mathbb{P}\big(Z_{\nu_1}=k|Z_{\tau_1}, L_1, \pi^L\big)=\mathbb{P}\big(Z_{\nu_1}=k|Z_{\tau_1}, L_1, \pi^L\big)\mathbf{I}_{\{L_1\geq k\}}.$$

Now, notice that on the event $\{L_1\geq k\}$, the term $\mathbb{P}\big(Z_{\nu_1}=k|Z_{\tau_1}, L_1, \pi^L\big)$ is bounded below by the probability of reaching state k from Z_{τ_1} in one step; that is,

$$\mathbb{P}\big(Z_{\nu_1}=k|Z_{\tau_1}, L_1, \pi^L\big)\geq \mathbb{P}\big(Z_{\nu_1}=k, \nu_1=\tau_1+1|Z_{\tau_1}, L_1, \pi^L\big)\mathbf{I}_{\{L_1\geq k\}}.$$

The right-hand side of the above inequality is bounded below by the probability that the first k individuals have exactly one offspring and the remaining $Z_{\tau_1}-k$ have no offspring; that is,

$$\mathbf{I}_{\{L_1\geq k\}}\prod_{r=1}^{k}\mathbb{P}\big(\xi_{\tau_1,r}^L=1|\Pi_{\tau_1}^L\big)\prod_{r=k+1}^{Z_{\tau_1}}\mathbb{P}\big(\xi_{\tau_1,r}^L=0|\Pi_{\tau_1}^L\big).$$

Once again, using that, conditional on the environment $\Pi_{\tau_1}^L$, the $\xi_{\tau_1,r}^L$ are i.i.d., this is equal to

$$\mathbf{I}_{\{L_1\geq k\}}\big(P_{\tau_1,1}^L\big)^k\big(P_{\tau_1,0}^L\big)^{Z_{\tau_1}-k}.$$

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Since $\mathbf{I}_{\{\Pi^L_{\tau_1}\in A\}}\le 1$ and $\{L_1\ge k\}\supset\{L_1=L_U\}$ because $k\le L_U$, the last term is bounded below by

$$\mathbf{I}_{\{L_1=L_U\}}\big(P^L_{\tau_1,1}\big)^k\big(P^L_{\tau_1,0}\big)^{Z_{\tau_1}-k}\mathbf{I}_{\{\Pi^L_{\tau_1}\in A\}}.$$

Finally, again using that the Π^L_n are i.i.d., and taking the expectation $\mathbb{E}_{\delta^L_i}[\cdot]$ as in (9), we obtain that $p^L_{ik}\ge \underline{p}^L_{ik}$, where

$$\underline{p}^L_{ik}:=\mathbb{P}\big(L_1=L_U\big)\mathbb{E}_{\delta^L_i}\Big[\big(P^L_{\tau_1,1}\big)^k\big(P^L_{\tau_1,0}\big)^{Z_{\tau_1}-k}\mathbf{I}_{\{\Pi^L_{\tau_1}\in A\}}\Big].$$

Notice that $\underline{p}^L_k$ is positive, because $\mathbb{P}(L_1=L_U)>0$, $\mathbb{P}_{E^L}\big(\Pi^L_0\in A\big)>0$, $P^L_{0,r}>0$ for $r=0$, 1, and $\Pi^L_0\in A$, and the environments Π^L_n are i.i.d. Finally, since S^L is finite, $\underline{p}^L:=\min_{i,k\in S^L}\underline{p}^L_{ik}>0$.

We now turn to the proof of (ii), which is similar to the proof of (i). Using (**H6**), let A and B be measurable subsets of $\mathcal{P}\times\mathcal{P}$ satisfying the following conditions:

(a) $\mathbb{P}_{E^U}\big(\Pi^U_0\in A\big)>0$ and $Q^U_{0,0}>0$, $P^U_{0,r}>0$ for all $r\in\mathbb{N}_0$ and $\Pi^U_0\in A$; and

(b) $\mathbb{P}_{E^U}\big(\Pi^U_0\in B\big)>0$ and $Q^U_{0,s}>0$ for some (fixed) $s\in\{1,\ldots,L_u\}$ and $\Pi^U_0\in B$.

Again using the law of total expectation, we obtain

$$p^U_{ik}=\mathbb{E}_{\delta^U_i}\big[\mathbb{P}\big(Z_{\tau_2}=k|Z_{\nu_1},U_2,\pi^U\big)\big]. \tag{10}$$

Since $\mathbb{P}\big(Z_{\tau_2}=k|Z_{\nu_1},U_2,\pi^U\big)=0$ on the event $\{U_2>k\}$, it follows that

$$\mathbb{P}\big(Z_{\tau_2}=k|Z_{\nu_1},U_2,\pi^U\big)=\sum_{z=0}^{L_U}\mathbb{P}\big(Z_{\tau_2}=k|Z_{\nu_1},U_2,\pi^U\big)\mathbf{I}_{\{U_2\le k\}}I_{\{Z_{\nu_1}=z\}}. \tag{11}$$

If $U_2\le k$ and $Z_{\nu_1}=z>0$, then $\mathbb{P}\big(Z_{\tau_2}=k|Z_{\nu_1},U_2,\pi^U\big)$ is bounded below by the probability that z individuals have a total of exactly k offspring and no immigration occurs; that is,

$$\begin{aligned}&\mathbb{P}\big(Z_{\tau_2}=k|Z_{\nu_1},U_2,\pi^U\big)\mathbf{I}_{\{U_2\le k\}}I_{\{Z_{\nu_1}=z\}}\\ \ge&\mathbb{P}\left(\sum_{r=1}^{z}\xi^U_{\nu_1,r}=k,\,I^U_{\nu_1}=0|Z_{\nu_1},U_2,\pi^U\right)\mathbf{I}_{\{U_2\le k\}}I_{\{Z_{\nu_1}=z\}}.\end{aligned}$$

The right-hand side of the above inequality is bounded below by the probability that the first $z_1:=(k_1+1)z-k$ individuals have $k_1:=\lfloor\frac{k}{z}\rfloor$ offspring and the last $z_2:=z-z_1$ individuals have $k_2:=k_1+1$ offspring (indeed $k_1z_1+k_2z_2=k$) and no immigration occurs—that is, by

$$\mathbb{P}\big(\cap_{r=1}^{z_1}\{\xi_{\nu_1,r}=k_1\},\cap_{r=z_1+1}^{z_2}\{\xi_{\nu_1,r}=k_2\},I^U_{\nu_1}=0|Z_{\nu_1},U_2,\pi^U\big)\mathbf{I}_{\{U_2\le k\}}I_{\{Z_{\nu_1}=z\}}.$$

Using that, conditional on the environment, $\Pi^U_{\nu_1}$, $\xi^U_{\nu_1,r}$ are i.i.d., the above is equal to

$$\Big(P^U_{\nu_1,k_1}\Big)^{z_1}\Big(P^U_{\nu_1,k_2}\Big)^{z_2}Q^U_{\nu_1,0}\mathbf{I}_{\{U_2\le k\}}I_{\{Z_{\nu_1}=z\}}. \tag{12}$$

Next, if $U_2\le k$ and $Z_{\nu_1}=z=0$, then $\mathbb{P}\big(Z_{\tau_2}=k|Z_{\nu_1},U_2,\pi^U\big)$ is bounded below by the probability $\mathbb{P}\big(Z_{\tau_2}=k,\tau_2=\nu_1+2|Z_{\nu_1},U_2,\pi^U\big)$. Now, this probability is bounded below by the

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probability that there are s immigrants at time ν_1+1, these immigrants have a total of exactly k offspring, and no immigration occurs at time ν_1+2—that is, by

$$\mathbb{P}\left(I^U_{\nu_1}=s,\ \sum_{r=1}^{s}\xi^U_{\nu_1+1,r}=k,\ I^U_{\nu_1+1}=0|Z_{\nu_1},\ U_2,\ \pi^U\right)\mathbf{I}_{\{U_2\le k\}}I_{\{Z_{\nu_1}=0\}}.$$

As before, this last probability is bounded below by the probability that $s_1:=(t_1+1)s-k$ individuals have $t_1:=\lfloor\frac{k}{s}\rfloor$ offspring and $s_2:=s-s_1$ individuals have $t_2:=t_1+1$ offspring. Thus, the above probability is bounded below by

$$Q^U_{\nu_1,s}\big(P^U_{\nu_1+1,t_1}\big)^{s_1}\big(P^U_{\nu_1+1,t_2}\big)^{s_2}Q^U_{\nu_1+1,0}\mathbf{I}_{\{U_2\le k\}}\mathbf{I}_{\{Z_{\nu_1}=0\}}. \tag{13}$$

Combining (11), (12), and (13) and using that $\mathbf{I}_{\{\Pi^U_{\nu_1}\in A\}},\ \mathbf{I}_{\{\Pi^U_{\nu_1+1}\in A\}},\ \mathbf{I}_{\{\Pi^U_{\nu_1}\in B\}}\le 1$, we obtain that $\mathbb{P}(Z_{\tau_2}=k|Z_{\nu_1},\ U_2,\ \pi^U)$ is bounded below by

$$\begin{aligned}&\mathbf{I}_{\{U_2\le k\}}\sum_{z=1}^{L_U}\mathbf{I}_{\{Z_{\nu_1}=z\}}\left(P^U_{\nu_1,k_1}\right)^{z_1}\left(P^U_{\nu_1,k_2}\right)^{z_2}Q^U_{\nu_1,0}\mathbf{I}_{\{\Pi^U_{\nu_1}\in A\}}\\ &+\mathbf{I}_{\{U_2\le k\}}\mathbf{I}_{\{Z_{\nu_1}=0\}}Q^U_{\nu_1,s}\left(P^U_{\nu_1+1,t_1}\right)^{s_1}\left(P^U_{\nu_1+1,t_2}\right)^{s_2}Q^U_{\nu_1+1,0}\mathbf{I}_{\{\Pi^U_{\nu_1}\in B\}}\mathbf{I}_{\{\Pi^U_{\nu_1+1}\in A\}}.\end{aligned}$$

Using that the Π^U_n are i.i.d. and taking the expectation $\mathbb{E}_{\delta^U_i}[\cdot]$ as in (10), we obtain that

$$p^U_{ik}\ge\mathbb{P}(U_1\le k)\sum_{z=0}^{L_U}H_k(z)\mathbb{P}_{\delta^U_i}(Z_{\nu_1}=z),$$

where $H_k:S^L\to\mathbb{R}$ is given by

$$H_k(z)=\begin{cases}\mathbb{E}\Big[Q^U_{\nu_1,s}\left(P^U_{\nu_1+1,t_1}\right)^{s_1}\left(P^U_{\nu_1+1,t_2}\right)^{s_2}Q^U_{\nu_1+1,0}\mathbf{I}_{\{\Pi^U_{\nu_1}\in B\}}\mathbf{I}_{\{\Pi^U_{\nu_1+1}\in A\}}\Big] & \text{if } z=0,\\ \mathbb{E}\Big[\left(P^U_{\nu_1,k_1}\right)^{z_1}\left(P^U_{\nu_1,k_2}\right)^{z_2}Q^U_{\nu_1,0}\mathbf{I}_{\{\Pi^U_{\nu_1}\in A\}}\Big] & \text{if } z\neq 0.\end{cases}$$

Since $\sum_{z=0}^{L_U}\mathbb{P}_{\delta^U_i}\big(Z_{\nu_1}=z\big)=1$, we conclude that $p^U_{ik}\ge\underline{p}^U_k$, where

$$\underline{p}^U_k:=\mathbb{P}(U_1\le k)\min_{z\in S^L}H_k(z)>0.$$

This concludes the proof of (ii). If, instead of 2, 3 holds, the proof is similar: one notices that for all $z\in S^L$, on the event $\{U_1\le k\}\cap\{Z_{\nu_1}=z\}$, there is a positive probability that at time ν_1+1 there are k immigrants and the z individuals have no offspring. A detailed proof can be obtained in the same manner as above. □

Before turning to the proof of Theorem 2.3, we introduce some notation. Let $T^L_{i,0}:=0$ and $T^L_{i,l}:=\inf\big\{j>T^L_{i,l-1}:Z_{\nu_j}=i\big\}$, $l\ge 1$, be the random times at which the Markov chain $\big\{Z_{\nu_j}\big\}_{j=0}^{\infty}$ enters state $i\in S^L$ when the initial state is $Z_{\nu_0}=i$. Similarly, we let $T^U_{i,0}:=1$ and $T^U_{i,l}:=\inf\{j>T^U_{i,l-1}:Z_{\tau_j}=i\}$, $l\ge 1$, be the random times at which the Markov chain $\big\{Z_{\tau_j}\big\}_{j=1}^{\infty}$ enters state $i\in S^U$ when the initial state is $Z_{\tau_1}=i$. The expected times of visiting state k starting from i are denoted by $f^L_{ik}:=\mathbb{E}_{\delta^L_i}\big[T^L_{k,1}\big]$ and $f^U_{ik}:=\mathbb{E}_{\delta^U_i}\big[T^U_{k,1}-1\big]$, respectively.

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Proof of Theorem 2.3. Lemma 3.1 implies that the state spaces of $\{Z_{\nu_j}\}_{j=0}^{\infty}$ and $\{Z_{\tau_j}\}_{j=1}^{\infty}$ are S^L and S^U, respectively. Next, to establish ergodicity of these Markov chains, it is sufficient to verify irreducibility, aperiodicity, and positive recurrence. Irreducibility and aperiodicity follow from Lemma 3.1 in both cases. Now, turning to positive recurrence, let $I_n^L(k) := \{(i_1, i_2, \ldots, i_{n-1}) : i_j \in S^L \setminus \{k\}\}$ for $k \in S^L$. Then, using the Markov property and Part (i) of Lemma 3.1, it follows that

$$f_{kk}^L = \sum_{n=1}^{\infty} \mathbb{P}_{\delta_k^L}\big(T_{k,1}^L \geq n\big) = \sum_{n=1}^{\infty} \mathbb{P}_{\delta_k^L}\Big(\cap_{j=1}^{n-1} \{Z_{\nu_j} \neq k\}\Big)$$

$$= \sum_{n=1}^{\infty} p_{ki_1} \prod_{I_n^L(k)} p_{i_{j-1}i_j} \leq \sum_{n=1}^{\infty} \Big(1 - \underline{p}^L\Big)^{n-1} < \infty.$$

Now from the finiteness of S^L it follows that $\{Z_{\nu_j}\}_{j=0}^{\infty}$ is uniformly ergodic. Next, as above, for all $k \in S^U$,

$$f_{kk}^U = \sum_{n=1}^{\infty} \mathbb{P}_{\delta_k^U}\big(T_{k,1}^U - 1 \geq n\big) \leq \sum_{n=1}^{\infty} \Big(1 - \underline{p}_k^U\Big)^{n-1} < \infty.$$

To complete the proof of uniform ergodicity of $\{Z_{\tau_j}\}_{j=1}^{\infty}$, we will verify the Doeblin condition for one-step transition: that is, for a probability distribution $q = \{q_k\}_{k \in S^U}$ and every set $A \subset S^U$ satisfying $\sum_{k \in A} q_k > \epsilon$,

$$\inf_{l \in S^U} \left(\sum_{k \in A} p_{lk}^U \right) > \delta.$$

Now, taking $q_k := \big(\underline{p}_k^U\big)/\big(\sum_{k \in S} \underline{p}_k^U\big)$, it follows from Lemma 3.1(ii) that

$$\inf_{l \in S^U} \left(\sum_{k \in A} p_{lk}^U \right) \geq \left(\sum_{k \in S} \underline{p}_k^U \right) \sum_{k \in A} q_k.$$

Choosing $\delta = \big(\sum_{k \in S} \underline{p}_k^U\big)\epsilon$ yields uniform ergodicity of $\{Z_{\tau_j}\}_{j=1}^{\infty}$. □

Remark 3.1. An immediate consequence of the above theorem is that $\{Z_{\nu_j}\}_{j=0}^{\infty}$ possesses a proper stationary distribution $\pi^L = \{\pi_k^L\}_{k \in S^L}$, where $\pi_k^L := 1/f_{kk}^L > 0$ and satisfies $\pi_k^L = \sum_{i \in S} \pi_i^L p_{ik}^L$ for all $k \in S^L$. Furthermore, $\lim_{j \to \infty} \sup_{l \in S^L} \|p_l^L(j) - \pi^L\| = 0$, where $\|\cdot\|$ denotes the total variation norm. Furthermore, under a finite-second-moment hypothesis, the central limit theorem holds for functions of Z_{ν_j}. A similar comment also holds for $\{Z_{\tau_j}\}_{j=1}^{\infty}$ with L replaced by U.

Remark 3.2. It is worth noticing that the stationary distributions π^L and π^U are connected using $\pi_i^L = \sum_{l \in S^U} \mathbb{P}_{\delta_l^U}\big(Z_{\nu_1} = i\big)\pi_l^U$ for all $i \in S^L$, since by time-homogeneity (Lemma A.1) we have $\mathbb{P}\big(Z_{\nu_{j+1}} = i | Z_{\tau_{j+1}} = l\big) = \mathbb{P}_{\delta_l^U}\big(Z_{\nu_1} = i\big)$. Now, if we take the limit as $j \to \infty$ in

$$\mathbb{P}\big(Z_{\nu_{j+1}} = i\big) = \sum_{l \in S^U} \mathbb{P}\big(Z_{\nu_{j+1}} = i | Z_{\tau_{j+1}} = l\big)\mathbb{P}\big(Z_{\tau_{j+1}} = l\big),$$

the above expression follows. Similarly, $\pi_k^U = \sum_{l \in S^L} \mathbb{P}_{\delta_l^L}\big(Z_{\tau_1} = k\big)\pi_l^L$ for all $k \in S^U$.

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Since the state space S^L is finite, π^L has moments of all orders. Proposition A.1 in Appendix A.3 shows that π^U has a finite first moment $\overline{\pi}^U := \sum_{k\in S^U} k\pi_k^U$.

4. Regenerative property of crossing times

In this section, we establish the law of large numbers and central limit theorem for the lengths of the supercritical and subcritical regimes $\{\Delta_j^U\}_{j=1}^\infty$ and $\{\Delta_j^L\}_{j=1}^\infty$. To this end, we will show that $\{\Delta_j^U\}_{j=1}^\infty$ and $\{\Delta_j^L\}_{j=1}^\infty$ are regenerative over the times $\{T_{i,l}^L\}_{l=0}^\infty$ and $\{T_{i,l}^U\}_{l=1}^\infty$, respectively. In our analysis we will also encounter the random variables $\overline{\Delta}_j^U := \Delta_j^U + \Delta_j^L$ and $\overline{\Delta}_j^L := \Delta_j^L + \Delta_{j+1}^U$. For $l \geq 1$ and $i \in S^L$, let $B_{i,l}^L := \left(K_{i,l}^L, \mathbf{\Delta}_{i,l}^L, \overline{\mathbf{\Delta}}_{i,l}^L\right)$, where

$$K_{i,l}^L := T_{i,l}^L - T_{i,l-1}^L,\ \mathbf{\Delta}_{i,l}^L := \left(\Delta^U_{T^L_{i,l-1}+1}, \ldots, \Delta^U_{T^L_{i,l}}\right) \quad \text{and} \quad \overline{\mathbf{\Delta}}_{i,l}^L := \left(\overline{\Delta}^U_{T^L_{i,l-1}+1}, \ldots, \overline{\Delta}^U_{T^L_{i,l}}\right).$$

The triple $B_{i,l}^L$ consists of the random time $K_{i,l}^L$ required for $\{Z_{\nu_j}\}_{j=0}^\infty$ to return for the lth time to state i, the lengths of all supercritical regimes Δ_j^U between the $(l-1)$th return and the lth return, and the lengths $\overline{\Delta}_j^U$ of both regimes in the same time interval. Similarly, for $l \geq 1$ and $i \in S^U$ we let $B_{i,l}^U := \left(K_{i,l}^U, \mathbf{\Delta}_{i,l}^U, \overline{\mathbf{\Delta}}_{i,l}^U\right)$, where

$$K_{i,l}^U := T_{i,l}^U - T_{i,l-1}^U,\ \mathbf{\Delta}_{i,l}^U := \left(\Delta^L_{T^U_{i,l-1}}, \ldots, \Delta^L_{T^U_{i,l}-1}\right) \quad \text{and} \quad \overline{\mathbf{\Delta}}_{i,l}^U := \left(\overline{\Delta}^L_{T^U_{i,l-1}}, \ldots, \overline{\Delta}^L_{T^U_{i,l}-1}\right).$$

The proof of the following lemma is included in Appendix A.4.

Lemma 4.1. *Assume (**H1**)–(**H4**). (i) If (**H5**)also holds and $Z_{\nu_0} = i \in S^L$, then $\{B_{i,l}^L\}_{l=1}^\infty$ are i.i.d. (ii) (**H6**) (or (**H7**)) holds and $Z_{\tau_1} = i \in S^U$, then $\{B_{i,l}^U\}_{l=1}^\infty$ are i.i.d.*

The proof of the following lemma, which is required in the proof of the Theorem 2.4, is also included in Appendix A.4. We need the following additional notation: $\overline{S}_n^U := \sum_{j=1}^n \overline{\Delta}_j^U$, $\overline{S}_n^L := \sum_{j=1}^n \overline{\Delta}_j^L$,

$$\overline{\sigma}^{2,U} := \mathbb{V}_{\pi^L}\left[\overline{\Delta}_1^U\right] + 2\sum_{j=1}^\infty \mathbb{C}_{\pi^L}\left[\overline{\Delta}_1^U, \overline{\Delta}_{j+1}^U\right],$$

$$\overline{\sigma}^{2,L} := \mathbb{V}_{\pi^U}\left[\overline{\Delta}_1^L\right] + 2\sum_{j=1}^\infty \mathbb{C}_{\pi^U}\left[\overline{\Delta}_1^L, \overline{\Delta}_{j+1}^L\right],$$

$$\mathbb{C}^U := \sum_{j=0}^\infty \mathbb{C}_{\pi^L}\left[\Delta_1^U, \overline{\Delta}_{j+1}^U\right] + \sum_{j=1}^\infty \mathbb{C}_{\pi^L}\left[\overline{\Delta}_1^U, \Delta_{j+1}^U\right], \quad \text{and}$$

$$\mathbb{C}^L := \sum_{j=0}^\infty \mathbb{C}_{\pi^U}\left[\Delta_1^L, \overline{\Delta}_{j+1}^L\right] + \sum_{j=1}^\infty \mathbb{C}_{\pi^U}\left[\overline{\Delta}_1^L, \Delta_{j+1}^L\right].$$

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Lemma 4.2. *Under the assumptions of Theorem 2.4, for all $i \in S^L$ the following hold:*

(i) $\mathbb{E}_{\delta_i^L}\left[S^U_{T^L_{i,1}}\right] = \left(\pi_i^L\right)^{-1}\mu^U$ and $\mathbb{E}_{\delta_i^L}\left[\overline{S}^U_{T^L_{i,1}}\right] = \left(\pi_i^L\right)^{-1}\left(\mu^U + \mu^L\right)$;

(ii) $\mathbb{V}_{\delta_i^L}\left[S^U_{T^L_{i,1}}\right] = \left(\pi_i^L\right)^{-1}\sigma^{2,U}$ and $\mathbb{V}_{\delta_i^L}\left[\overline{S}^U_{T^L_{i,1}}\right] = \left(\pi_i^L\right)^{-1}\overline{\sigma}^{2,U}$; and

(iii) $\mathbb{C}_{\delta_i^L}\left[S^U_{T^L_{i,1}}, \overline{S}^U_{T^L_{i,1}}\right] = \left(\pi_i^L\right)^{-1}\mathbb{C}^U$.

The above statements also hold with U replaced by L.

Proposition A.2 in Appendix A.5 shows that $\sigma^{2,T}$ and $\overline{\sigma}^{2,T}$ are positive and finite, and $|\mathbb{C}^T| < \infty$. We are now ready to prove Theorem 2.4. The proof relies on decomposing S_n^U and S_n^L into i.i.d. cycles using Lemma 4.1. Specifically, conditionally on $Z_{\nu_0} = i \in S^L$ (resp. $Z_{\tau_1} = i \in S^U$), the random variables $\left\{S^U_{T^L_{i,l}} - S^U_{T^L_{i,l-1}}\right\}_{l=1}^{\infty}$ $\left(\text{resp. } \left\{S^L_{T^U_{i,l}-1} - S^L_{T^U_{i,l-1}-1}\right\}_{l=1}^{\infty}\right)$ are i.i.d.

Proof of Theorem 2.4. We begin by proving (i). For $i \in S^L$ and $n \in \mathbb{N}$, let

$$N_i^L(n) := \sum_{l=1}^{\infty} \mathbf{I}_{\{T^L_{i,l} \le n\}}$$

be the number of times $T^L_{i,l}$ is in $\{0, 1, \ldots, n\}$. Conditionally on $Z_{\nu_0} = i$, notice that $N_i^L(n)$ is a renewal process (recall that $T^L_{i,0} = 0$). We recall that $K^L_{i,l} = T^L_{i,l} - T^L_{i,l-1}$ and let

$$K^{*,L}_{i,n} := n - T_{i,N_i^L(n)}, \qquad R^L_{i,l} := S^U_{T^L_{i,l}} - S^U_{T^L_{i,l-1}}, \quad \text{and} \quad R^{*,L}_{i,n} := S^U_n - S^U_{T^L_{i,N_i^L(n)}}.$$

Using the decomposition

$$\frac{1}{n} S_n^U = \frac{N_i^L(n)}{n}\left(\frac{1}{N_i^L(n)} \sum_{l=1}^{N_i^L(n)} R^L_{i,l} + \frac{1}{N_i^L(n)} R^{*,L}_{i,n}\right) \tag{14}$$

and the fact that $\left\{R^L_{i,l}\right\}_{l=1}^{\infty}$ are i.i.d. and $\lim_{n\to\infty} N_i^L(n) = \infty$ a.s., we obtain using the law of large numbers for random sums and Lemma 4.2(i) that

$$\lim_{n\to\infty} \frac{1}{N_i^L(n)} \sum_{l=1}^{N_i^L(n)} R^L_{i,l} = \mathbb{E}_{\delta_i^L}\left[S^U_{T^L_{i,1}}\right] = (\pi_i^L)^{-1}\mu^U \text{ a.s.} \tag{15}$$

Also,

$$\limsup_{n\to\infty} \frac{1}{N_i^L(n)} R^{*,L}_{i,n} \le \lim_{n\to\infty} \frac{1}{N_i^L(n)} R^L_{i,N_i^L(n)+1} = 0 \text{ a.s.}$$

Finally, using the key renewal theorem (Corollary 2.11 of Serfozo [27]) and Remark 3.1, we have

$$\lim_{n\to\infty} \frac{N_i^L(n)}{n} = \frac{1}{\mathbb{E}_{\delta_i^L}[T^L_{i,1}]} = \pi_i^L \text{ a.s.} \tag{16}$$

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Using (15) and (16) in (14), we obtain the strong law of large numbers for S_n^U. Turning to the central limit theorem, we let $\underline{R}_{i,l}^L := R_{i,l}^L - \mu^U K_{i,l}^L$ and $\underline{R}_{i,n}^{*,L} := R_{i,n}^{*,L} - \mu^U K_{i,n}^{*,L}$. Conditionally on $Z_{\nu_0} = i$, using the decomposition in (14) and centering, we obtain

$$\frac{1}{\sqrt{n}}\left(S_n^U - n\mu^U\right) = \sqrt{\frac{N_i^L(n)}{n}}\left(\frac{1}{\sqrt{N_i^L(n)}}\sum_{l=1}^{N_i^L(n)} \underline{R}_{i,l}^L + \frac{1}{\sqrt{N_i^L(n)}}\underline{R}_{i,n}^{*,L}\right),$$

where $\{\underline{R}_{i,l}^L\}_{l=1}^{\infty}$ are i.i.d. with mean 0 and variance

$$\mathbb{V}_{\delta_i^L}\left[S_{T_{i,1}^L}^U - \mu^U T_{i,1}^L\right] = \left(\pi_i^L\right)^{-1}\sigma^{2,U} \tag{17}$$

by Lemma 4.2. Finally, using the central limit theorem for i.i.d. random sums and (16), it follows that

$$\sqrt{\frac{N_i^L(n)}{n}}\left(\frac{1}{\sqrt{N_i^L(n)}}\sum_{l=1}^{N_i^L(n)} \underline{R}_{i,l}^L\right) \xrightarrow[n\to\infty]{d} N\left(0, \sigma^{2,U}\right).$$

To complete the proof notice that

$$\left|\frac{1}{\sqrt{N_i^L(n)}}\underline{R}_{i,n}^{*,L}\right| \le \frac{1}{\sqrt{N_i^L(n)}}\left|\underline{R}_{i,N_i^L(n)+1}^L\right| \xrightarrow[n\to\infty]{p} 0.$$

The proof for S_n^L is similar. □

When studying the proportion of time the process spends in the supercritical and subcritical regimes, we will need the above theorem with n replaced by a random time $\tilde{N}(n)$.

Remark 4.1. Theorem 2.4 holds if n is replaced by a random time $\tilde{N}(n)$, where $\lim_{n\to\infty}\tilde{N}(n) = \infty$ a.s.

5. Proportion of time spent in supercritical and subcritical regimes

We recall that $\chi_n^U = \mathbf{I}_{\cup_{j=1}^{\infty}[\nu_{j-1},\tau_j)}(n)$ is 1 if the process is in the supercritical regime and 0 otherwise, and similarly $\chi_n^L = 1 - \chi_n^U$. Also, $\theta_n^U = \frac{1}{n}C_n^U$ is the proportion of time the process spends in the supercritical regime up to time $n-1$; the quantity θ_n^L is defined similarly. The limit theorems for θ_n^U and θ_n^L will invoke the i.i.d. blocks developed in Section 4. Let $\boldsymbol{S}_n^U := \left(S_n^U, \overline{S}_n^U\right)^\top$, $\boldsymbol{\mu}^U := \left(\mu^U, \mu^U + \mu^L\right)^\top$, $\boldsymbol{\mu}^L := \left(\mu^L, \mu^U + \mu^L\right)^\top$, and

$$\Sigma^U := \begin{pmatrix}\sigma^{2,U} & \mathbb{C}^U \\ \mathbb{C}^U & \overline{\sigma}^{2,U}\end{pmatrix}, \qquad \Sigma^L := \begin{pmatrix}\sigma^{2,L} & \mathbb{C}^L \\ \mathbb{C}^L & \overline{\sigma}^{2,L}\end{pmatrix}.$$

We note that while S_n^U represents the length of the first n supercritical regimes, $\overline{S}_n^U$ is the total time taken for the process to complete the first n cycles.

Lemma 5.1. *Under the conditions of Theorem 2.5,* $\frac{1}{\sqrt{n}}\left(\boldsymbol{S}_n^U - n\boldsymbol{\mu}^U\right) \xrightarrow[n\to\infty]{d} N\left(\mathbf{0}, \Sigma^U\right)$, *and* $\frac{1}{\sqrt{n}}\left(\boldsymbol{S}_n^L - n\boldsymbol{\mu}^L\right) \xrightarrow[n\to\infty]{d} N\left(\mathbf{0}, \Sigma^L\right)$.

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Proof of Lemma 5.1. The proof is similar to that of Theorem 2.4. We let

$$\boldsymbol{R}_{i,l}^{L} := \boldsymbol{S}_{T_{i,l}^{L}}^{U} - \boldsymbol{S}_{T_{i,l-1}^{L}}^{U}, \qquad \boldsymbol{R}_{i,n}^{*,L} := \boldsymbol{S}_{n}^{U} - \boldsymbol{S}_{T_{i,N_i^L(n)}^{L}}^{U},$$

$$\underline{\boldsymbol{R}}_{i,l}^{L} := \boldsymbol{R}_{i,l}^{L} - K_{i,l}^{L}\boldsymbol{\mu}^{U}, \qquad \underline{\boldsymbol{R}}_{i,n}^{*,L} := \boldsymbol{R}_{i,n}^{*,L} - K_{i,n}^{*,L}\boldsymbol{\mu}^{U}.$$

Conditionally on $Z_{\nu_0} = i$, we write

$$\frac{1}{\sqrt{n}}(\boldsymbol{S}_n^U - n\boldsymbol{\mu}^U) = \sqrt{\frac{N_i^L(n)}{n}}\left(\frac{1}{\sqrt{N_i^L(n)}}\sum_{l=1}^{N_i^L(n)} \underline{\boldsymbol{R}}_{i,l}^{L} + \frac{1}{\sqrt{N_i^L(n)}}\underline{\boldsymbol{R}}_{i,n}^{*,L}\right).$$

Now, by Lemma 4.1 and Lemma 4.2, $\{\underline{\boldsymbol{R}}_{i,l}^{L}\}_{l=1}^{\infty}$ are i.i.d. with mean $\mathbf{0} = (0, 0)^\top$ and covariance matrix $(\pi_i^L)^{-1}\Sigma^U$. Using the key renewal theorem, we conclude that

$$\sqrt{\frac{N_i^L(n)}{n}}\left(\frac{1}{\sqrt{N_i^L(n)}}\sum_{l=1}^{N_i^L(n)} \underline{\boldsymbol{R}}_{i,l}^{L}\right) \xrightarrow[n\to\infty]{d} N\big(\mathbf{0},\, \Sigma^U\big)$$

and

$$\frac{1}{\sqrt{N_i^L(n)}}|\underline{\boldsymbol{R}}_{i,n}^{*,L}| \xrightarrow[n\to\infty]{p} 0.$$

The proof of (ii) is similar. □

Remark 5.1. Lemma 5.1 also holds with n replaced by a random time $\tilde{N}(n)$ such that $\lim_{n\to\infty}\tilde{N}(n) = \infty$ a.s. The next lemma concerns the number of crossings of upper and lower thresholds, namely, $\tilde{N}^U(n) := \sup\{j \geq 0 : \tau_j \leq n\}$ and $\tilde{N}^L(n) := \sup\{j \geq 0 : \nu_j \leq n\}$, where $n \in \mathbb{N}_0$.

Lemma 5.2. *Under the conditions of Theorem* 2.5*,*

(i) $\lim_{n\to\infty}\frac{\tilde{N}^U(n)}{n+1} = \frac{1}{\mu^U+\mu^L}$ *and* $\lim_{n\to\infty}\frac{\tilde{N}^L(n)}{n+1} = \frac{1}{\mu^U+\mu^L}$ *a.s.;*

(ii) $\lim_{n\to\infty}\frac{C_{n+1}^U}{\tilde{N}^U(n)} = \mu^U$ *and* $\lim_{n\to\infty}\frac{C_{n+1}^L}{\tilde{N}^L(n)} = \mu^L$ *a.s.*

Proof of Lemma 5.2. We begin by proving (i). We recall that $\tau_0 = -1$ and $\tau_j < \nu_j < \tau_{j+1}$ a.s. for all $j \geq 0$, yielding that

$$\tilde{N}^L(n) \leq \tilde{N}^U(n) \leq \tilde{N}^L(n) + 1.$$

Since τ_j and ν_j are finite a.s., we obtain that $\lim_{n\to\infty}\tilde{N}^U(n) = \infty$ and $\lim_{n\to\infty}\tilde{N}^L(n) = \infty$ a.s. Part (i) follows if we show that

$$\lim_{n\to\infty}\frac{\tilde{N}^L(n)}{n+1} = \frac{1}{\mu^U + \mu^L} \text{ a.s.}$$

To this end, we notice that $\nu_{\tilde{N}^L(n)} \leq n \leq \nu_{\tilde{N}^L(n)+1}$, and for $n \geq \nu_1$,

$$\frac{\nu_{\tilde{N}^L(n)}}{\tilde{N}^L(n)} \leq \frac{n}{\tilde{N}^L(n)} \leq \frac{\nu_{\tilde{N}^L(n)+1}}{\tilde{N}^L(n)+1}\,\frac{\tilde{N}^L(n)+1}{\tilde{N}^L(n)}.$$

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Clearly, $\lim_{n\to\infty} \frac{\tilde{N}^L(n)+1}{\tilde{N}^L(n)} = 1$ a.s. By Remark 4.1 with $\tilde{N}(n) = \tilde{N}^U(n)$,

$$\lim_{n\to\infty} \frac{\nu_{\tilde{N}^L(n)}}{\tilde{N}^L(n)} = \lim_{n\to\infty} \frac{1}{\tilde{N}^L(n)} \sum_{j=1}^{\tilde{N}^L(n)} \overline{\Delta}_j^U = \mu^U + \mu^L \text{ a.s.}$$

Thus, we obtain

$$\lim_{n\to\infty} \frac{\tilde{N}^L(n)}{n+1} = \frac{1}{\mu^U + \mu^L} \text{ a.s.} \tag{18}$$

Turning to (ii), we notice that

$$C^U_{n+1} = \sum_{j=0}^{n} \chi^U_j = \sum_{j=1}^{\tilde{N}^U(n)} \Delta^U_j + \sum_{l=\nu_{\tilde{N}^U(n)}}^{n} 1,$$

where $\sum_{l=r}^{n} = 0$ for $r > n$. Remark 4.1 with $\tilde{N}(n) = \tilde{N}^U(n)$ yields that

$$\lim_{n\to\infty} \frac{1}{\tilde{N}^U(n)} \sum_{j=1}^{\tilde{N}^U(n)} \Delta^U_j = \mu^U \text{ a.s. and}$$

$$\limsup_{n\to\infty} \frac{1}{\tilde{N}^U(n)} \sum_{l=\nu_{\tilde{N}^U(n)}}^{n} 1 \leq \lim_{n\to\infty} \frac{1}{\tilde{N}^U(n)} \Delta^U_{\tilde{N}^U(n)+1} = 0 \text{ a.s.}$$

Thus, we obtain that $\lim_{n\to\infty} \frac{C^U_{n+1}}{\tilde{N}^U(n)} = \mu^U$ a.s. Similarly, $\lim_{n\to\infty} \frac{C^L_{n+1}}{\tilde{N}^L(n)} = \mu^L$ a.s. □

Our next result is concerned with the joint distribution of the last time when the process is in a specific regime and the proportion of time the process spends in that regime, under the assumptions of Theorem 2.5. Let

$$\overline{\Sigma}^U := \begin{pmatrix} \sigma^{2,U}\left(\mu^U + \mu^L\right) & -\frac{\mathbb{C}^U}{\mu^U+\mu^L} \\ -\frac{\mathbb{C}^U}{\mu^U+\mu^L} & \frac{\overline{\sigma}^{2,U}}{\left(\mu^U+\mu^L\right)^3} \end{pmatrix} \quad \text{and} \quad \overline{\Sigma}^L := \begin{pmatrix} \sigma^{2,L}\left(\mu^U + \mu^L\right) & -\frac{\mathbb{C}^L}{\mu^U+\mu^L} \\ -\frac{\mathbb{C}^L}{\mu^U+\mu^L} & \frac{\overline{\sigma}^{2,L}}{\left(\mu^U+\mu^L\right)^3} \end{pmatrix}.$$

Lemma 5.3. *Under the conditions of Theorem 2.5,*

$$\sqrt{n+1} \begin{pmatrix} \frac{C^U_{n+1}}{\tilde{N}^U(n)} - \mu^U \\ \frac{\tilde{N}^U(n)}{n+1} - \frac{1}{\mu^U+\mu^L} \end{pmatrix} \xrightarrow[n\to\infty]{d} N\left(\mathbf{0}, \overline{\Sigma}^U\right) \quad \text{and}$$

$$\sqrt{n+1} \begin{pmatrix} \frac{C^L_{n+1}}{\tilde{N}^L(n)} - \mu^L \\ \frac{\tilde{N}^L(n)}{n+1} - \frac{1}{\mu^U+\mu^L} \end{pmatrix} \xrightarrow[n\to\infty]{d} N\left(\mathbf{0}, \overline{\Sigma}^L\right).$$

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Proof of Lemma 5.3. We only prove the statement for C^U_{n+1} and $\tilde{N}^U(n)$, since the other case is similar. We write

$$\sqrt{\tilde{N}^U(n)}\begin{pmatrix}\frac{C^U_{n+1}}{\tilde{N}^U(n)}-\mu^U\\ \frac{n+1}{\tilde{N}^U(n)}-\left(\mu^U+\mu^L\right)\end{pmatrix}$$

$$=\frac{1}{\sqrt{\tilde{N}^U(n)}}\sum_{j=1}^{\tilde{N}^U(n)}\begin{pmatrix}\Delta^U_j-\mu^U\\ \overline{\Delta}^U_j-\left(\mu^U+\mu^L\right)\end{pmatrix}+\frac{1}{\sqrt{\tilde{N}^U(n)}}\sum_{l=\nu_{\tilde{N}^U(n)}}^{n}\begin{pmatrix}1\\1\end{pmatrix},$$

and using Lemma 5.1 and Remark 5.1 with $\tilde{N}(n)=\tilde{N}^U(n)$, we obtain that

$$\sqrt{\tilde{N}^U(n)}\begin{pmatrix}\frac{C^U_{n+1}}{\tilde{N}^U(n)}-\mu^U\\ \frac{n+1}{\tilde{N}^U(n)}-\left(\mu^U+\mu^L\right)\end{pmatrix}\xrightarrow[n\to\infty]{d}N\left(\mathbf{0},\,\Sigma^U\right).$$

Next, we apply the delta method with $g:\mathbb{R}^2\to\mathbb{R}^2$ given by $g(x,\,y)=(x,\,1/y)$ and obtain that

$$\sqrt{\tilde{N}^U(n)}\begin{pmatrix}\frac{C^U_{n+1}}{\tilde{N}^U(n)}-\mu^U\\ \frac{\tilde{N}^U(n)}{n+1}-\frac{1}{\mu^U+\mu^L}\end{pmatrix}\xrightarrow[n\to\infty]{d}N\left(\mathbf{0},\,\Sigma^U_2\right),$$

where

$$\Sigma^U_2=J_g\left(\boldsymbol{\mu}^U\right)\Sigma^U J_g\left(\boldsymbol{\mu}^U\right)^\top=\begin{pmatrix}\sigma^{2,U} & -\frac{\mathbb{C}^U}{\left(\mu^U+\mu^L\right)^2}\\ -\frac{\mathbb{C}^U}{\left(\mu^U+\mu^L\right)^2} & \frac{\overline{\sigma}^{2,U}}{\left(\mu^U+\mu^L\right)^4}\end{pmatrix}$$

and $J_g(\cdot)$ is the Jacobian matrix of $g(\cdot)$. Using Lemma 5.2(i), we obtain that

$$\sqrt{n+1}\begin{pmatrix}\frac{C^U_{n+1}}{\tilde{N}^U(n)}-\mu^U\\ \frac{\tilde{N}^U(n)}{n+1}-\frac{1}{\mu^U+\mu^L}\end{pmatrix}\xrightarrow[n\to\infty]{d}N\left(\mathbf{0},\,\overline{\Sigma}^U\right).$$

□

We are now ready to prove Theorem 2.5. Recall that $\theta^U_n=\frac{C^U_n}{n}$, $\theta^L_n=\frac{C^L_n}{n}$, $\theta^U=\frac{\mu^U}{\mu^U+\mu^L}$, and $\theta^L=\frac{\mu^L}{\mu^L+\mu^U}$, and let $\theta^{k,U}$ and $\theta^{k,L}$ be the kth powers of θ^U and θ^L, respectively.

Proof of Theorem 2.5. Almost sure convergence of θ^T_n follows from Lemma 5.3 upon noticing that

$$\frac{C^T_{n+1}}{n+1}=\left(\frac{C^T_{n+1}}{\tilde{N}^T(n)}\right)\left(\frac{\tilde{N}^T(n)}{n+1}\right).$$

Using Lemma 5.2 and the decomposition

$$\sqrt{n+1}\left(\frac{C^T_{n+1}}{n+1}-\frac{\mu^T}{\mu^U+\mu^L}\right)$$

$$=\frac{\tilde{N}^T(n)}{n+1}\cdot\sqrt{n+1}\left(\frac{C^T_{n+1}}{\tilde{N}^T(n)}-\mu^T\right)+\mu^T\cdot\sqrt{n+1}\left(\frac{\tilde{N}^T(n)}{n+1}-\frac{1}{\mu^U+\mu^L}\right),$$

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it follows that $\sqrt{n+1}(\theta_{n+1}^T - \theta^T)$ is asymptotically normal with mean zero and variance

$$\eta^{2,T} := \frac{1}{\mu^T}\left(\sigma^{2,T}\theta^T - 2\mathbb{C}^T\theta^{2,T} + \overline{\sigma}^{2,T}\theta^{3,T}\right). \tag{19}$$

□

Corollary 5.1. *Under the conditions of Theorem 2.4, for* $T \in \{L, U\}$,

$$\sqrt{\tilde{N}^T(n)}\left(\frac{C_{n+1}^T}{\tilde{N}^T(n)} - \mu^T\right) \xrightarrow[n\to\infty]{d} N\left(0, \sigma^{2,T}\right).$$

Proof of Corollary 5.1. We only prove the case $T = U$. We write

$$\sqrt{\tilde{N}^U(n)}\left(\frac{C_{n+1}^U}{\tilde{N}^U(n)} - \mu^U\right) = \frac{1}{\sqrt{\tilde{N}^U(n)}}\sum_{j=1}^{\tilde{N}^U(n)}\left(\Delta_j^U - \mu^U\right) + \frac{1}{\sqrt{\tilde{N}^U(n)}}\sum_{l=\nu_{\tilde{N}^U(n)}}^{n} 1.$$

Taking the limit in the above equation and using Remark 4.1 yields the result. □

6. Estimating the mean of the offspring distribution

We recall that $\tilde{\chi}_n^U = \chi_n^U \mathbf{I}_{\{Z_n\geq 1\}}$, $\tilde{C}_n^U = \sum_{j=1}^n \tilde{\chi}_{j-1}^U$, and for the subcritical regime we set $\tilde{\chi}_n^L := \chi_n^L$ and $\tilde{C}_n^L := C_n^L$. We also recall that the offspring mean estimates of the BPRET $\{Z_n\}_{n=0}^\infty$ in the supercritical and subcritical regimes are given by

$$M_n^U = \frac{1}{\tilde{C}_n^U}\sum_{j=1}^n \frac{Z_j - I_{j-1}^U}{Z_{j-1}}\tilde{\chi}_{j-1}^U \quad \text{and} \quad M_n^L = \frac{1}{\tilde{C}_n^L}\sum_{j=1}^n \frac{Z_j}{Z_{j-1}}\tilde{\chi}_{j-1}^L.$$

The decomposition

$$M_n^T = M^T + \frac{1}{\tilde{C}_n^T}\left(M_{n,1}^T + M_{n,2}^T\right) \tag{20}$$

will be used in the proof of Theorem 2.6 and involves the martingale structure of $M_{n,i}^T := \sum_{j=1}^n D_{j,i}^T$, where

$$D_{j,1}^T := \left(\overline{P}_{j-1}^T - M^T\right)\tilde{\chi}_{j-1}^T \quad \text{and} \quad D_{j,2}^T := \frac{\tilde{\chi}_{j-1}^T}{Z_{j-1}}\sum_{i=1}^{Z_{j-1}}\left(\xi_{j-1,i}^T - \overline{P}_{j-1}^T\right). \tag{21}$$

Specifically, let $\mathcal{G}_n$ be the σ-algebra generated by the random environments $\left\{\Pi_j^T\right\}_{j=0}^n$; $\mathcal{H}_{n,1}$ the σ-algebra generated by $\mathcal{F}_n$ and $\mathcal{G}_{n-1}$; and $\mathcal{H}_{n,2}$ the σ-algebra generated by $\mathcal{F}_n$, $\mathcal{G}_{n-1}$, and the offspring distributions $\{\xi_{j,i}^T\}_{i=0}^\infty$, $j = 0, 1, \ldots, n-1$. Hence, Z_n, $\tilde{\chi}_n^T$, and Π_{n-1}^T are $\mathcal{H}_{n,1}$-measurable, whereas Π_n^T is not $\mathcal{H}_{n,1}$-measurable. We also denote by $\tilde{\mathcal{H}}_{n,1}$ the σ-algebra generated by $\mathcal{F}_{n-1}$ and $\mathcal{G}_{n-1}$, and by $\tilde{\mathcal{H}}_{n,2}$ the σ-algebra generated by $\mathcal{F}_{n-1}$, $\mathcal{G}_{n-1}$, and $\{\xi_{j,i}^T\}_{i=0}^\infty$, $j = 0, 1, \ldots, n-1$. Hence, Z_{n-1}, $\tilde{\chi}_{n-1}^T$, and Π_{n-1}^T are all $\tilde{\mathcal{H}}_{n,1}$-measurable but not $\tilde{\mathcal{H}}_{n-1,1}$-measurable. We establish in Proposition A.3 in Appendix A.6 that

$$\left\{(M_{n,1}^T, \mathcal{H}_{n,i})\right\}_{n=1}^\infty \quad \text{and} \quad \left\{\left(M_{n,2}^T, \mathcal{H}_{n,2}\right)\right\}_{n=1}^\infty$$

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are mean-zero martingale sequences. Additionally,

$$\mathbb{E}\big[\big(M_{n,1}^T\big)^2\big] = V_1^T \mathbb{E}\big[\tilde{C}_n^T\big] \quad \text{and} \quad \mathbb{E}\big[\big(M_{n,2}^T\big)^2\big] = V_2^T \mathbb{E}[\tilde{A}_n^T],$$

where

$$\tilde{A}_n^U := \sum_{j=1}^{n} \frac{\tilde{\chi}_{j-1}^U}{Z_{j-1}}$$

is the sum of $\frac{1}{Z_j}$ over supercritical time steps up to time $n-1$, discarding times at which Z_j is zero, and

$$\tilde{A}_n^L := A_n^L := \sum_{j=1}^{n} \frac{\chi_{j-1}^L}{Z_{j-1}}$$

is the sum of $\frac{1}{Z_j}$ over subcritical time steps up to time $n-1$. Proposition A.3 contains other two martingales involving the terms $D_{j,1}^T$ and $D_{j,2}^T$ in (21) and related moment bounds, which will be used in the proof of Theorem 2.6. As a first step, we derive the limit of the variances $\mathbb{E}\big[\big(M_{n,1}^T\big)^2\big]$ and $\mathbb{E}\big[\big(M_{n,2}^T\big)^2\big]$ when rescaled by n. By Proposition A.3, this entails studying the limit behavior of the quantities $\frac{1}{n}\tilde{C}_n^T$ and $\frac{1}{n}\tilde{A}_n^T$. To this end, we build i.i.d. blocks as in Section 4. For $l \geq 1$ and $i \in S^L$, let $\overline{B}_{i,l}^L := \Big(K_{i,l}^L, \tilde{\mathbf{\Delta}}_{i,l}^L, \tilde{\mathbf{\Gamma}}_{i,l}^L\Big)$, where

$$K_{i,l}^L = T_{i,l}^L - T_{i,l-1}^L, \qquad \tilde{\mathbf{\Delta}}_{i,l}^L := \Big(\tilde{\Delta}_{T_{i,l-1}^L+1}^U, \ldots, \tilde{\Delta}_{T_{i,l}^L}^U\Big), \qquad \tilde{\mathbf{\Gamma}}_{i,l}^L := \Big(\tilde{\Gamma}_{T_{i,l-1}^L+1}^U, \ldots, \tilde{\Gamma}_{T_{i,l}^L}^U\Big),$$

$$\tilde{\Delta}_{j+1}^U := \sum_{k=\nu_j+1}^{\tau_{j+1}} \tilde{\chi}_{k-1}^U, \qquad \tilde{\Gamma}_{j+1}^U := \sum_{k=\nu_j+1}^{\tau_{j+1}} \frac{\tilde{\chi}_{k-1}^U}{Z_{k-1}}.$$

The triple $B_{i,l}^L$ consists of the random time $K_{i,l}^L$ required for $\{Z_{\nu_j}\}_{j=0}^{\infty}$ to return for the lth time to state i, the lengths of all supercritical regimes $\tilde{\Delta}_j^U$ between the $(l-1)$th return and the lth return, and the sum of Z_j-inverse over supercritical regimes, disregarding the times when the process hits zero. Similarly, for $l \geq 1$ and $i \in S^U$ we let $\overline{B}_{i,l}^U := \big(K_{i,l}^U, \mathbf{\Delta}_{i,l}^U, \mathbf{\Gamma}_{i,l}^U\big)$, where

$$K_{i,l}^U = T_{i,l}^U - T_{i,l-1}^U, \qquad \mathbf{\Delta}_{i,l}^U = \Big(\Delta_{T_{i,l-1}^U}^L, \ldots, \Delta_{T_{i,l}^U-1}^L\Big), \qquad \mathbf{\Gamma}_{i,l}^U := \Big(\Gamma_{T_{i,l-1}^U}^L, \ldots, \Gamma_{T_{i,l}^U-1}^L\Big),$$

$$\Gamma_j^L := \sum_{k=\tau_j+1}^{\nu_j} \frac{\chi_{k-1}^L}{Z_{k-1}}.$$

Notice that, since $\tilde{C}_n^L = C_n^L$, Theorem 2.5 already yields that $\lim_{n\to\infty} \frac{C_n^L}{n} = \frac{\mu^L}{\mu^U+\mu^L}$. We need the following slight modification of Lemma 4.1, whose proof is similar and hence omitted.

Lemma 6.1. *Assume (**H1**)–(**H4**). (i) If (**H5**) holds and $Z_{\nu_0} = i \in S^L$, then $\big\{\overline{B}_{i,l}^L\big\}_{l=1}^{\infty}$ are i.i.d. (**H6**) (or (**H7**)) holds and $Z_{\tau_1} = i \in S^U$, then $\{\overline{B}_{i,l}^U\}_{l=1}^{\infty}$ are i.i.d.*

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Proposition 6.1. *Suppose that (**H1**)–(**H6**) (or (**H7**)) hold and μ^U, $\mu^L < \infty$. Then*

(i) $\lim_{n\to\infty} \frac{\tilde{C}_n^U}{\tilde{N}^L(n)} = \tilde{\mu}^U$ *and* $\lim_{n\to\infty} \frac{\tilde{C}_n^U}{n} = \frac{\tilde{\mu}^U}{\mu^U+\mu^L}$ *a.s.;*

(ii) $\lim_{n\to\infty} \frac{\tilde{A}_n^U}{\tilde{N}^L(n)} = \tilde{A}^U$ *and* $\lim_{n\to\infty} \frac{\tilde{A}_n^U}{n} = \frac{\tilde{A}^U}{\mu^U+\mu^L}$ *a.s.;*

(iii) $\lim_{n\to\infty} \frac{A_n^L}{\tilde{N}^U(n)} = A^L$ *and* $\lim_{n\to\infty} \frac{A_n^L}{n} = \frac{A^L}{\mu^U+\mu^L}$ *a.s.*

Since $\frac{\tilde{C}_n^T}{n}$ *and* $\frac{\tilde{A}_n^T}{n}$ *are non-negative and bounded by one, Proposition 6.1 implies convergence in mean of these quantities.*

Proof of Proposition 6.1. By Lemma 5.2(i) it is enough to show the first part of the statements (i)–(iii). Since the proofs of the other cases are similar, we only prove (i). We recall that for $i \in S^L$ and $j \in \mathbb{N}$,

$$N_i^L(j) = \sum_{l=1}^{\infty} \mathbf{I}_{\{T_{i,l}^L \le j\}}$$

is the number of times $T_{i,l}^L$ is in $\{0, 1, \ldots, j\}$. We define

$$\overline{N}_i^L(n) := N_i^L\big(\tilde{N}^L(n)\big), \qquad \tilde{D}_{i,l}^L := \tilde{C}_{\nu_{T_{i,l}^L}}^U - \tilde{C}_{\nu_{T_{i,l-1}^L}}^U, \quad \text{and} \quad \tilde{D}_{i,n}^{*,L} := \tilde{C}_n^U - \tilde{C}_{\nu_{T_{i,\overline{N}_i^L(n)}^L}}^U.$$

Conditionally on $Z_{\nu_0} = i$, $T_{i,0}^L = 0$, and we write

$$\frac{\tilde{C}_n^U}{\tilde{N}^L(n)} = \frac{\overline{N}_i^L(n)}{\tilde{N}^L(n)} \left(\frac{1}{\overline{N}_i^L(n)} \sum_{l=1}^{\overline{N}_i^L(n)} \tilde{D}_{i,l}^L + \frac{1}{\overline{N}_i^L(n)} \tilde{D}_{i,n}^{*,L} \right).$$

Lemma 6.1 implies that $\{\tilde{D}_{i,l}^L\}_{l=1}^{\infty}$ are i.i.d. with expectation given by

$$\mathbb{E}_{\delta_i^L}\left[\tilde{C}_{\nu_{T_{i,1}^L}}^U \right] = \big(\pi_i^L\big)^{-1} \tilde{\mu}^U,$$

using Proposition 1.69 of Serfozo [27]. Since $\lim_{n\to\infty} \overline{N}_i^L(n) = \infty$ a.s., we obtain that

$$\lim_{n\to\infty} \frac{1}{\overline{N}_i^L(n)} \sum_{l=1}^{\overline{N}_i^L(n)} \tilde{D}_{i,l}^L = (\pi_i^L)^{-1} \tilde{\mu}^U \text{ a.s. and } \lim_{n\to\infty} \frac{1}{\overline{N}_i^L(n)} \tilde{D}_{i,n}^{*,L} = 0 \text{ a.s.,}$$

since

$$\tilde{D}_{i,n}^{*,L} \le \tilde{D}_{i,\overline{N}_i^L(n)+1}^L.$$

Finally, it holds that $\lim_{n\to\infty} \frac{\overline{N}_i^L(n)}{\tilde{N}^L(n)} = \pi_i^L$ a.s. □

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We next establish that, when rescaled by their standard deviations, the terms $M_{n,i}^T$, where $i = 1, 2$ and $T \in \{L, U\}$, are jointly asymptotically normal. To this end, let

$$\overline{\boldsymbol{M}}_n^T := \left(\frac{M_{n,1}^T}{\sqrt{\mathbb{E}\left[\left(M_{n,1}^T\right)^2\right]}}, \frac{M_{n,2}^T}{\sqrt{\mathbb{E}\left[\left(M_{n,2}^T\right)^2\right]}} \right)^\top \text{ and } \overline{\boldsymbol{M}}_n := \left(\left(\overline{\boldsymbol{M}}_n^U\right)^\top, \left(\overline{\boldsymbol{M}}_n^L\right)^\top \right)^\top.$$

Lemma 6.2. *Under the assumption of Theorem 2.6(ii),* $\overline{\boldsymbol{M}}_n \xrightarrow[n\to\infty]{d} N(\boldsymbol{0}, I)$.

Proof of Lemma 6.2. By the Cramér–Wold theorem (see Theorem 29.4 of Billingsley [28]), it is enough to show that for $t_i^T \in \mathbb{R}$, where $i = 1, 2$ and $T \in \{L, U\}$,

$$\sum_{T\in\{L,U\}} \sum_{i=1}^{2} t_i^T \frac{M_{n,i}^T}{\sqrt{\mathbb{E}\left[\left(M_{n,i}^T\right)^2\right]}} \xrightarrow[n\to\infty]{d} N\left(0, \sum_{T\in\{L,U\}} \sum_{i=1}^{2} \left(t_i^T\right)^2\right). \tag{22}$$

Using Proposition A.3, we see that

$$\left\{ \left(\sum_{T\in\{L,U\}} \sum_{i=1}^{2} t_i^T M_{n,i}^T, \mathcal{H}_{n,2} \right) \right\}_{n=1}^{\infty}$$

is a mean-zero martingale sequence. In particular,

$$\left\{ \left(\sum_{T\in\{L,U\}} \sum_{i=1}^{2} t_i^T \frac{M_{j,i}^T}{\sqrt{\mathbb{E}\left[\left(M_{n,i}^T\right)^2\right]}}, \mathcal{H}_{j,2} \right) \right\}_{j=1}^{n}$$

is a mean-zero martingale array. We will apply Theorem 3.2 of Hall and Heyde [29] with $k_n = n$,

$$X_{nl} = \sum_{T\in\{L,U\}} \sum_{i=1}^{2} t_i^T \frac{D_{l,i}^T}{\sqrt{\mathbb{E}\left[\left(M_{n,i}^T\right)^2\right]}},$$

$S_{nj} = \sum_{l=1}^{j} X_{nl}$, $\mathcal{F}_{nj} = \mathcal{H}_{j,2}$, and $B^2 = \sum_{T\in\{L,U\}} \sum_{i=1}^{2} \left(t_i^T\right)^2$; from this we will obtain (22). To this end, we need to verify the following conditions:

(i) $\mathbb{E}\left[\left(S_{nj}\right)^2\right] < \infty$,

(ii) $\max_{l=1,\ldots,n} |X_{nl}| \xrightarrow[n\to\infty]{p} 0$,

(iii) $\sum_{l=1}^{n} X_{nl}^2 \xrightarrow[n\to\infty]{p} B^2$, and

(iv) $\sup_{n\in\mathbb{N}} \mathbb{E}\left[\max_{l=1,\ldots,n} X_{nl}^2\right] < \infty$.

Using Proposition A.3 (iv), we have $\mathbb{E}\left[\left(M_{j,i_1}^{T_1}\right)\left(M_{j,i_2}^{T_2}\right)\right] = 0$ if either $T_1 \neq T_2$ or $i_1 \neq i_2$, and since $\mathbb{E}\left[\left(M_{j,i}^T\right)^2\right]$ are non-decreasing in j, we obtain that

$$\mathbb{E}\left[\left(\sum_{T\in\{L,U\}}\sum_{i=1}^{2} t_i^T \frac{M_{j,i}^T}{\sqrt{\mathbb{E}\big[\big(M_{n,i}^T\big)^2\big]}}\right)^2\right] = \sum_{T\in\{L,U\}}\sum_{i=1}^{2} \big(t_i^T\big)^2 \frac{\mathbb{E}\big[\big(M_{j,i}^T\big)^2\big]}{\mathbb{E}\big[\big(M_{n,i}^T\big)^2\big]}$$

$$\leq \sum_{T\in\{L,U\}}\sum_{i=1}^{2} \big(t_i^T\big)^2 < \infty,$$

which yields Condition (i). Using again that $\mathbb{E}\big[\big(D_{l,i_1}^{T_1}\big)\big(D_{l,i_2}^{T_2}\big)\big] = 0$ if either $T_1 \neq T_2$ or $i_1 \neq i_2$ and $\mathbb{E}\big[\big(M_{n,i}^T\big)^2\big] = \sum_{l=1}^{n} \mathbb{E}\big[\big(D_{l,i}^T\big)^2\big]$, we obtain that

$$\mathbb{E}\left[\max_{l=1,\ldots,n}\left(\sum_{T\in\{L,U\}}\sum_{i=1}^{2} t_i^T \frac{D_{l,i}^T}{\sqrt{\mathbb{E}\big[\big(M_{n,i}^T\big)^2\big]}}\right)^2\right] \leq \mathbb{E}\left[\sum_{l=1}^{n}\left(\sum_{T\in\{L,U\}}\sum_{i=1}^{2} t_i^T \frac{D_{l,i}^T}{\sqrt{\mathbb{E}\big[\big(M_{n,i}^T\big)^2\big]}}\right)^2\right]$$

$$= \sum_{T\in\{L,U\}}\sum_{i=1}^{2} \big(t_i^T\big)^2,$$

yielding Condition (iv). Turning to Condition (ii), assuming without loss of generality that $t_i^T \neq 0$; using that

$$\left(\sum_{T\in\{L,U\}}\sum_{i=1}^{2} t_i^T \frac{D_{l,i}^T}{\sqrt{\mathbb{E}\big[\big(M_{n,i}^T\big)^2\big]}}\right)^2 \leq 4 \sum_{T\in\{L,U\}}\sum_{i=1}^{2} \big(t_i^T\big)^2 \frac{\big(D_{l,i}^T\big)^2}{\sqrt{\mathbb{E}\big[\big(M_{n,i}^T\big)^2\big]}},$$

we obtain that for all $\epsilon > 0$,

$$\mathbb{P}\left(\max_{l=1,\ldots,n}\left|\sum_{T\in\{L,U\}}\sum_{i=1}^{2} t_i^T \frac{D_{l,i}^T}{\sqrt{\mathbb{E}\big[\big(M_{n,i}^T\big)^2\big]}}\right| \geq \epsilon\right)$$

$$\leq \sum_{l=1}^{n} \mathbb{P}\left(\left(\sum_{T\in\{L,U\}}\sum_{i=1}^{2} t_i^T \frac{D_{l,i}^T}{\sqrt{\mathbb{E}\big[\big(M_{n,i}^T\big)^2\big]}}\right)^2 \geq \epsilon^2\right)$$

$$\leq \sum_{T\in\{L,U\}}\sum_{i=1}^{2}\sum_{l=1}^{n} \mathbb{P}\left(\big(D_{l,i}^T\big)^2 \geq \left(\frac{\epsilon}{4t_i^T}\right)^2 \mathbb{E}\big[\big(M_{n,i}^T\big)^2\big]\right).$$

For $i = 1$, we use that $\tilde{\chi}_{l-1}^T \in \{0, 1\}$ and obtain

$$\sum_{l=1}^{n} \mathbb{P}\left(\big(D_{l,1}^T\big)^2 \geq \left(\frac{\epsilon}{4t_1^T}\right)^2 \mathbb{E}\big[\big(M_{n,1}^T\big)^2\big]\right)$$

$$\leq \sum_{l=1}^{n} \mathbb{P}\left(\big(D_{l,1}^T\big)^2 \geq \left(\frac{\epsilon}{4t_1^T}\right)^2 \mathbb{E}\big[\big(M_{n,1}^T\big)^2\big] \big| \tilde{\chi}_{l-1}^T = 1\right)$$

$$= \frac{n}{\mathbb{E}\big[\big(M_{n,1}^T\big)^2\big]} \mathbb{E}\big[\big(M_{n,1}^T\big)^2\big] \mathbb{P}\left(\left(\overline{P}_0^T - M^T\right)^2 \geq \left(\frac{\epsilon}{4t_1^T}\right)^2 \mathbb{E}\big[\big(M_{n,1}^T\big)^2\big]\right).$$

It follows from Proposition A.3(i) and Proposition 6.1(i) that

$$\lim_{n\to\infty} \frac{n}{\mathbb{E}\big[\big(M_{n,1}^T\big)^2\big]} = \frac{1}{\tilde{\theta}^T V_1^T} < \infty, \tag{23}$$

where, for $T=L$, $\tilde{\theta}^L := \theta^L$, and since $V_1^T < \infty$,

$$\lim_{n\to\infty} \mathbb{E}\big[\big(M_{n,1}^T\big)^2\big]\mathbb{P}\bigg(\Big(\overline{P}_0^T - M^T\Big)^2 \geq \bigg(\frac{\epsilon}{4t_1^T}\bigg)^2 \mathbb{E}\big[\big(M_{n,1}^T\big)^2\big]\bigg) = 0,$$

yielding that

$$\lim_{n\to\infty} \sum_{l=1}^{n} \mathbb{P}\bigg(\big(D_{l,1}^T\big)^2 \geq \bigg(\frac{\epsilon}{4t_1^T}\bigg)^2 \mathbb{E}\big[\big(M_{n,1}^T\big)^2\big]\bigg) = 0.$$

For $i=2$, we use that if $\tilde{\chi}_{l-1}^T = 1$ then $Z_{l-1} \geq 1$ and obtain that

$$\sum_{l=1}^{n} \mathbb{P}\bigg(\big(D_{l,2}^T\big)^2 \geq \bigg(\frac{\epsilon}{4t_2^T}\bigg)^2 \mathbb{E}\big[\big(M_{n,2}^T\big)^2\big]\bigg)$$

$$\leq n \sup_{z\in\mathbb{N}} \mathbb{P}\Bigg(\bigg(\frac{1}{z}\sum_{i=1}^{z}\Big(\xi_{0,i}^T - \overline{P}_0^T\Big)\bigg)^2 \geq \bigg(\frac{\epsilon}{4t_2^T}\bigg)^2 \mathbb{E}\big[\big(M_{n,2}^T\big)^2\big]\Bigg).$$

Next, using Proposition A.3(ii) and Proposition 6.1(ii)–(iii), we have that

$$\lim_{n\to\infty} \frac{n}{\mathbb{E}\big[\big(M_{n,2}^T\big)^2\big]} \leq \frac{\mu^U + \mu^L}{V_2^T \tilde{A}^T} < \infty, \tag{24}$$

where $\tilde{A}^L := A^L$. Since by Jensen's inequality

$$\mathbb{E}\Bigg[\bigg|\frac{1}{z}\sum_{i=1}^{z}\big(\xi_{0,i}^T - \overline{P}_0^T\big)\bigg|^{2+\delta}\Bigg] \leq \mathbb{E}\big[\Lambda_{0,2}^{T,2+\delta}\big] < \infty,$$

using the Markov inequality, we obtain

$$\sum_{n=0}^{\infty} \sup_{z\in\mathbb{N}} \mathbb{P}\Bigg(\bigg(\frac{1}{z}\sum_{i=1}^{z}\Big(\xi_{0,i}^T - \overline{P}_0^T\Big)\bigg)^2 \geq \bigg(\frac{\epsilon}{4t_2^T}\bigg)^2 \mathbb{E}\big[\big(M_{n,2}^T\big)^2\big]\Bigg) < \infty,$$

which yields that

$$\lim_{n\to\infty} n \sup_{z\in\mathbb{N}} \mathbb{P}\Bigg(\bigg(\frac{1}{z}\sum_{i=1}^{z}\big(\xi_{0,i}^T - \overline{P}_0^T\big)\bigg)^2 \geq \bigg(\frac{\epsilon}{4t_2^T}\bigg)^2 \mathbb{E}\big[\big(M_{n,2}^T\big)^2\big]\Bigg) = 0.$$

For (iii), we decompose

$$\sum_{l=1}^{n}\Bigg(\sum_{T\in\{L,U\}}\sum_{i=1}^{2} t_i^T \frac{D_{l,i}^T}{\sqrt{\mathbb{E}\big[\big(M_{n,i}^T\big)^2\big]}}\Bigg)^2 - \sum_{T\in\{L,U\}}\sum_{i=1}^{2}\big(t_i^T\big)^2$$

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as

$$\sum_{T\in\{L,U\}}\sum_{i=1}^{2}(t_i^T)^2\frac{\sum_{l=1}^n(D_{l,i}^T)^2-\mathbb{E}\big[(M_{n,i}^T)^2\big]}{\mathbb{E}\big[(M_{n,i}^T)^2\big]}+\sum_{T_1\neq T_2\text{ or }i_1\neq i_2}t_{i_1}^{T_1}t_{i_2}^{T_2}\frac{\sum_{l=1}^n(D_{l,i_1}^{T_1})(D_{l,i_2}^{T_2})}{\sqrt{\mathbb{E}\big[(M_{n,i_1}^{T_1})^2\big]\mathbb{E}\big[(M_{n,i_2}^{T_2})^2\big]}}\tag{25}$$

and show that each of the above terms converges to zero with probability one. Since $\mathbb{E}\big[(M_{n,i}^T)^2\big]=\sum_{l=1}^n\mathbb{E}[(D_{l,i}^T)^2]$, we use Proposition A.3(iii) and obtain that

$$\left\{\left(\sum_{l=1}^n((D_{l,i}^T)^2-\mathbb{E}\big[(D_{l,i}^T)^2\big]),\ \tilde{\mathcal{H}}_{n,i}\right)\right\}_{n=1}^{\infty}$$

are mean-zero martingale sequences, and for $s=1+\delta/2$,

$$\mathbb{E}\left[\left|(D_{l,i}^T)^2-\mathbb{E}\big[(D_{l,i}^T)^2\big]\right|^s|\tilde{\mathcal{H}}_{l-1,i}\right]\leq 2^s\mathbb{E}\big[\Lambda_{0,i}^{T,2s}\big]\mathbb{E}\big[\tilde{\chi}_{l-1}^T\big]<\infty.$$

We use (23) and (24), and apply Theorem 2.18 of Hall and Heyde [29] with $S_n=\sum_{l=1}^n((D_{l,i}^T)^2-\mathbb{E}\big[(D_{l,i}^T)^2\big])$, $X_l=(D_{l,i}^T)^2-\mathbb{E}\big[(D_{l,i}^T)^2\big]$, $\mathcal{F}_n=\tilde{\mathcal{H}}_{n,i}$, $U_n=\mathbb{E}\big[(M_{n,i}^T)^2\big]$, and $p=s$, where $i=1,2$, to obtain the convergence of the first term in (25). For the second term we proceed similarly. Specifically, using Proposition A.3(iv) and the Cauchy–Schwartz inequality, we obtain that $\{(\sum_{l=1}^n(D_{l,i_1}^{T_1})(D_{l,i_2}^{T_2}),\ \tilde{\mathcal{H}}_{n,2})\}_{n=0}^{\infty}$ is a mean-zero martingale sequence, and for $s=1+\delta/2$,

$$\mathbb{E}\left[|\left(D_{l,i_1}^{T_1}\right)\left(D_{l,i_2}^{T_2}\right)|^s|\tilde{\mathcal{H}}_{l-1,2}\right]\leq\mathbb{E}\left[\Lambda_{0,i_1}^{T_1,s}\Lambda_{0,i_2}^{T_2,s}\right]\leq\sqrt{\mathbb{E}\left[\Lambda_{0,i_1}^{T_1,2+\delta}\right]\mathbb{E}\left[\Lambda_{0,i_2}^{T_2,2+\delta}\right]}<\infty.$$

Finally, we apply Theorem 2.18 of Hall and Heyde [29] with $S_n=\sum_{l=1}^n\left(D_{l,i_1}^{T_1}\right)\left(D_{l,i_2}^{T_2}\right)$, $X_l=\left(D_{l,i_1}^{T_1}\right)\left(D_{l,i_2}^{T_2}\right)$, $\mathcal{F}_n=\tilde{\mathcal{H}}_{n,2}$, $U_n=\sqrt{\mathbb{E}\left[\left(M_{n,i_1}^{T_1}\right)^2\right]\mathbb{E}\big[(M_{n,i_2}^{T_2})^2\big]}$, and $p=s$, from which we obtain convergence of the second term in (25). □

We are now ready to prove the main result of the section.

Proof of Theorem 2.6. Using Proposition A.3(i)–(ii) and $\tilde{\chi}_{j-1}^T\leq 1$, we obtain that for $i=1,2$, $\{(M_{n,i}^T,\ \mathcal{H}_{n,i})\}_{n=1}^{\infty}$ are martingales and

$$\sum_{j=1}^{\infty}\frac{1}{j^s}\mathbb{E}\big[|D_{j,i}^T|^s|\mathcal{H}_{j-1,i}\big]\leq\mathbb{E}\big[\Lambda_{0,i}^{T,s}\big]\sum_{j=1}^{\infty}\frac{1}{j^s}<\infty.$$

We apply Theorem 2.18 of Hall and Heyde [29] with $S_n=M_{n,i}^T$, $X_j=D_{j,i}^T$, where $i=1,2$ and $T\in\{L,U\}$, and with $U_n=n$, $p=s$, and $\mathcal{F}_n=\mathcal{H}_{n,1}$ for $i=1$ and $\mathcal{F}_n=\mathcal{H}_{n,2}$ for $i=2$, From this we obtain that $\lim_{n\to\infty}\frac{1}{n}M_{n,i}^T=0$ a.s. From this, Theorem 2.5, and Proposition 6.1(i), we obtain that $\lim_{n\to\infty}\frac{1}{C_n^T}M_{n,i}^T=0$ a.s. Using (20) we conclude that $\lim_{n\to\infty}M_n^T=M^T$ a.s. Turning to the central limit theorem, Lemma 6.2(iii) yields that

$$\overline{\boldsymbol{M}}_n=\left(\frac{M_{n,1}^U}{\sqrt{\mathbb{E}\big[(M_{n,1}^U)^2\big]}},\frac{M_{n,2}^U}{\sqrt{\mathbb{E}\big[(M_{n,2}^U)^2\big]}},\frac{M_{n,1}^L}{\sqrt{\mathbb{E}\big[(M_{n,1}^L)^2\big]}},\frac{M_{n,2}^L}{\sqrt{\mathbb{E}\big[(M_{n,2}^L)^2\big]}}\right)^{\top}$$

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is asymptotically normal with mean zero and identity covariance matrix. Let D_n^2 be the 4×4 diagonal matrix

$$D_n^2 := \text{Diag}\left(\frac{n\mathbb{E}\big[\big(M_{n,1}^U\big)^2\big]}{\big(\tilde{C}_n^U\big)^2}, \frac{n\mathbb{E}\big[\big(M_{n,2}^U\big)^2\big]}{\big(\tilde{C}_n^U\big)^2}, \frac{n\mathbb{E}\big[\big(M_{n,1}^L\big)^2\big]}{\big(\tilde{C}_n^L\big)^2}, \frac{n\mathbb{E}\big[\big(M_{n,2}^L\big)^2\big]}{\big(\tilde{C}_n^L\big)^2}\right).$$

By Proposition A.3(i)–(ii) and Proposition 6.1, $D_n\overline{\boldsymbol{M}}_n$ is asymptotically normal with mean zero and covariance matrix

$$\tilde{D}^2 := \text{Diag}\left(\frac{V_1^U}{\tilde{\theta}^U}, \frac{\tilde{A}^U V_2^U}{\tilde{\theta}^U \tilde{\mu}^U}, \frac{V_1^L}{\theta^L}, \frac{A^L V_2^L}{\theta^L \mu^L}\right).$$

Using the continuous mapping theorem, it follows that

$$\sqrt{n}(\boldsymbol{M}_n - \boldsymbol{M}) \xrightarrow[n\to\infty]{d} N(\boldsymbol{0}, \Sigma).$$

□

7. Discussion and concluding remarks

In this paper we have developed the BPRE with thresholds to describe periods of growth and decrease in the population size arising in several applications, including COVID dynamics. Even though the model is non-Markov, we identify Markov subsequences and use them to understand the length of time the process spends in the supercritical and subcritical regimes. Furthermore, using the regeneration technique, we also study the rate of growth (resp. decline) of the process in the supercritical (resp. subcritical) regime. It is possible to start the process using the subcritical BPRE and then move to the supercritical regime; this introduces only minor changes, and the qualitative results remain the same. Finally, we note that without the incorporation of immigration in the supercritical regime, the process will become extinct with probability one, and hence the cyclical path behavior may not be observed.

An interesting question concerns the choice of strongly subcritical BPRE for the subcritical regime. It is folklore that the generation sizes of moderately and weakly subcritical processes can increase for long periods of time, and in that case the time needed to cross the lower threshold will have a heavier tail. This could lead to a lack of identifiability of the supercritical and subcritical regimes. Similar issues arise when a subcritical BPRE is replaced by a critical BPRE or when immigration is allowed in both regimes. Since a subcritical BPRE with immigration converges in distribution to a proper limit law [30], we may fail to observe a clear period of decrease. The path properties of these alternatives could be useful for modeling other dynamics observed (see [9, 12]). Mathematical issues arising from these alternatives would involve different techniques from those used in this paper.

We end this section with a brief discussion concerning the moment conditions in Theorem 2.6. It is possible to reduce the conditions $\mathbb{E}\big[\Lambda_{0,i}^{T,2+\delta}\big] < \infty$ to a finite-second-moment hypothesis. This requires an extension of Lemma 4.1 to joint independence of blocks in $B_{i,l}^L$, $B_{i,l}^U$, offspring random variables, environments, and immigration over cycles. The proof will require the Markov property of the pair $\big\{\big(Z_{\nu_{j-1}}, Z_{\tau_j}\big)\big\}_{j=1}^{\infty}$ and its uniform ergodicity. The joint Markov property will also yield a joint central limit theorem for the length and proportion of time spent in the supercritical and subcritical regimes. The proof is similar to that of Theorem 2.3

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and Lemma 4.1, but is more cumbersome, with an increased notational burden. The numerical experiments suggest that the estimators of the mean parameters of the supercritical and subcritical regimes are not affected by the choice of various distributions. A thorough statistical analysis of the robustness of the estimators and an analysis of the datasets are beyond the scope of this paper and will be undertaken elsewhere.

Appendix A. Auxiliary results

This section contains detailed descriptions and proofs of auxiliary results used in the paper. We begin with a detailed description of the probability space for the BPRET.

A.1. Probability space

In this subsection we describe in detail the random variables used to define the BPRET, as well as the underlying probability space. The thresholds $\{(U_j, L_j)\}_{j=1}^{\infty}$ are i.i.d. random vectors with support $S_B^U \times S_B^L$, where $S_B^U = \mathbb{N} \cap [L_U + 1, \infty)$, $S_B^L = \mathbb{N} \cap [L_0, L_U]$, and $1 \le L_0 \le L_U$ are fixed integers, defined on the probability space $(\Omega_B, \mathcal{F}_B, \mathbb{P}_B)$. Next, $\Pi^L = \{\Pi_n^L\}_{n=0}^{\infty}$ and $\Pi^U = \{\Pi_n^U\}_{n=0}^{\infty}$ are subcritical and supercritical environmental sequences that are defined on probability spaces $(\Omega_{E^L}, \mathcal{F}_{E^L}, \mathbb{P}_{E^L})$ and $(\Omega_{E^U}, \mathcal{F}_{E^U}, \mathbb{P}_{E^U})$. Specifically, $\Pi_n^U = \left(P_n^U, Q_n^U\right)$ and $\Pi_n^L = P_n^L$, where $P_n^U = \{P_{n,r}^U\}_{r=0}^{\infty}$, $P_n^L = \{P_{n,r}^L\}_{r=0}^{\infty}$, and $Q_n^U = \{Q_{n,r}^U\}_{r=0}^{\infty}$ are probability distributions in $\mathcal{P}$. Let $(\Omega_U, \mathcal{F}_U, \mathbb{P}_U)$ and $(\Omega_L, \mathcal{F}_L, \mathbb{P}_L)$ denote probability spaces corresponding to the supercritical BPRE with immigration and the subcritical BPRE. Hence, the environment sequence $\Pi^U = \{\Pi_n^U\}_{n=0}^{\infty}$, the offspring sequence $\{\xi_{n,i}^U\}_{i=1}^{\infty}$, and the immigration sequence $\{I_n^U\}_{n=0}^{\infty}$ are random variables on $(\Omega_U, \mathcal{F}_U, \mathbb{P}_U)$. Similarly, $\Pi^L = \{\Pi_n^L\}_{n=0}^{\infty}$ and $\{\xi_{n,i}^L\}_{i=1}^{\infty}$, $n \ge 0$, are random variables on $(\Omega_L, \mathcal{F}_L, \mathbb{P}_L)$. We point out here that the probability spaces $(\Omega_U, \mathcal{F}_U, \mathbb{P}_U)$ and $(\Omega_{E^U}, \mathcal{F}_{E^U}, \mathbb{P}_{E^U})$ are linked; that is, for all integrable functions $H:\Omega_U \to \mathbb{R}$,

$$\int H(z, \Pi^U) d\mathbb{P}_U(z, \Pi^U) = \int \int H(z, \Pi^U) d\mathbb{P}_U(z|\Pi^U) d\mathbb{P}_{E^U}(\Pi^U).$$

Similar comments also hold with U replaced by L in the above. All of the random variables described above are defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_B \times \Omega_U \times \Omega_L, \mathcal{F}_B \otimes \mathcal{F}_U \otimes \mathcal{F}_L, \mathbb{P}_B \times \mathbb{P}_U \times \mathbb{P}_L)$.

A.2. Time-homogeneity of $\{Z_{\nu_j}\}_{j=0}^{\infty}$ and $\left\{Z_{\tau_j}\right\}_{j=1}^{\infty}$

Lemma A.1. *Assume (**H1**) and (**H2′**). For all $i \in S^L$, $k \in S^U$, and $j \in \mathbb{N}_0$, the following holds:*

$$\text{(i) } \mathbb{P}\left(Z_{\tau_{j+1}} = k | Z_{\nu_j} = i, \nu_j < \infty\right) = \mathbb{P}_{\delta_i^L}\left(Z_{\tau_1} = k\right) \quad \text{and}$$

$$\mathbb{P}\left(Z_{\tau_{j+1}} = k | \tau_{j+1} < \infty, Z_{\nu_j} = i\right) = \mathbb{P}_{\delta_i^L}\left(Z_{\tau_1} = k | \tau_1 < \infty\right), \quad \text{and}$$

$$\text{(ii) } \mathbb{P}\left(Z_{\nu_{j+1}} = i | Z_{\tau_{j+1}} = k, \tau_{j+1} < \infty\right) = \mathbb{P}_{\delta_k^U}\left(Z_{\nu_1} = i | \tau_1 < \infty\right) \quad \text{and}$$

$$\mathbb{P}\left(Z_{\nu_{j+1}} = i | \nu_{j+1} < \infty, Z_{\tau_{j+1}} = k\right) = \mathbb{P}_{\delta_k^U}\left(Z_{\nu_1} = i | \nu_1 < \infty\right).$$

*If additionally (**H3**) holds, then (iii) τ_j and ν_j are finite a.s.,*

$$\mathbb{P}\left(Z_{\tau_{j+1}} = k | Z_{\nu_j} = i\right) = \mathbb{P}_{\delta_i^L}\left(Z_{\tau_1} = k\right), \quad \text{and}$$

$$\mathbb{P}\left(Z_{\nu_{j+1}} = i | Z_{\tau_{j+1}} = k\right) = \mathbb{P}_{\delta_k^U}\left(Z_{\nu_1} = i\right).$$

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Proof of Lemma A.1. We only prove (i) and (iii). Since $\mathbb{P}\big(Z_{\tau_{j+1}}=k|Z_{\nu_j}=i,\,\nu_j<\infty\big)$ is equal to

$$\sum_{s=1}^{\infty}\sum_{u=L_U+1}^{\infty}\mathbb{P}\big(Z_{\nu_j+s}=k,\,Z_{\nu_j+s-1}<u,\,\ldots,\,Z_{\nu_j+1}<u|Z_{\nu_j}=i,\,\nu_j<\infty\big),$$

it is enough to show that for all $s\geq 1$ and $u\geq L_U+1$,

$$\begin{aligned}&\mathbb{P}\big(Z_{\nu_j+s}=k,\,Z_{\nu_j+s-1}<u,\,\ldots,\,Z_{\nu_j+1}<u|Z_{\nu_j}=i,\,\nu_j<\infty\big)\\&=\mathbb{P}_{\delta_i^L}(Z_s=k,\,Z_{s-1}<u,\,\ldots,\,Z_1<u).\end{aligned}\tag{26}$$

To this end, we condition on $\Pi^U_{\nu_j+l}=\big(p^U_l,\,q^U_l\big)$ and $\Pi^U_l=\big(p^U_l,\,q^U_l\big)$, where $l=0,\,1,\,\ldots,\,s-1$. Since, given $\Pi^U_{\nu_j+l}=\big(p^U_l,\,q^U_l\big)$ and $\Pi^U_l=\big(p^U_l,\,q^U_l\big)$, both the sequences $\big\{\xi^U_{\nu_j+l,i}\big\}_{i=1}^{\infty}$, $\{\xi^U_{l,i}\}_{i=1}^{\infty}$ and the random variables $I^U_{\nu_j+l}$, I^U_l are i.i.d., we obtain from (1) that

$$\begin{aligned}&\mathbb{P}\big(Z_{\nu_j+s}=k,\,Z_{\nu_j+s-1}<u,\,\ldots,\,Z_{\nu_j+1}<u|Z_{\nu_j}=i,\,\nu_j<\infty,\,\big\{\Pi^U_{\nu_j+l}\big\}_{l=0}^{s-1}=\big\{\big(p^U_l,\,q^U_l\big)\big\}_{l=0}^{s-1}\big)\\&=\mathbb{P}_{\delta_i^L}\big(Z_s=k,\,Z_{s-1}<u,\,\ldots,\,Z_1<u|\big\{\Pi^U_l\big\}_{l=0}^{s-1}=\big\{\big(p^U_l,\,q^U_l\big)\big\}_{l=0}^{s-1}\big).\end{aligned}$$

By taking the expectation with respect to $\Pi^U=\{\Pi^U_n\}_{n=0}^{\infty}$ and using that the Π^U_n are i.i.d., we obtain (26). Next, we notice that

$$\mathbb{P}(Z_{\tau_{j+1}}=k|\tau_{j+1}<\infty,\,Z_{\nu_j}=i)=\frac{\mathbb{P}\big(Z_{\tau_{j+1}}=k|Z_{\nu_j}=i,\,\nu_j<\infty\big)}{\mathbb{P}\big(\tau_{j+1}<\infty|Z_{\nu_j}=i,\,\nu_j<\infty\big)},\ \text{where}$$

$$\mathbb{P}\big(\tau_{j+1}<\infty|Z_{\nu_j}=i,\,\nu_j<\infty\big)=\sum_{k=L_U+1}^{\infty}\mathbb{P}\big(Z_{\tau_{j+1}}=k|Z_{\nu_j}=i,\,\nu_j<\infty\big)$$

is positive because $M^U>1$. It follows from Part (i) that $\mathbb{P}\big(Z_{\tau_{j+1}}=k|\tau_{j+1}<\infty,\,Z_{\nu_j}=i\big)=\mathbb{P}_{\delta_i^L}(Z_{\tau_1}=k|\tau_1<\infty)$. Finally, (iii) follows from (i) and (ii) using (3) and (4). □

A.3. Finiteness of $\overline{\pi}^U$

We show that the stationary distribution π^U of the Markov chain $\big\{Z_{\tau_j}\big\}_{j=1}^{\infty}$ has a finite first moment $\overline{\pi}^U$.

Proposition A.1. *Under (**H1**)–(**H4**), (**H6**) (or (**H7**)), and (**H9**), $\overline{\pi}^U<\infty$.*

Proof of Proposition A.1. Using that $\pi^U=\{\pi^U_k\}_{k\in S^U}$ is the stationary distribution of the Markov chain $\{Z_{\tau_j}\}_{j=1}^{\infty}$, for all $k\in S^U$ we write $\pi^U_k=\mathbb{P}_{\pi^U}(Z_{\tau_2}=k)=\mathbb{E}[\mathbb{P}_{\pi^U}(Z_{\tau_2}=k|U_2)]$. Next, we notice that

$$\mathbb{P}_{\pi^U}\big(Z_{\tau_2}=k|U_2\big)=\sum_{n=3}^{\infty}\mathbb{P}_{\pi^U}\big(Z_{\tau_2}=k|\tau_2=n,\,U_2\big)\mathbb{P}_{\pi^U}\big(\tau_2=n|U_2\big).$$

Now, using that the event $\{\tau_2=n\}$ is same as

$$\{Z_n\geq U_2\}\cap\cap_{k=\nu_1+1}^{n-1}\{Z_k<U_2\}\cap\{\nu_1\leq n-1\},$$

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the right-hand side of the above inequality is bounded above by

$$\max_{i=0,1,\ldots,U_2-1}\sum_{n=3}^{\infty}\mathbb{P}\big(Z_n=k|Z_n\geq U_2,\,U_2,\,Z_{n-1}=i,\,\cap_{k=\nu_1+1}^{n-2}\{Z_k<U_2\},\,\nu_1\leq n-1\big)$$
$$\times\,\mathbb{P}_{\pi^U}(\tau_2=n|U_2).$$

Since BPRE is a time-homogeneous Markov chain, it follows that

$$\mathbb{P}\big(Z_n=k|Z_n\geq U_2,\,U_2,\,Z_{n-1}=i,\,\cap_{k=\nu_1+1}^{n-2}\{Z_k<U_2\},\,\nu_1\leq n-1\big)$$
$$=\mathbb{P}_{\delta_i^L}(Z_1=k|Z_1\geq U_2,\,U_2),$$

where we also use the fact that the process starts in the supercritical regime. Now, using the fact that U_1 and U_2 are i.i.d., it follows that

$$\pi_k^U\leq\mathbb{E}\Big[\max_{i=0,1,\ldots,U_1-1}\mathbb{P}_{\delta_i^L}(Z_1=k|Z_1\geq U_1,\,U_1)\Big].$$

Since

$$\mathbb{P}_{\delta_i^L}(Z_1=k|Z_1\geq U_1,\,U_1)\leq\frac{\mathbb{P}_{\delta_i^L}(Z_1=k)}{\mathbb{P}_{\delta_i^L}(Z_1\geq U_1|U_1)},$$

using the Fubini–Tonelli theorem, we obtain that

$$\overline{\pi}^U\leq\mathbb{E}\Bigg[\max_{i=0,1,\ldots,U_1-1}\frac{\sum_{k\in S^U}k\mathbb{P}_{\delta_i^L}(Z_1=k)}{\mathbb{P}_{\delta_i^L}(Z_1\geq U_1|U_1)}\Bigg]\leq\mathbb{E}\Bigg[\max_{i=0,1,\ldots,U_1-1}\frac{\mathbb{E}_{\delta_i^L}[Z_1]}{\mathbb{P}_{\delta_i^L}(Z_1\geq U_1|U_1)}\Bigg].$$

Now, for all $i=0,\,1,\,\ldots,\,U_1-1$, we have that

$$\mathbb{P}_{\delta_i^L}(Z_1\geq U_1|U_1)\geq\mathbb{P}(I_0^U\geq U_1|U_1).$$

Finally, using the assumptions (H2), (H3), and (H9), we conclude that

$$\overline{\pi}^U\leq\mathbb{E}\Bigg[\frac{(U_1-1)M^U+N^U}{\mathbb{P}(I_0^U\geq U_1|U_1)}\Bigg]\leq\mathbb{E}\Bigg[\frac{U_1}{\mathbb{P}(I_0^U\geq U_1|U_1)}\Bigg]\max\big(M^U,\,N^U\big)<\infty.$$

□

A.4. Proofs of Lemma 4.1 and Lemma 4.2

Proof of Lemma 4.1. We begin by proving (i). It is sufficient to show that for $n\in\mathbb{N}_0$ and $k\in\mathbb{N}$,

$$\mathbb{P}_{\delta_i^L}\big(B_{i,n+1}^L=\big(k,\boldsymbol{d}^L,\boldsymbol{d}^L+\boldsymbol{d}^U\big)|\boldsymbol{B}_{i,n}^L\big)=\mathbb{P}_{\delta_i^L}\big(B_{i,1}^L=\big(k,\boldsymbol{d}^L,\boldsymbol{d}^L+\boldsymbol{d}^U\big)\big),\tag{27}$$

where $\boldsymbol{d}^L=\big(d_1^L,\,\ldots,\,d_k^L\big)$, $\boldsymbol{d}^U=\big(d_1^U,\,\ldots,\,d_k^U\big)$, $d_j^L,\,d_j^U\in\mathbb{N}$, and $\boldsymbol{B}_{i,n}^L:=\big\{B_{i,l}^L\big\}_{l=1}^n$. For simplicity set $X_j:=Z_{\nu_j}$. We recall that $\overline{\Delta}_j^U=\Delta_j^U+\Delta_j^L$ and notice

$$\mathbb{P}_{\delta_i^L}\big(B_{i,n+1}^L=\big(k,\boldsymbol{d}^L,\boldsymbol{d}^L+\boldsymbol{d}^U\big)|\boldsymbol{B}_{i,n}^L\big)=$$
$$\mathbb{P}_{\delta_i^L}\big(K_{i,n+1}^L=k,\,\cap_{j=1}^k\big\{\Delta_{T_{i,n}+j}^U=d_j^U,\,\Delta_{T_{i,n}+j}^L=d_j^L\big\}|X_{T_{i,n}^L}=i,\,\boldsymbol{B}_{i,n}^L\big)=$$
$$\mathbb{P}_{\delta_i^L}\big(X_{T_{i,n}+k}=i,\,\cap_{j=1}^{k-1}\big\{X_{T_{i,n}+j}\neq i\big\},\,\cap_{j=1}^k\big\{\Delta_{T_{i,n}+j}^U=d_j^U,\,\Delta_{T_{i,n}+j}^L=d_j^L\big\}|X_{T_{i,n}^L}=i,\,\boldsymbol{B}_{i,n}^L\big).$$

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We now compute the last term of the above equation. Specifically, by proceeding as in the proof of Lemma A.1 (involving conditioning on the environments), we obtain that for $n, k \in \mathbb{N}$, $x_j \in S^L$, and $x_0 = i$,

$$\mathbb{P}_{\delta_i^L}\Big(\cap_{j=1}^k \Big\{X_{T_{i,n}^L+j} = x_j,\, \Delta^U_{T_{i,n}^L+j} = d_j^U,\, \Delta^L_{T_{i,n}^L+j} = d_j^L\Big\} | X_{T_{i,n}^L} = i, \boldsymbol{B}_{i,n}^L\Big)$$

$$= \prod_{j=1}^k \mathbb{P}\Big(X_{T_{i,n}^L+j} = x_j,\, \Delta^U_{T_{i,n}^L+j} = d_j^U,\, \Delta^L_{T_{i,n}^L+j} = d_j^L | X_{T_{i,n}^L+j-1} = x_{j-1}\Big)$$

$$= \prod_{j=1}^k \mathbb{P}\Big(X_j = x_j,\, \Delta_j^U = d_j^U,\, \Delta_j^L = d_j^L | X_{j-1} = x_{j-1}\Big)$$

$$= \mathbb{P}\Big(\cap_{j=1}^k \Big\{X_j = x_j,\, \Delta_j^U = d_j^U,\, \Delta_j^L = d_j^L\Big\} | X_0 = i\Big).$$

Now, by summing over $x_k \in \{i\}$ and $x_j \in S^L \setminus \{i\}$, we obtain that

$$\mathbb{P}_{\delta_i^L}\big(X_{T_{i,n}+k} = i, \cap_{j=1}^{k-1}\big\{X_{T_{i,n}+j} \neq i\big\}, \cap_{j=1}^k\big\{\Delta^U_{T_{i,n}+j} = d_j^U,\, \Delta^L_{T_{i,n}+j} = d_j^L\big\} | X_{T_{i,n}^L} = i, \boldsymbol{B}_{i,n}^L\big)$$

$$= \mathbb{P}_{\delta_i^L}\big(X_k = i, \cap_{j=1}^{k-1}\big\{X_j \neq i\big\}, \cap_{j=1}^k\big\{\Delta_j^U = d_j^U,\, \Delta_j^L = d_j^L\big\}\big).$$

The last term in the above is

$$\mathbb{P}_{\delta_i^L}\big(T_{i,1}^L = k, \cap_{j=1}^k\big\{\Delta_j^U = d_j^U,\, \Delta_j^L = d_j^L\big\}\big) = \mathbb{P}_{\delta_i^L}\big(B_{i,1}^L = \big(k, \boldsymbol{d}^L, \boldsymbol{d}^L + \boldsymbol{d}^U\big)\big).$$

We thus obtain (27). The proof of (ii) is similar. □

Proof of Lemma 4.2. The first part of (i) follows from Proposition 1.69 of Serfozo [27] with $X_j = Z_{\nu_j}$, $\pi = \pi^L = \{\pi_i^L\}_{i \in S^L}$, and $V_j = \Delta_{j+1}^U$. For the second part of (i) we use the above proposition with $V_j = \overline{\Delta}_{j+1}^U$ and obtain that

$$\mathbb{E}_{\delta_i^L}\Big[\overline{S}^U_{T_{i,1}^L}\Big] = \big(\pi_i^L\big)^{-1}\mathbb{E}_{\pi^L}\Big[\overline{\Delta}_1^U\Big] = \big(\pi_i^L\big)^{-1}\Big(\mathbb{E}_{\pi^L}\big[\Delta_1^L\big] + \mu^U\Big).$$

Remark (3.2) yields that

$$\mathbb{E}_{\pi^L}\big[\Delta_1^L\big] = \sum_{k \in S^L}\sum_{l \in S^U} \mathbb{E}_{\delta_l^U}\big[\Delta_1^L\big] P_{\delta_k^L}(Z_{\tau_1} = l]\pi_k^L = \mu^L.$$

We now prove the first part of (ii). Since, conditionally on $Z_{\nu_0} = i$, Δ_1^U and $\Delta^U_{T_{i,1}^L+1}$ have the same distribution, using (i) we have that

$$\mathbb{V}_{\delta_i^L}\Big[S^U_{T_{i,1}^L}\Big] = \mathbb{E}_{\delta_i^L}\left[\left(\sum_{j=1}^{T_{i,1}^L}\big(\Delta_{j+1}^U - \mu^U\big)\right)^2\right] = \mathbb{E}_{\delta_i^L}\left[\sum_{j=1}^{T_{i,1}^L}\big(\Delta_{j+1}^U - \mu^U\big)^2\right] + 2C_i^U,$$

where

$$C_i^U := \mathbb{E}_{\delta_i^L}\left[\sum_{j=1}^{T_{i,1}^L}\big(\Delta_{j+1}^U - \mu^U\big)\sum_{l=j+1}^{T_{i,1}^L}\big(\Delta_{l+1}^U - \mu^U\big)\right].$$

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Next, we apply Proposition 1.69 of Serfozo [27] with $X_j = Z_{\nu_j}$, $\pi = \pi^L = \{\pi_i^L\}_{i\in S^L}$, and $V_j = \left(\Delta_{j+1}^U - \mu^U\right)^2$ and obtain that

$$\mathbb{E}_{\delta_i^L}\left[\sum_{j=1}^{T_{i,1}^L}\left(\Delta_{j+1}^U - \mu^U\right)^2\right] = \left(\pi_i^L\right)^{-1}\mathbb{E}_{\pi^L}\left[\left(\Delta_1^U - \mu^U\right)^2\right].$$

Then we compute

$$\begin{aligned}
C_i^U &= \mathbb{E}_{\delta_i^L}\left[\sum_{j=1}^{\infty}\mathbf{I}_{\{T_{i,1}^L \geq j\}}\mathbb{E}_{\delta_i^L}\left[\left(\Delta_{j+1}^U - \mu^U\right)\sum_{l=j+1}^{T_{i,1}^L}\left(\Delta_{l+1}^U - \mu^U\right)|T_{i,1}^L \geq j\right]\right] \\
&= \mathbb{E}_{\delta_i^L}\left[\sum_{j=1}^{\infty}\mathbf{I}_{\{T_{i,1}^L \geq j\}}\sum_{k\in S^L}\mathbf{I}_{\{Z_{\nu_j}=k\}}g^L(k)\right] \\
&= \sum_{k\in S^L} g^L(k)\mathbb{E}_{\delta_i^L}\left[\sum_{j=1}^{T_{i,1}^L}\mathbf{I}_{\{Z_{\nu_j}=k\}}\right],
\end{aligned}$$

where $g^L : S^L \to \mathbb{R}$ is given by

$$\begin{aligned}
g^L(k) &= \mathbb{E}_{\delta_i^L}\left[\left(\Delta_{j+1}^U - \mu^U\right)\sum_{l=j+1}^{T_{i,1}^L}\left(\Delta_{l+1}^U - \mu^U\right)|T_{i,1}^L \geq j, Z_{\nu_j} = k\right] \\
&= \sum_{l=j+1}^{\infty}\mathbb{E}_{\delta_i^L}\left[\left(\Delta_{j+1}^U - \mu^U\right)\left(\Delta_{l+1}^U - \mu^U\right)\mathbf{I}_{\{T_{i,1}^L \geq l\}}|T_{i,1}^L \geq j, Z_{\nu_j} = k\right].
\end{aligned}$$

Using Theorem 1.54 of Serfozo [27], we obtain

$$\mathbb{E}_{\delta_i^L}\left[\sum_{j=1}^{T_{i,1}^L}\mathbf{I}_{\{Z_{\nu_j}=k\}}\right] = \left(\pi_i^L\right)^{-1}\pi_k^L,$$

which yields

$$C_i^U = \left(\pi_i^L\right)^{-1}\sum_{k\in S^L} g^L(k)\pi_k^L.$$

Now, using Lemma 4.1, we see that, conditionally on $j \leq T_{i,1}^L < l$, $\left(\Delta_{l+1}^U - \mu^U\right)$ is independent of $\left(\Delta_{j+1}^U - \mu^U\right)$. If $Z_{\nu_j} \sim \pi^L$, then using stationarity (see Remark 3.1),

$$\mathbb{E}\left[\left(\Delta_{l+1}^U - \mu^U\right)|j \leq T_{i,1}^L < l, Z_{\nu_j} \sim \pi^L\right] = \mathbb{E}\left[\left(\Delta_{l+1}^U - \mu^U\right)|Z_{\nu_l} \sim \pi^L\right] = 0.$$

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Therefore,

$$\sum_{l=j+1}^{\infty}\sum_{k\in S^L}\pi_k^L\mathbb{E}_{\delta_i^L}\Big[\big(\Delta_{j+1}^U-\mu^U\big)\big(\Delta_{l+1}^U-\mu^U\big)\mathbf{I}_{\{T_{i,1}^L<l\}}|T_{i,1}^L\ge j, Z_{\nu_j}=k\Big]=$$

$$\sum_{l=j+1}^{\infty}\mathbb{E}_{\delta_i^L}\Big[\big(\Delta_{j+1}^U-\mu^U\big)\mathbb{E}\big[\big(\Delta_{l+1}^U-\mu^U\big)|j\le T_{i,1}^L<l, Z_{\nu_j}\sim\pi^L\big]\mathbf{I}_{\{T_{i,1}^L<l\}}|T_{i,1}^L\ge j, Z_{\nu_j}\sim\pi^L\Big]$$

equals 0. Adding the above to $\sum_{k\in S^L} g^L(k)\pi_k^L$, we conclude that

$$\begin{aligned}\sum_{k\in S^L} g^L(k)\pi_k^L &= \sum_{l=j+1}^{\infty}\sum_{k\in S^L}\pi_k^L\mathbb{E}_{\delta_i^L}\Big[\big(\Delta_{j+1}^U-\mu^U\big)\big(\Delta_{l+1}^U-\mu^U\big)|T_{i,1}^L\ge j, Z_{\nu_j}=k\Big]\\ &=\sum_{l=1}^{\infty}\sum_{k\in S^L}\pi_k^L\mathbb{E}_{\delta_k^L}\Big[\big(\Delta_1^U-\mu^U\big)\big(\Delta_{l+1}^U-\mu^U\big)\Big]\\ &=\sum_{l=1}^{\infty}\mathbb{C}_{\pi^L}\big[\Delta_1^U,\,\Delta_{l+1}^U\big].\end{aligned}$$

The second part of (ii) and (iii) are obtained similarly. □

A.5. Finiteness of μ^T, $\sigma^{2,T}$, and $\overline{\sigma}^{2,T}$

We establish positivity and finiteness of $\sigma^{2,T}$ and $\overline{\sigma}^{2,T}$, where $T\in\{L, U\}$. Lemma A.2 below is used to control the covariance terms in $\sigma^{2,T}$ and $\overline{\sigma}^{2,T}$. We recall that uniform ergodicity of the Markov chains $\{Z_{\nu_j}\}_{j=0}^{\infty}$ and $\{Z_{\tau_j}\}_{j=1}^{\infty}$ is equivalent to the existence of constants $C_T\ge 0$ and $\rho_T\in(0, 1)$ such that $\sup_{l\in S_T}\|p_l^T(j)-\pi^T\|\le C_T\rho_T^j$.

Lemma A.2. *Assume (**H1**)–(**H4**).The following holds:*

(i) *If (**H5**) holds and $w_i\in\mathbb{R}$, $i\in S^L$, then*

$$\sum_{j=1}^{\infty}\left|\sum_{i,k\in S^L} w_kw_i\pi_k^Lp_{ki}^L(j)-\left(\sum_{k\in S^L}w_k\pi_k^L\right)\left(\sum_{i\in S^L}w_i\pi_i^L\right)\right|\le 2C_L^{1/2}\frac{\rho_L^{1/2}}{1-\rho_L^{1/2}}\left(\sum_{k\in S^L}w_k^2\pi_k^L\right).$$

(ii) *If (**H6**) (or (**H7**))holds and $w_k\in\mathbb{R}$, $k\in S^U$, then*

$$\sum_{j=1}^{\infty}\left|\sum_{i,k\in S^U} w_kw_i\pi_k^Up_{ki}^U(j)-\left(\sum_{k\in S^U}w_k\pi_k^U\right)\left(\sum_{i\in S^U}w_i\pi_i^U\right)\right|\le 2C_U^{1/2}\frac{\rho_U^{1/2}}{1-\rho_U^{1/2}}\left(\sum_{k\in S^U}w_k^2\pi_k^U\right).$$

The proof of the above lemma can be constructed along the lines of Theorem 17.2.3 of Ibragimov and Linnik [31] with $p=q=1/2$; it involves repeated use of the Cauchy–Schwarz inequality and the stationarity in Remark 3.1.

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Proof of Lemma A.2. Since the proof of (ii) is similar, we only prove (i). We proceed along the lines of the proof of Theorem 17.2.3 of Ibragimov and Linnik [31]. Using the Cauchy–Schwarz inequality, we have that

$$\begin{aligned}
&\Bigg|\sum_{i,k\in S^L} w_k w_i \pi_k^L p_{ki}^L(j) - \Bigg(\sum_{k\in S^L} w_k \pi_k^L\Bigg)\Bigg(\sum_{i\in S^L} w_i \pi_i^L\Bigg)\Bigg| \\
&= \Bigg|\sum_{k\in S^L} w_k \big(\pi_k^L\big)^{1/2} \sum_{i\in S^L} w_i \big(p_{ki}^L(j) - \pi_i^L\big)\big(\pi_k^L\big)^{1/2}\Bigg| \\
&\le \Bigg(\sum_{k\in S^L} (w_k)^2 \pi_k^L\Bigg)^{1/2} \Bigg(\sum_{k\in S^L} \pi_k^L \Bigg(\sum_{i\in S^L} w_i \big(p_{ki}^L(j) - \pi_i^L\big)\Bigg)^2\Bigg)^{1/2}.
\end{aligned}$$

Using the Cauchy–Schwarz inequality again, we obtain that

$$\begin{aligned}
\Bigg(\sum_{i\in S^L} w_i\big(p_{ki}^L(j) - \pi_i^L\big)\Bigg)^2 &\le \Bigg(\sum_{i\in S^L} |w_i| \big(p_{ki}^L(j) + \pi_i^L\big)^{1/2} |p_{ki}^L(j) - \pi_i^L|^{1/2}\Bigg)^2 \\
&\le \Bigg(\sum_{i\in S^L} (w_i)^2 \big(p_{ki}^L(j) + \pi_i^L\big)\Bigg)\Bigg(\sum_{i\in S^L} |p_{ki}^L(j) - \pi_i^L|\Bigg).
\end{aligned}$$

Since $\sum_{k\in S^L} p_{ki}^L(j)\pi_k^L = \pi_i^L$ by Remark 3.1, we deduce that

$$\begin{aligned}
&\Bigg(\sum_{k\in S^L} \pi_k^L \Bigg(\sum_{i\in S^L} w_i \big(p_{ki}^L(j) - \pi_i^L\big)\Bigg)^2\Bigg)^{1/2} \\
&\le \Bigg(\sum_{k\in S^L} \pi_k^L \Bigg(\sum_{i\in S^L} (w_i)^2 \big(p_{ki}^L(j) + \pi_i^L\big)\Bigg)\Bigg(\sum_{i\in S^L} |p_{ki}^L(j) - \pi_i^L|\Bigg)\Bigg)^{1/2} \\
&\le \Bigg(2\sum_{i\in S^L} (w_i)^2 \pi_i^L\Bigg)^{1/2} \sup_{k\in S^L} \Bigg(\sum_{i\in S^L} |p_{ki}^L(j) - \pi_i^L|\Bigg)^{1/2}.
\end{aligned}$$

Using that $\sup_{l\in S^L} \|p_l^L(j) - \pi^L\| \le C_L \rho_L^j$, we obtain that

$$\begin{aligned}
&\sup_{k\in S^L} \Bigg(\sum_{i\in S^L} |p_{ki}^L(j) - \pi_i^L|\Bigg) \\
&\le \sup_{k\in S^L} \Bigg(\sum_{i\in S^L\,:\,p_{ki}^L(j)-\pi_i^L>0} \big(p_{ki}^L(j) - \pi_i^L\big)\Bigg) + \sup_{k\in S^L} \Bigg(\sum_{i\in S^L\,:\,p_{ki}^L(j)-\pi_i^L<0} \big(\pi_i^L - p_{ki}^L(j)\big)\Bigg) \\
&\le \sup_{k\in S^L} \Bigg(\sum_{i\in S^L\,:\,p_{ki}^L(j)-\pi_i^L>0} p_{ki}^L(j) - \sum_{i\in S^L\,:\,p_{ki}^L(j)-\pi_i^L>0} \pi_i^L\Bigg)
\end{aligned}$$

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$$+\sup_{k\in S^L}\left(\sum_{i\in S^L:\,p^L_{ki}(j)-\pi^L_i<0}\pi^L_i-\sum_{i\in S^L:\,p^L_{ki}(j)-\pi^L_i<0}p^L_{ki}(j)\right)$$
$$\le 2C_L\rho_L^j.$$

We have thus shown that

$$\left|\sum_{i,k\in S^L}w_kw_i\pi^L_kp^L_{ki}(j)-\left(\sum_{k\in S^L}w_k\pi^L_k\right)\left(\sum_{i\in S^L}w_i\pi^L_i\right)\right|\le 2C_L^{1/2}\rho_L^{j/2}\left(\sum_{k\in S^L}w_k^2\pi^L_k\right),$$

which yields that

$$\sum_{j=1}^{\infty}\left|\sum_{i,k\in S^L}w_kw_i\pi^L_kp^L_{ki}(j)-\left(\sum_{k\in S^L}w_k\pi^L_k\right)\left(\sum_{i\in S^L}w_i\pi^L_i\right)\right|\le 2C_L^{1/2}\frac{\rho_L^{1/2}}{1-\rho_L^{1/2}}\left(\sum_{k\in S^L}w_k^2\pi^L_k\right).$$

□

We are now ready to study the finiteness of means and variances μ^T, $\sigma^{2,T}$, and $\overline{\sigma}^{2,T}$, where $T\in\{L, U\}$.

Proposition A.2. *Assume (**H1**)–(**H4**). (i) If (**H5**) and (**H8**) also hold, then $\mu^U<\infty$ and $0<\sigma^{2,U}<\infty$. (ii) Next, if (**H6**) (or (**H7**)) and (**H9**) hold, then $\mu^L<\infty$ and $0<\sigma^{2,L}<\infty$. (iii) Additionally, under the assumptions in (i) and (ii), we have $0<\overline{\sigma}^{2,U}, \overline{\sigma}^{2,L}<\infty$.*

It is easy to see that Proposition A.2 implies that $|\mathbb{C}^U|$ and $|\mathbb{C}^L|$ are finite.

Proof of Proposition A.2. We begin by proving (i). For all $i\in S^L$ it holds that

$$\mathbb{E}_{\delta^L_i}[\tau_1]=\sum_{n=1}^{\infty}\mathbb{P}_{\delta^L_i}(\tau_1\ge n)\le\sum_{n=0}^{\infty}\mathbb{P}_{\delta^L_i}(\tilde{Z}_n<U_1),$$

where $\{\tilde{Z}_n\}_{n=0}^{\infty}$ is a supercritical BPRE with immigration having environmental sequence $\Pi^U=\{\Pi^U_n\}_{n=0}^{\infty}$ and, conditionally on Π^U_n, offspring distributions $\{\xi^U_{n,i}\}_{i=0}^{\infty}$ and immigration distribution I^U_n. Using the fact that $\lim_{n\to\infty}\tilde{Z}_n=\infty$ a.s. and $\mathbb{E}[U_1]<\infty$, we see that $\lim_{n\to\infty}\tilde{Z}_n\mathbb{P}_{\delta^L_i}(\tilde{Z}_n<U_1|\tilde{Z}_n)=0$ a.s. Since $\lim_{n\to\infty}\frac{\tilde{Z}_n}{(M^L)^n}>0$, we obtain $\lim_{n\to\infty}(M^L)^n\mathbb{P}_{\delta^L_i}(\tilde{Z}_n<U_1|\tilde{Z}_n)=0$ a.s., which yields $\lim_{n\to\infty}(M^L)^n\mathbb{P}_{\delta^L_i}(\tilde{Z}_n<U_1)=0$. Therefore, there exists $\tilde{C}$ such that

$$\mathbb{P}_{\delta^L_i}(\tilde{Z}_n<U_1)\le\tilde{C}(M^L)^n. \tag{28}$$

Now, using the fact that S^L is finite and $\mathbb{E}[U_1]<\infty$, it follows that

$$\mu^U=\sum_{i\in S^L}\mathbb{E}_{\delta^L_i}[\tau_1]\pi^L_i\le\left(\max_{i\in S^L}C_i\right)\frac{\mathbb{E}[U_1]}{1-\gamma}<\infty.$$

Turning to the finiteness of $\sigma^{2,U}$, replacing n by $\lfloor\sqrt{n}\rfloor$ in (28), one obtains that

$$\mathbb{E}_{\delta^L_k}\left[\tau_1^2\right]=\sum_{n=1}^{\infty}\mathbb{P}_{\delta^L_k}\left(\tau_1\ge\sqrt{n}\right)\le\sum_{n=0}^{\infty}\mathbb{P}_{\delta^L_k}(\tilde{Z}_{\lfloor\sqrt{n}\rfloor}<U_1)<\infty.$$

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This together with the finiteness of μ^U yields $\mathbb{V}_{\pi^L}[\Delta_1^U] < \infty$. Turning to the covariance terms in $\sigma^{2,U}$, we apply Lemma A.2(i) with $w_i = \mathbb{E}_{\delta_i^L}[\tau_1 - \mu^U]$, and using $\sum_{i \in S^L} w_i \pi_i^L = 0$, we obtain that

$$\sum_{j=1}^{\infty} |\mathbb{C}_{\pi^L}\left[\Delta_1^U, \Delta_{j+1}^U\right]| = \sum_{j=1}^{\infty} |\sum_{k \in S^L} \mathbb{E}_{\delta_k^L}\left[\tau_1 - \mu^U\right]\pi_k^L \sum_{i \in S^L} \mathbb{E}_{\delta_i^L}\left[\tau_1 - \mu^U\right] p_{ki}^L(j)|$$

$$\leq 2C_L^{1/2}\left(\frac{\rho_L^{1/2}}{1-\rho_L^{1/2}}\right)\left(\sum_{k \in S^L} \left(\mathbb{E}_{\delta_k^L}[\tau_1 - \mu^U]\right)^2 \pi_k^L\right) < \infty.$$

We conclude that $\sigma^{2,U}$ is finite. For $i \in S^U$, it holds that

$$\mathbb{E}_{\delta_i^U}\left[\Delta_1^L\right] = \sum_{n=0}^{\infty} \mathbb{P}_{\delta_i^U}(\Delta_1^L > n) \leq i \sum_{n=0}^{\infty} \left(M^L\right)^n = \frac{i}{1-M^L},$$

where the inequality follows from the upper bound on the extinction time of the process in the subcritical regime. Hence, using Proposition A.1 it follows that $\mu^L \leq \frac{\overline{\pi}^U}{1-M^L} < \infty$.

Next we show that $\sigma^{2,L}$ is finite. As before, we obtain that for all $i \in S^U$,

$$\mathbb{E}_{\delta_i^U}\left[\left(\Delta_1^L\right)^2\right] \leq \sum_{n=0}^{\infty} \mathbb{P}_{\delta_i^U}(\Delta_1^L > \lfloor\sqrt{n}\rfloor] \leq i \sum_{n=0}^{\infty} \left(M^L\right)^{\lfloor\sqrt{n}\rfloor}, \tag{29}$$

yielding that

$$\mathbb{E}_{\pi^U}\left[\left(\Delta_1^L\right)^2\right] \leq \overline{\pi}^U \sum_{n=0}^{\infty} \left(M^L\right)^{\lfloor\sqrt{n}\rfloor} < \infty. \tag{30}$$

This together with the finiteness of $\mu^L < \infty$ implies that $\mathbb{V}_{\pi^L}\left[\Delta_1^L\right] < \infty$. Turning to covariances, we apply Lemma A.2(ii) with $w_i = \mathbb{E}_{\delta_i^U}\left[\Delta_1^L - \mu^L\right]$, and using the fact that $\sum_{i \in S^U} w_i \pi_i^U = 0$, we obtain that

$$\sum_{j=1}^{\infty} |\mathbb{C}_{\pi^U}[\Delta_1^L, \Delta_{j+1}^L]| \leq 2C_U^{1/2}\left(\frac{\rho_U^{1/2}}{1-\rho_U^{1/2}}\right)\left(\sum_{k \in S^U} \left(\mathbb{E}_{\delta_k^U}[\Delta_1^L - \mu^L]\right)^2 \pi_k^U\right) < \infty,$$

yielding the finiteness of $\sigma^{2,L}$. Turning to (iii), we compute

$$\mathbb{V}_{\pi^L}\left[\overline{\Delta}_1^U\right] \leq 2\left(\mathbb{V}_{\pi^L}\left[\Delta_1^U\right] + \mathbb{V}_{\pi^L}\left[\Delta_1^L\right]\right),$$

where, by Part (i), $\mathbb{V}_{\pi^L}\left[\Delta_1^U\right] < \infty$, and using Remark 3.2,

$$\mathbb{V}_{\pi^L}\left[\Delta_1^L\right] = \sum_{k \in S^U} \sum_{i \in S^L} \mathbb{V}_{\delta_k^U}\left[\Delta_1^L\right] \mathbb{P}_{\delta_i^L}\left(Z_{\tau_1} = k\right)\pi_i^L = \mathbb{V}_{\pi^U}\left[\Delta_1^L\right] < \infty.$$

Turning to the covariance, we again apply Lemma A.2 (i) with $w_i = \mathbb{E}_{\delta_i^L}\left[\overline{\Delta}_1^U - \left(\mu^U + \mu^L\right)\right]$ and conclude that also

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$$\sum_{j=1}^{\infty}\mathbb{C}_{\pi^L}\Big[\overline{\Delta}_1^U,\overline{\Delta}_{j+1}^U\Big]\le 2C_L^{1/2}\left(\frac{\rho_L^{1/2}}{1-\rho_L^{1/2}}\right)\left(\sum_{k\in S^L}\left(\mathbb{E}_{\delta_k^L}\Big[\overline{\Delta}_1^U-(\mu^U+\mu^L)\Big]\right)^2\pi_k^L\right).$$

The finiteness of the right-hand side yields $\overline{\sigma}^{2,U}<\infty$. The proof that $\overline{\sigma}^{2,L}<\infty$ is similar.

We finally establish that $\sigma^{2,U}$, $\sigma^{2,L}$, $\overline{\sigma}^{2,U}$, and $\overline{\sigma}^{2,L}$ are positive. We first show that, conditionally on $Z_0\sim\delta_i^L$ and $Z_{\tau_1}\sim\delta_k^U$, Δ_1^U and Δ_1^L are non-degenerate. To this end, suppose for the sake of contradiction that

$$1=\mathbb{P}_{\delta_i^L}\big(\Delta_1^U=\mu^U\big)=\sum_{u=L_U+1}^{\infty}\mathbb{P}_{\delta_i^L}\big(Z_{\mu^U}\ge u, Z_{\mu^U-1}<u,\ldots,Z_1<u\big)\mathbb{P}(U_1=u).$$

Since U_1 has support S_B^U, we obtain that $\mathbb{P}_{\delta_i^L}(Z_{\mu^U}\ge u, Z_{\mu^U-1}<u,\ldots,Z_1<u)=1$ for all $u\in S_B^U$. In particular,

$$\mathbb{E}_{\delta_i^L}\big[Z_{\mu^U-1}\big]<u\le\mathbb{E}_{\delta_i^L}\big[Z_{\mu^U}\big]=M^U\mathbb{E}_{\delta_i^L}\big[Z_{\mu^U-1}\big]+N^U.$$

By taking both $u=L_U+1$ and $u\ge M^U(L_U+1)+N^U$ in the above equation, we obtain that both $\mathbb{E}_{\delta_i^L}[Z_{\mu^U-1}]<L_U+1$ and $\mathbb{E}_{\delta_i^L}[Z_{\mu^U-1}]\ge L_U+1$. Similarly, if

$$1=\mathbb{P}_{\delta_k^U}\big(\Delta_1^L=\mu^L\big)=\sum_{l=L_0}^{L_U}\mathbb{P}_{\delta_k^U}\big(Z_{\mu^L+\tau_1}\le l, Z_{\mu^L+\tau_1-1}>l,\ldots,Z_{\tau_1+1}>l\big)\mathbb{P}\big(L_1=l\big),$$

then using that L_1 has support $\mathbb{N}\cap[L_0, L_U]$, we obtain that

$$\mathbb{P}_{\delta_k^U}(Z_{\mu^L+\tau_1}\le l, Z_{\mu^L+\tau_1-1}>l,\ldots,Z_{\tau_1+1}>l)=1\qquad\text{for all } L_0\le l\le L_U.$$

In particular,

$$\big(M^L\big)^{\mu^L}k\le l<\big(M^L\big)^{(\mu^L-1)}k.$$

By taking both $l=L_0$ and $l=L_U$, we obtain that $L_0>M^LL_U$, which contradicts (**H4**). We deduce that

$$\sum_{j=0}^{T_{i,1}^L-1}\big(\Delta_{j+1}^U-\mu^U\big)$$

is non-degenerate, and similarly,

$$\sum_{j=0}^{T_{i,1}^U-1}\big(\Delta_{j+1}^L-\mu^L\big),\qquad\sum_{j=0}^{T_{i,1}^L-1}\Big(\overline{\Delta}_{j+1}^U-\big(\mu^U+\mu^L\big)\Big),\qquad\text{and}\qquad\sum_{j=0}^{T_{i,1}^L-1}\Big(\overline{\Delta}_{j+1}^L-(\mu^L+\mu^U)\Big)$$

are non-degenerate. Using Lemma 4.2 below, we conclude that

$$\sigma^{2,U}=\pi_i^L\mathbb{V}_{\delta_i^L}\Big[S_{T_{i,1}^L}^U-\mu^UT_{i,1}^L\Big]=\pi_i^L\mathbb{E}_{\delta_i^L}\left[\left(\sum_{j=0}^{T_{i,1}^L-1}\big(\Delta_{j+1}^U-\mu^U\big)\right)^2\right]>0,$$

and similarly $\sigma^{2,L}>0$, $\overline{\sigma}^{2,U}>0$, and $\overline{\sigma}^{2,L}>0$. □

A.6. Martingale structure of $\mathbf{M_{n,i}^T}$

We recall that $M_{n,i}^T := \sum_{j=1}^n D_{j,i}^T$, where

$$D_{j,1}^T = \left(\overline{P}_{j-1}^T - M^T\right)\tilde{\chi}_{j-1}^T \quad \text{and} \quad D_{j,2}^T = \frac{\tilde{\chi}_{j-1}^T}{Z_{j-1}} \sum_{i=1}^{Z_{j-1}} \left(\xi_{j-1,i}^T - \overline{P}_{j-1}^T\right).$$

Also,

$$\tilde{A}_n^T = \sum_{j=1}^n \frac{\tilde{\chi}_{j-1}^T}{Z_{j-1}},$$

and for $s \geq 0$,

$$\Lambda_{j,1}^{T,s} = \left|\overline{P}_j^T - M^T\right|^s \quad \text{and} \quad \Lambda_{j,2}^{T,s} = \mathbb{E}\left[\left|\xi_{j,i}^T - \overline{P}_j^T\right|^s \middle| \Pi_j^T\right].$$

Proposition A.3. *The following statements hold:*

(i) *For $i = 1, 2$, $\{(M_{n,1}^T, \mathcal{H}_{n,i})\}_{n=1}^\infty$ is a mean-zero martingale sequence, and for all $s \geq 0$, we have $\mathbb{E}\big[|D_{j,1}^T|^s|\mathcal{H}_{j-1,i}\big] = \mathbb{E}\big[\Lambda_{0,1}^{T,s}\big]\tilde{\chi}_{j-1}^T$ a.s. In particular,*

$$\mathbb{E}\big[\big(D_{j,1}^T\big)^2|\mathcal{H}_{j-1,i}\big] = V_1^T\tilde{\chi}_{j-1}^T \text{ a.s., and } \mathbb{E}\big[\big(M_{n,1}^T\big)^2\big] = V_1^T\mathbb{E}\big[\tilde{C}_n^T\big].$$

(ii) *$\big\{(M_{n,2}^T, \mathcal{H}_{n,2})\big\}_{n=1}^\infty$ is a mean-zero martingale sequence satisfying*

$$\mathbb{E}\big[\big(D_{j,2}^T\big)^2|\mathcal{H}_{j-1,2}\big] = V_2^T\frac{\tilde{\chi}_{j-1}^T}{Z_{j-1}} \text{a.s., and } \mathbb{E}\big[\big(M_{n,2}^T\big)^2\big] = V_2^T\mathbb{E}\big[\tilde{A}_n^T\big].$$

Additionally, for all $s \geq 1$, $\mathbb{E}\big[|D_{j,2}^T|^s|\mathcal{H}_{j-1,2}\big] \leq \mathbb{E}\big[\Lambda_{0,2}^{T,s}\big]\tilde{\chi}_{j-1}^T$ a.s.

(iii) *For $i = 1, 2$, $\big\{\big(\sum_{j=1}^n \big(\big(D_{j,i}^T\big)^2 - \mathbb{E}\big[\big(D_{j,i}^T\big)^2\big]\big), \tilde{\mathcal{H}}_{n,i}\big)\big\}_{n=1}^\infty$ are mean-zero martingale sequences, and for all $s \geq 1$,*

$$\mathbb{E}\big[|\big(D_{j,i}^T\big)^2 - \mathbb{E}\big[\big(D_{j,i}^T\big)^2\big]|^s|\tilde{\mathcal{H}}_{j-1,i}\big] \leq 2^s\mathbb{E}\big[\Lambda_{0,i}^{T,2s}\big]\mathbb{E}\big[\tilde{\chi}_{j-1}^T\big].$$

(iv) *For all $T_1, T_2 \in \{L, U\}$ and $i_1, i_2 \in \{1, 2\}$ such that either $T_1 \neq T_2$ or $i_1 \neq i_2$, it holds that $\mathbb{E}\big[\big(D_{j,i_1}^{T_1}\big)\big(D_{l,i_2}^{T_2}\big)\big] = 0$ for all $j, l = 1, \ldots, n$ and $\mathbb{E}\big[\big(M_{n,i_1}^{T_1}\big)\big(M_{n,i_2}^{T_2}\big)\big] = 0$. In particular, $\big\{\big(\sum_{j=1}^n \big(D_{j,i_1}^{T_1}\big)\big(D_{j,i_2}^{T_2}\big), \tilde{\mathcal{H}}_{n,2}\big)\big\}_{n=0}^\infty$ is a mean-zero martingale sequence, and for all $s \geq 1$,*

$$\mathbb{E}\big[|\big(D_{j,i_1}^{T_1}\big)\big(D_{j,i_2}^{T_2}\big)|^s|\tilde{\mathcal{H}}_{j-1,2}\big] \leq \mathbb{E}\big[\Lambda_{0,i_1}^{T_1,s}\Lambda_{0,i_2}^{T_2,s}\big]\mathbb{E}\big[\tilde{\chi}_{j-1}^{T_1}\tilde{\chi}_{j-1}^{T_2}\big].$$

Proof of Proposition A.3. We begin by proving (i) with $i = 1$. We notice that $\big(M_{n,1}^T, \mathcal{H}_{n,i}\big)$ is a martingale, since $M_{n,1}^T$ is $\mathcal{H}_{n,1}$-measurable and

$$\mathbb{E}\big[M_{n,1}^T|\mathcal{H}_{n-1,1}\big] = M_{n-1,1}^T + \mathbb{E}\Big[\overline{P}_{n-1}^T - M^T\Big]\tilde{\chi}_{n-1}^T = M_{n-1,1}^T.$$

It follows that $\mathbb{E}\big[M_{n,1}^T\big] = \mathbb{E}\big[M_{1,1}^T\big] = 0$. Next, notice that for $s \geq 0$,

$$\mathbb{E}\big[|D_{j,1}^T|^s|\mathcal{H}_{j-1,1}\big] = \mathbb{E}\big[|\overline{P}_{j-1}^T - M^T|^s\big]\tilde{\chi}_{j-1}^T = \mathbb{E}\big[\Lambda_{0,1}^{T,s}\big]\tilde{\chi}_{j-1}^T \text{ a.s.}$$

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In particular, if $s=2$, then $\mathbb{E}\big[\big(D^T_{j,1}\big)^2|\mathcal{H}_{j-1,1}\big]=V^T_1\tilde{\chi}^T_{j-1}$ a.s., and the martingale property yields that

$$\mathbb{E}[\big(M^T_{n,1}\big)^2=\mathbb{E}\left[\sum_{j=1}^{n}\mathbb{E}\big[\big(D^T_{j,1}\big)^2|\mathcal{H}_{j-1,1}\big]\right]=V^T_1\mathbb{E}[\tilde{C}^T_n].$$

Finally, we notice that, since the $D^T_{j,1}$ do not depend on the offspring distributions $\{\xi^T_{j,i})\}_{i=0}^{\infty}$, Part (i) holds with $\mathcal{H}_{n,1}$ replaced by $\mathcal{H}_{n,2}$.

We now turn to the proof of (ii). We notice that $M^T_{n,2}$ is $\mathcal{H}_{n,2}$-measurable, and using that $\mathbb{E}[\xi^T_{n-1,i}-\overline{P}^T_{n-1}|\Pi^T_{n-1}]=0$, we obtain that

$$\mathbb{E}\big[M^T_{n,2}|\mathcal{H}_{n-1,2}\big]=M^T_{n-1,2}+\frac{\tilde{\chi}^T_{n-1}}{Z_{n-1}}\sum_{i=1}^{Z_{n-1}}\mathbb{E}[\mathbb{E}[\xi^T_{n-1,i}-\overline{P}^T_{n-1}|\Pi^T_{n-1}]]=M^T_{n-1,2},$$

yielding the martingale property. It follows that $\mathbb{E}\big[M^T_{n,2}\big]=\mathbb{E}\big[M^T_{1,2}\big]=0$, as

$$\mathbb{E}\big[M^T_{1,2}|\mathcal{H}_{0,2}\big]=\frac{\tilde{\chi}^T_0}{Z_0}\sum_{i=1}^{Z_0}\mathbb{E}[\mathbb{E}[\xi^T_{0,i}-\overline{P}^T_0|\Pi^T_0]]=0.$$

We now compute

$$\mathbb{E}\big[\big(D^T_{j,2}\big)^2|\mathcal{H}_{j-1,2}\big]=\frac{\tilde{\chi}^T_{j-1}}{Z^2_{j-1}}\mathbb{E}\left[\mathbb{E}\left[\left(\sum_{i=1}^{Z_{j-1}}(\xi^T_{j-1,i}-\overline{P}^T_{j-1})\right)^2|\mathcal{H}_{j-1,2},\,\Pi^T_{j-1}\right]|\mathcal{H}_{j-1,2}\right].$$

Using that, conditionally on the environment Π^T_{j-1}, $\{\xi^T_{j-1,i}\}_{i=1}^{\infty}$ are i.i.d. with variance $\overline{\overline{P}}^T_{j-1}$, we obtain that

$$\mathbb{E}\left[\left(\sum_{i=1}^{Z_{j-1}}\xi^T_{j-1,i}-\overline{P}^T_{j-1}\right)^2|\mathcal{H}_{j-1,2},\,\Pi^T_{j-1}\right]=\sum_{i=1}^{Z_{j-1}}\mathbb{E}\left[\big(\xi^T_{j-1,i}-\overline{P}^T_{j-1}\big)^2|\mathcal{H}_{j-1,2},\,\Pi^T_{j-1}\right]$$
$$=Z_{j-1}\overline{\overline{P}}^T_{j-1}.$$

We conclude that

$$\mathbb{E}\Big[\Big(D^T_{j,2}\Big)^2|\mathcal{H}_{j-1,2}\Big]=V^T_2\frac{\tilde{\chi}^T_{j-1}}{Z_{j-1}}\text{ a.s.}$$

and

$$\mathbb{E}\Big[\Big(M^T_{n,2}\Big)^2\Big]=\mathbb{E}\left[\sum_{j=1}^{n}\mathbb{E}\big[\big(D^T_{j,2}\big)^2|\mathcal{H}_{j-1,2}\big]\right]=V^T_2\mathbb{E}[\tilde{A}^T_n].$$

Additionally, Jensen's inequality yields that for $s\geq 1$,

$$\mathbb{E}\big[\big|D^T_{j,2}\big|^s|\mathcal{H}_{j-1,2}\big]\leq\mathbb{E}\left[\frac{\tilde{\chi}^T_{j-1}}{Z_{j-1}}\sum_{i=1}^{Z_{j-1}}\big|\xi^T_{j-1,i}-\overline{P}^T_{j-1}\big|^s|\mathcal{H}_{j-1,2}\right]=\mathbb{E}\big[\Lambda^{T,s}_{0,2}\big]\tilde{\chi}^T_{j-1}\text{ a.s.}$$

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For (iii), we notice that $\sum_{j=1}^{n}\big((D^T_{j,i})^2-\mathbb{E}\big[(D^T_{j,i})^2\big]\big)$ is $\tilde{\mathcal{H}}_{n,i}$-measurable, and since $\tilde{\chi}^T_{n-1}$, Z_{n-1}, Π^T_{n-1}, and $\{\xi^T_{n-1,i}\}_{i=0}^{\infty}$ are not $\tilde{\mathcal{H}}_{n-1,i}$-measurable, we have that $\mathbb{E}\big[(D^T_{n,i})^2|\tilde{\mathcal{H}}_{n-1,i}\big]=\mathbb{E}\big[(D^T_{n,i})^2\big]$ and

$$\mathbb{E}\left[\sum_{j=1}^{n}\big((D^T_{j,i})^2-\mathbb{E}\big[(D^T_{j,i})^2\big]\big)|\tilde{\mathcal{H}}_{n-1,i}\right]=\sum_{j=1}^{n-1}\big((D^T_{j,i})^2-\mathbb{E}\big[(D^T_{j,i})^2\big]\big).$$

Again using the convexity of the function $|\cdot|^s$ for $s\geq 1$, we get that

$$\mathbb{E}\big[|(D^T_{j,i})^2-\mathbb{E}\big[(D^T_{j,i})^2\big]|^s|\mathcal{H}_{j-1,i}\big]\leq 2^{s-1}\big(\mathbb{E}\big[(D^T_{j,i})^{2s}\big]+\big(\mathbb{E}\big[(D^T_{j,i})^2\big]\big)^s\big)\leq 2^s\mathbb{E}\big[(D^T_{j,i})^{2s}\big].$$

If $i=1$, then by conditioning on $\tilde{\chi}^T_{j-1}$ we have that $\mathbb{E}\big[(D^T_{j,1})^{2s}\big]=\mathbb{E}\big[\Lambda^{T,2s}_{0,1}\big]\mathbb{E}\big[\tilde{\chi}^T_{j-1}\big]$. If $i=2$, then we apply Jensen's inequality and obtain that

$$\mathbb{E}\big[(D^T_{j,2})^{2s}\big]\leq\mathbb{E}\big[\Lambda^{T,2s}_{j-1,2}\tilde{\chi}^T_{j-1}\big]=\mathbb{E}\big[\Lambda^{T,2s}_{0,2}\big]\mathbb{E}\big[\tilde{\chi}^T_{j-1}\big].$$

Turning to (iv), we show that for all $T_1, T_2\in\{L, U\}$ and $i_1, i_2\in\{1, 2\}$ such that either $T_1\neq T_2$ or $i_1\neq i_2$, it holds that $\mathbb{E}\big[(D^{T_1}_{j,i_1})(D^{T_2}_{l,i_2})\big]=0$ for all $j, l=1, \ldots, n$. This also yields that

$$\mathbb{E}\big[(M^{T_1}_{n,i_1})(M^{T_2}_{n,i_2})\big]=\sum_{j=1}^{n}\sum_{l=1}^{n}\mathbb{E}\big[(D^{T_1}_{j,i_1})(D^{T_2}_{l,i_2})\big]=0.$$

First, if $l=j$ and $T_1\neq T_2$, then $\mathbb{E}\big[(D^{T_1}_{j,i_1})(D^{T_2}_{j,i_2})\big]=0$ because $\tilde{\chi}^{T_1}_{j-1}\tilde{\chi}^{T_2}_{j-1}=0$. Next, if $l=j$ and $i_1\neq i_2$ (say $i_1=1$ or $i_2=2$), then by conditioning on $\mathcal{H}_{j-1,1}$ and $\Pi^{T_2}_{j-1}$ and using the fact that

$$\mathbb{E}\Big[\xi^{T_2}_{j-1,i}-\overline{P}^{T_2}_{j-1}|\mathcal{H}_{j-1,1}, \Pi^{T_2}_{j-1}\Big]=0\text{ a.s.}$$

and that $\Pi^{T_1}_{j-1}$, $\tilde{\chi}^{T_1}_{j-1}$, $\tilde{\chi}^{T_2}_{j-1}$, and Z_{j-1} are $\mathcal{H}_{j-1,1}$-measurable, we obtain that

$$\mathbb{E}\big[(D^{T_1}_{j,1})(D^{T_2}_{j,2})\big]=\mathbb{E}\left[\Big(\overline{P}^{T_1}_{j-1}-M^{T_1}\Big)\tilde{\chi}^{T_1}_{j-1}\frac{\tilde{\chi}^{T_2}_{j-1}}{Z_{j-1}}\sum_{i=1}^{Z_{j-1}}\mathbb{E}\Big[\xi^{T_2}_{j-1,i}-\overline{P}^{T_2}_{j-1}|\mathcal{H}_{j-1,1}, \Pi^{T_2}_{j-1}\Big]\right]=0.$$

Finally, if $l\neq j$ (say $l>j$), then by conditioning on $\tilde{\mathcal{H}}_{l-1,2}$ and using that D_{j,i_1} is $\tilde{\mathcal{H}}_{l-1,2}$-measurable and $\mathbb{E}[(D_{l,i_2})|\tilde{\mathcal{H}}_{l-1,2}]=\mathbb{E}[D_{l,i_2}]=0$, we obtain that

$$\mathbb{E}\big[(D_{j,i_1})(D_{l,i_2})\big]=\mathbb{E}\big[D_{j,i_1}\mathbb{E}\big[D_{l,i_2}|\tilde{\mathcal{H}}_{l-1,2}\big]\big]=0.$$

We have that $\{\big(\sum_{j=1}^{n}(D^{T_1}_{j,i_1})(D^{T_2}_{j,i_2}), \tilde{\mathcal{H}}_{n,2}\}_{n=0}^{\infty}$ is a mean-zero martingale sequence, since $\sum_{j=1}^{n}(D^{T_1}_{j,i_1})(D^{T_2}_{j,i_2})$ is $\tilde{\mathcal{H}}_{n,2}$-measurable and

$$\mathbb{E}\big[(D^{T_1}_{j,i_1})(D^{T_2}_{j,i_2})|\tilde{\mathcal{H}}_{j-1,2}\big]=\mathbb{E}\big[(D^{T_1}_{j,i_1})(D^{T_2}_{j,i_2})\big]=0$$

if either $T_1\neq T_2$ or $i_1\neq i_2$. If $T_1\neq T_2$, then both $\mathbb{E}\big[|(D^{T_1}_{j,i_1})(D^{T_2}_{j,i_2})|^s|\tilde{\mathcal{H}}_{j-1,2}\big]=0$ and $\mathbb{E}\big[\tilde{\chi}^{T_1}_{j-1}\tilde{\chi}^{T_2}_{j-1}\big]=0$. Finally, if $T_1=T_2$ and $i_1\neq i_2$ (say $i_1=1$ and $i_2=2$), then by Jensen's inequality

$$\mathbb{E}\big[|(D^{T_1}_{j,i_1})(D^{T_2}_{j,i_2})|^s|\Pi^{T_1}_{j-1}, \mathcal{F}_{j-1}\big]\leq|(D^{T_1}_{j,i_1})|^s\Lambda^{T,s}_{j-1,i_2}\tilde{\chi}^{T_2}_{j-1},$$

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which yields that

$$\mathbb{E}\big[|\big(D^{T_1}_{j,i_1}\big)\big(D^{T_2}_{j,i_2}\big)|^s|\mathcal{F}_{j-1}\big] \le \mathbb{E}\big[\Lambda^{T_1,s}_{0,i_1}\Lambda^{T_2,s}_{0,i_2}\big]\tilde{\chi}^{T_1}_{j-1}\tilde{\chi}^{T_2}_{j-1}$$

and

$$\mathbb{E}\big[|\big(D^{T_1}_{j,i_1}\big)\big(D^{T_2}_{j,i_2}\big)|^s|\tilde{\mathcal{H}}_{j-1,2}\big] \le \mathbb{E}\big[\Lambda^{T_1,s}_{0,i_1}\Lambda^{T_2,s}_{0,i_2}\big]\mathbb{E}\big[\tilde{\chi}^{T_1}_{j-1}\tilde{\chi}^{T_2}_{j-1}\big].$$

□

Appendix B. Numerical experiments

In this section we describe numerical experiments to illustrate the evolution of the process under different distributional assumptions. We also study the empirical distribution of the lengths of supercritical and subcritical regimes and illustrate how the process changes when U_j and L_j exhibit an increasing trend. We emphasize that these experiments illustrate the behavior of the estimates of the parameters of the BPRET when using a finite number of generations in a single synthetic dataset. In the numerical experiments 1–4 below, we set $L_0 = 10^2$, $L_U = 10^4$, $n \in \{0, 1, \ldots, 10^4\}$, $U_j \sim L_U + \texttt{Zeta}(3)$, $L_j \sim \texttt{Unif}_d(L_0, 10L_0)$, and we use different distributions for Z_0, $I^U_0 \sim Q^U_0$, $\xi^U_{0,1} \sim P^U_0$, and $\xi^L_{0,1} \sim P^L_0$ as follows:

	$Z_0 - L_0$	I^U_0	$\xi^U_{0,1}$	$\xi^L_{0,1}$
Exp. 1	$\texttt{Pois}(1; L_U - L_0)$	$\texttt{Pois}(\Lambda^I)$, $\Lambda^I \sim \texttt{Unif}(0, 10)$,	$\texttt{Pois}(\Lambda^U)$, $\Lambda^U \sim \texttt{Unif}(0.9, 2.1)$,	$\texttt{Pois}(\Lambda^L)$, $\Lambda^L \sim \texttt{Unif}(0.5, 1.1)$,
Exp. 2	$\texttt{Pois}(1; L_U - L_0)$	$\texttt{Pois}(\Lambda^I)$, $\Lambda^I \sim \texttt{Gamma}(5, 1)$,	$\texttt{Pois}(\Lambda^U)$, $\Lambda^U \sim \texttt{Gamma}(2, 1)$,	$\texttt{Pois}(\Lambda^L)$, $\Lambda^L \sim \texttt{Gamma}(0.8, 1)$,
Exp. 3	$\texttt{Nbin}(1, 1; L_U - L_0)$	$\texttt{Nbin}(R^I, O^I)$, $R^I \sim 1 + \texttt{Pois}(1)$, $O^I \sim \texttt{Unif}(0, 10)$,	$\texttt{Nbin}(R^U, O^U)$, $R^U \sim 1 + \texttt{Pois}(1)$, $O^U \sim \texttt{Unif}(0.9, 2.1)$	$\texttt{Nbin}(R^L, O^L)$, $R^L \sim 1 + \texttt{Pois}(1)$, $O^L \sim \texttt{Unif}(0.5, 1.1)$
Exp. 4	$\texttt{Nbin}(1, 1; L_U - L_0)$	$\texttt{Nbin}(R^I, O^I)$, $R^I \sim 1 + \texttt{Pois}(1)$, $O^I \sim \texttt{Gamma}(5, 1)$,	$\texttt{Nbin}(R^U, O^U)$, $R^U \sim 1 + \texttt{Pois}(1)$, $O^U \sim \texttt{Gamma}(2, 1)$,	$\texttt{Nbin}(R^L, O^L)$, $R^L \sim 1 + \texttt{Pois}(1)$, $O^L \sim \texttt{Gamma}(0.8, 1)$,

In the above description, we have used the notation $\texttt{Unif}(a, b)$ for the uniform distribution over the interval (a,b) and $\texttt{Unif}_d(a, b)$ for the uniform distribution over integers between a and b. $\texttt{Zeta}(s)$ is the zeta distribution with exponent $s > 1$. $\texttt{Pois}(\lambda)$ is the Poisson distribution with parameter λ, while $\texttt{Pois}(\lambda;b)$ is the Poisson distribution truncated to values not larger than b. Similarly, $\texttt{Nbin}(r, o)$ is the negative binomial distribution with predefined number of successful trials r and mean o, while $\texttt{Nbin}(r, o;b)$ is the negative binomial distribution truncated to values not larger than b. Finally, $\texttt{Gamma}(\alpha, \beta)$ is the gamma distribution with shape parameter α and rate parameter β. In these experiments, there were between 400 and 700 crossings of the thresholds, depending on the distributional assumptions. The results of the numerical experiments 1–4 are shown in Figure 2.

We next turn our attention to the construction of confidence intervals for the means in the supercritical and subcritical regimes. The values of $M^T = \mathbb{E}[\overline{P}^T_0]$, $V^T_1 = \mathbb{V}[\overline{P}^T_0]$, and $V^T_2 = \mathbb{E}[\overline{\overline{P}}^T_0]$ in Experiments 1–4 can be deduced from the underlying distributions and are summarized below. The values of V^U_2 and V^L_2 in Experiments 3–4 are rounded to three decimal digits.

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	M^U	V_1^U	V_2^U	M^L	V_1^L	V_2^L
Exp. 1	1.5	0.12	1.5	0.8	0.03	0.8
Exp. 2	2	2	2	0.8	0.8	0.8
Exp. 3	1.5	0.12	2.998	0.8	0.03	1.224
Exp. 4	2	2	5.793	0.8	0.8	1.710

In the next table, we provide the estimators M_n^U, $C_n^U/\tilde{N}^L(n)$, $\tilde{C}_n^U/\tilde{N}^L(n)$, and $\tilde{A}_n^U/\tilde{N}^L(n)$ of $\mathbb{E}\left[\overline{P}_0^U\right]$, μ^U, $\tilde{\mu}^U$, and $\tilde{A}^U$, respectively. Notice that

$$V_{n,1}^U := \frac{1}{\tilde{C}_n^U} \sum_{j=1}^n \left(\frac{Z_j - I_{j-1}^U}{Z_{j-1}} - M_n^U \right)^2 \tilde{\chi}_{j-1}^U \quad \text{and}$$

$$V_{n,2}^U := \frac{1}{\tilde{C}_n^U} \sum_{j=1}^n \frac{1}{Z_{j-1}} \sum_{i=1}^{Z_{j-1}} \left(\xi_{j-1,i}^U - \frac{Z_j - I_{j-1}^U}{Z_{j-1}} \right)^2 \tilde{\chi}_{j-1}^U$$

are used to estimate V_1^U and V_2^U. As in the proof of Theorem 2.4, it is easy to see that $V_{n,1}^U$ and $V_{n,2}^U$ are consistent estimators of V_1^U and V_2^U. Similar comments hold when U is replaced by L.

	M_n^U	$V_{n,1}^U$	$V_{n,2}^U$	$C_n^U/\tilde{N}^L(n)$	$\tilde{C}_n^U/\tilde{N}^L(n)$	$\tilde{A}_n^U/\tilde{N}^L(n)$
Exp. 1	1.496	0.122	1.499	9.282	9.282	0.010
Exp. 2	2.009	2.051	2.003	10.807	10.804	0.105
Exp. 3	1.507	0.119	3.016	9.069	9.069	0.011
Exp. 4	2.018	2.157	5.929	10.899	10.893	0.112
	M_n^L	$V_{n,1}^L$	$V_{n,2}^L$	$C_n^L/\tilde{N}^U(n)$	$\tilde{C}_n^L/\tilde{N}^U(n)$	$A_n^L/\tilde{N}^U(n)$
Exp. 1	0.799	0.031	0.800	14.063	14.063	0.009
Exp. 2	0.809	0.805	0.809	5.118	5.118	0.002
Exp. 3	0.799	0.030	1.214	14.028	14.028	0.010
Exp. 4	0.822	0.945	1.869	5.136	5.136	0.002

Using the above estimators in Theorem 2.6, we obtain the following confidence intervals for M_n^U and M_n^L. We also provide confidence intervals for the estimator M_n defined below, which *does not take into account different regimes*. Specifically,

$$M_n := \frac{1}{\sum_{j=1}^n \mathbf{I}_{\{Z_{j-1} \geq 1\}}} \sum_{j=1}^n \frac{Z_j - I_{j-1}^T}{Z_{j-1}} \mathbf{I}_{\{Z_{j-1} \geq 1\}},$$

where I_{j-1}^T is equal to I_{j-1}^U if $T = U$ and 0 otherwise.

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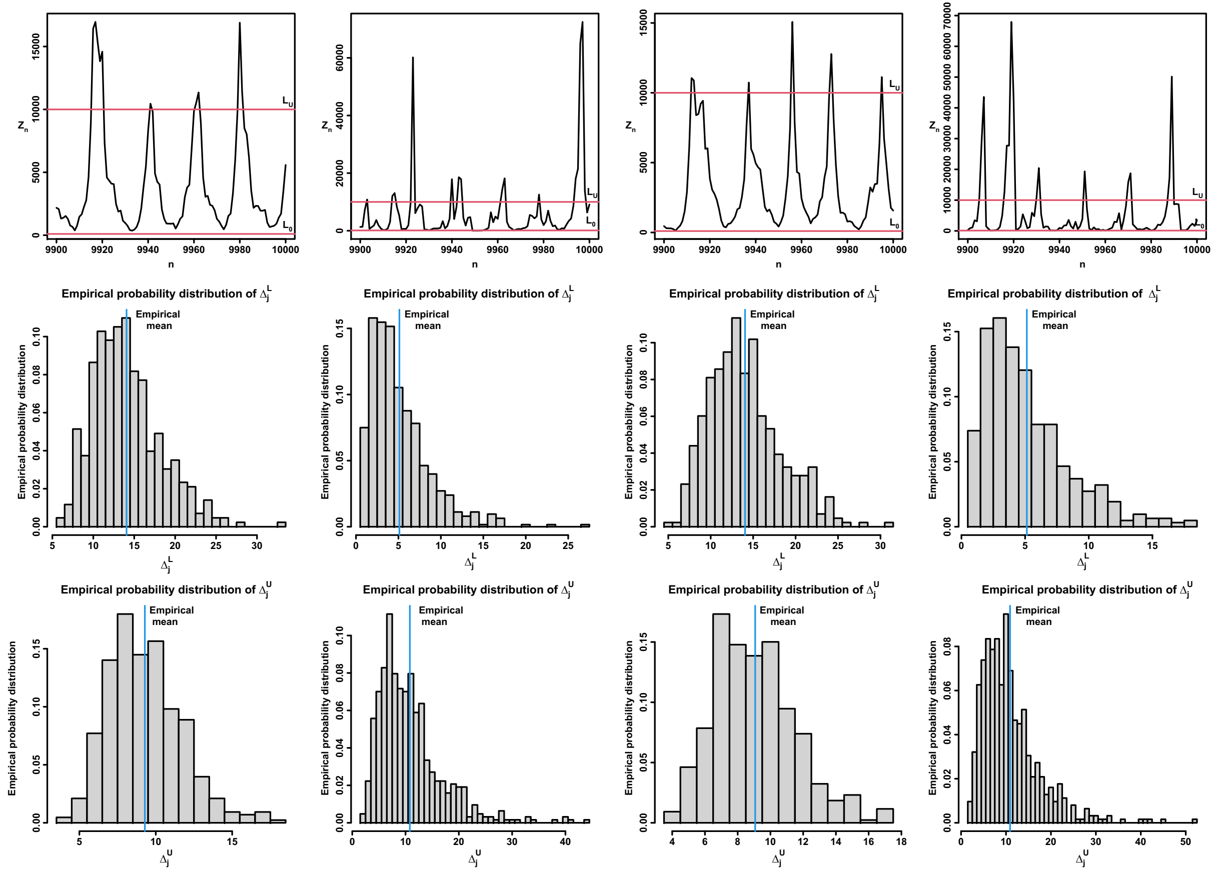

FIGURE 2. The four columns give the results of the numerical experiments 1, 2, 3, and 4. The first row shows the process Z_n for $n = 10^4 - 10^2, \ldots, 10^4$. The second and third rows show the empirical probability distributions of $\left\{\Delta_j^L\right\}$ and $\left\{\Delta_j^U\right\}$, respectively.

	95% CI using M_n^U	95% CI using M_n^L	95% CI using M_n
Exp. 1	(1.485, 1.507)	(0.795, 0.804)	(1.068, 1.085)
Exp. 2	(1.975, 2.043)	(0.778, 0.840)	(1.596, 1.651)
Exp. 3	(1.496, 1.518)	(0.794, 0.803)	(1.068, 1.085)
Exp. 4	(1.982, 2.053)	(0.788, 0.855)	(1.607, 1.664)

Next, we investigate the behavior of the process when the thresholds L_j and U_j increase with j. To this end, we let $L_0 = 10^2$, $L_U = 10^4$, and $n \in \{0, 1, \ldots, 10^3\}$ and take initial distribution Z_0, immigration distribution I_0^U, and offspring distributions $\xi_{0,1}^U$ and $\xi_{0,1}^L$ as in Experiment 1. We consider four different distributions for L_j and U_j, as follows:

	$U_j - L_U$	L_j
Exp. 5	$\mathtt{Zeta}(3)$	$\mathtt{Unif}_d(L_0, 10L_0)$
Exp. 6	$\mathtt{Zeta}(3)$	$\mathtt{Unif}_d(L_{j,1}, L_{j,2})$, where $L_{j,1} = \min(L_0 + 100(j-1), L_U)$, $L_{j,2} = \min(10L_0 + 100(j-1), L_U)$
Exp. 7	$\mathtt{Zeta}(3) + 500(j-1)$	$\mathtt{Unif}_d(L_0, 10L_0)$
Exp. 8	$\mathtt{Zeta}(3) + 500(j-1)$	$\mathtt{Unif}_d(L_{j,1}, L_{j,2})$, where $L_{j,1} = \min(L_0 + 100(j-1), L_U)$, $L_{j,2} = \min(10L_0 + 100(j-1), L_U)$

The results of Experiments 5–8 are shown in Figure 3. From the plots, we see that the number of cases after crossing the upper thresholds is between 10^4 and $2 \cdot 10^4$, whereas when the thresholds increase they almost reach the $6 \cdot 10^4$ mark. Also, the number of regimes up to time $n = 10^3$ decreases, as it takes more time to reach a larger threshold. As a consequence, the overall number of cases also increases.

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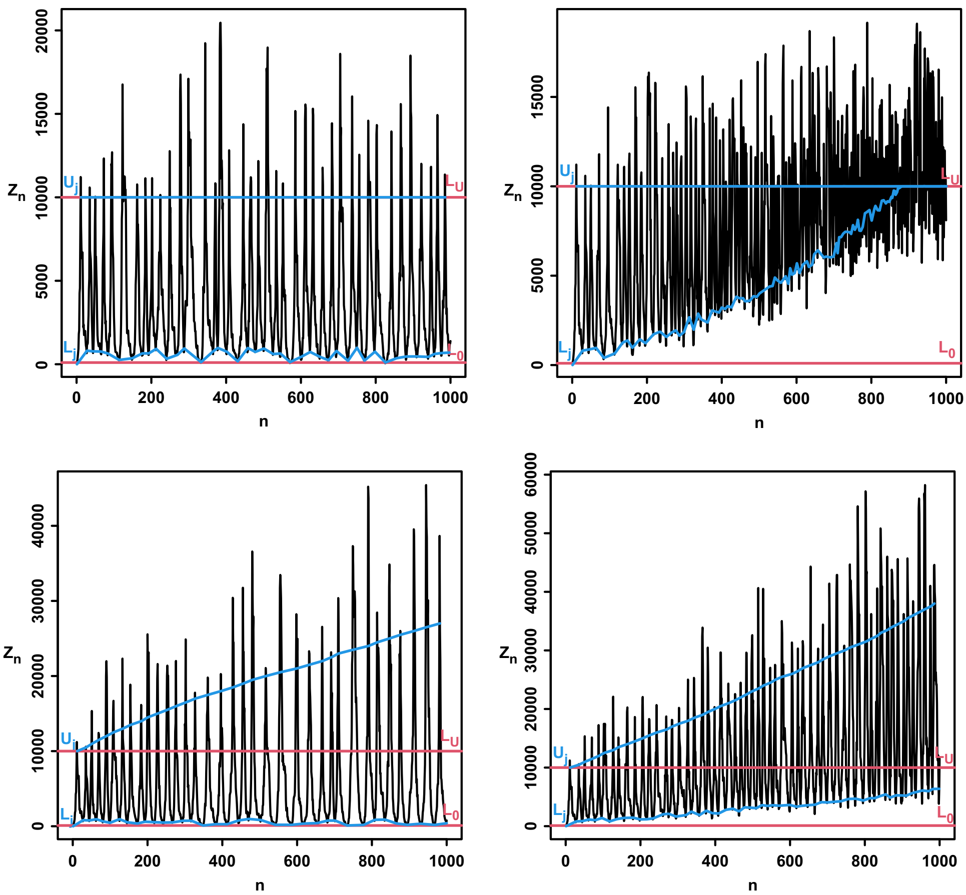

FIGURE 3. From left to right, Experiments 5, 6, 7, and 8. The process Z_n for $n \in \{0, 1, \ldots, 10^3\}$ (in black), horizontal lines at L_0 and L_U (in red), and the thresholds U_j and L_j (in blue).

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References

[1] JAGERS, P. (1975). *Branching Processes with Biological Applications*. John Wiley, New York.

[2] HACCOU, P., JAGERS, P. AND VATUTIN, V. A. (2007). *Branching Processes: Variation, Growth, and Extinction of Populations*. Cambridge University Press.

[3] HANLON, B. AND VIDYASHANKAR, A. N. (2011). Inference for quantitation parameters in polymerase chain reactions via branching processes with random effects. *J. Amer. Statist. Assoc.* **106**, 525–533.

[4] KIMMEL, M. AND AXELROD, D. E. (2015). *Branching Processes in Biology*. Springer, New York.

[5] YANEV, N. M., STOIMENOVA, V. K. AND ATANASOV, D. V. (2020). *Stochastic modeling and estimation of COVID-19 population dynamics*. Preprint. Available at https://arxiv.org/abs/2004.00941.

[6] ATANASOV, D., STOIMENOVA, V. AND YANEV, N. M. (2021). Branching process modelling of COVID-19 pandemic including immunity and vaccination. *Stoch. Quality Control* **36**, 157–164.

[7] FALCÓ, C. AND CORRAL, Á. (2022). Finite-time scaling for epidemic processes with power-law superspreading events. *Phys. Rev. E* **105**, article no. 064122.

[8] SUN, H., KRYVEN, I. AND BIANCONI, G. (2022). Critical time-dependent branching process modelling epidemic spreading with containment measures. *J. Phys. A* **55**, article no. 224006.

[9] KLEBANER, F. C. (1993). Population-dependent branching processes with a threshold. *Stoch. Process. Appl.* **46**, 115–127.

[10] TESCHL, G. (2012). *Ordinary Differential Equations and Dynamical Systems*. American Mathematical Society, Providence, RI.

[11] PERKO, L. (2013). *Differential Equations and Dynamical Systems*. Springer, New York.

[12] IANNELLI, M. AND PUGLIESE, A. (2014). *An Introduction to Mathematical Population Dynamics*. Springer, Cham.

[13] TYSON, R., HAINES, S. AND HODGES, K. E. (2010). Modelling the Canada lynx and snowshoe hare population cycle: the role of specialist predators. *Theoret. Ecol.* **3**, 97–111.

[14] HEMPEL, K. AND EARN, D. J. D. (2015). A century of transitions in New York City's measles dynamics. *J. R. Soc. Interface* **12**, article no. 20150024.

[15] ATHREYA, K. B. AND KARLIN, S. (1971). On branching processes with random environments: I: Extinction probabilities. *Ann. Math. Statist.* **42**, 1499–1520.

[16] KERSTING, G. AND VATUTIN, V. A. (2017). *Discrete Time Branching Processes in Random Environment*. John Wiley, London.

[17] ATHREYA, K. B. AND SCHUH, H.-J. (2016). A Galton–Watson process with a threshold. *J. Appl. Prob.* **53**, 614–621.

[18] KLEBANER, F. C. (1984). On population-size-dependent branching processes. *Adv. Appl. Prob.* **16**, 30–55.

[19] MAYSTER, P. (2005). Alternating branching processes. *J. Appl. Prob.* **42**, 1095–1108.

[20] JAGERS, P. AND KLEBANER, F. C. (2011). Population-size-dependent, age-structured branching processes linger around their carrying capacity. *J. Appl. Prob.* **48**, 249–260.

[21] JAGERS, P. AND ZUYEV, S. (2020). Populations in environments with a soft carrying capacity are eventually extinct. *J. Math. Biol.* **81**, 845–851.

[22] JAGERS, P. AND ZUYEV, S. (2021). Amendment to: populations in environments with a soft carrying capacity are eventually extinct. *J. Math. Biol.* **83**, article no. 3.

[23] DION, J. P. AND ESTY, W. W. (1979). Estimation problems in branching processes with random environments. *Ann. Statist.* 680–685.

[24] HEYDE, C. C. (1971). Some central limit analogues for supercritical Galton–Watson processes. *J. Appl. Prob.* **8**, 52–59.

[25] LAMPERTI, J. (1960). Criteria for the recurrence or transience of stochastic process. I. *J. Math. Anal. Appl.* **1**, 314–330.

[26] LAMPERTI, J. (1963). Criteria for stochastic processes II: passage-time moments. *J. Math. Anal. Appl.* **7**, 127–145.

[27] SERFOZO, R. (2009). *Basics of Applied Stochastic Processes*. Springer, Berlin, Heidelberg.

[28] BILLINGSLEY, P. (2013). *Convergence of Probability Measures*. John Wiley, New York.

[29] HALL, P. AND HEYDE, C. C. (1980). *Martingale Limit Theory and Its Application*. Academic Press, New York.

[30] ROITERSHTEIN, A. (2007). A note on multitype branching processes with immigration in a random environment. *Ann. Prob.* **4**, 1573–1592.

[31] IBRAGIMOV, I. A. AND LINNIK, Y. V. (1971). *Independent and Stationary Sequences of Random Variables*. Wolters-Noordhoff, Groningen.