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CONSTRUCTION OF SOME IRREDUCIBLE SUBGROUPS OF E_8 AND E_6

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Abstract

We construct two embeddings of finite groups into groups of Lie type. These embeddings have the interesting property that the finite subgroup acts irreducibly on a minimal module for the group of Lie type. We present our constructions as examples of a general method that obtains embeddings into groups of Lie type.

1. *Introduction*

This paper establishes existence of two embeddings:

$$^2F_4(2) < E_8(3) \qquad \text{and} \qquad L_2(8).3 < E_6(\mathbb{C}).$$

These embeddings are of particular interest because in both cases the subgroup acts irreducibly in a minimal projective representation of the overgroup. The two embeddings complete the classification by Liebeck and Seitz of Lie primitive finite subgroups that are irreducible on a minimal module for an exceptional algebraic group (see [**6**]).

We give computer constructions of the embeddings. In each case, we start with a natural invariant Lie algebra for the subgroup and construct an invariant subalgebra of appropriate exceptional Lie type. This gives the desired embedding. Our approach is similar to the computer construction used in [**3**], and is based on the strategy introduced in [**9**].

The second embedding $L_2(8).3 < E_6(\mathbb{C})$ is independently constructed by Aschbacher as Result 29.18 in the unpublished article 'The maximal subgroups of E_6' (1986). This previous construction is noted in [**1**], where the character of the embedding is reported incorrectly. We prove that there can be no embedding with the character given in [**1**].

2. *Invariant Lie subalgebras*

In this section, we give a general description of our method for the construction of embeddings of subgroups into overgroups of Lie type. Our method is to construct an action of the subgroup on the Lie algebra of the overgroup. We begin with a natural action of the subgroup on a larger Lie algebra, and we search for invariant subalgebras to locate the desired action. The following standard pair of theorems shows that there are natural invariant Lie algebras with appropriate types of subalgebras.

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THEOREM 2.1. *Suppose that a finite group G embeds in a finite Chevalley group E. Let $\mathcal{E}$ be the Lie algebra of E and let $(\ ,\ )$ be the Killing form of $\mathcal{E}$. Then $\mathcal{E}/Z(\mathcal{E})$ is isomorphic to a G-invariant Lie subalgebra of the Lie algebra of derivations of $(\ ,\ )$.*

We use the term 'Chevalley group' to mean a quotient of a simply connected universal Chevalley group. The term 'Chevalley Lie algebra' refers to the Lie algebra of a Chevalley group. A linear transformation τ is a derivation of a bilinear form $(\ ,\ )$ if, for all vectors v and w, $(v\tau, w) + (v, w\tau) = 0$.

Proof. The adjoint representation ρ of $\mathcal{E}$ gives an E-invariant embedding of $\rho(\mathcal{E})$ into the Lie algebra of linear transformations of $\mathcal{E}$. The Killing form is associative. Hence the transformations in $\rho(\mathcal{E})$ are derivations of $(\ ,\ )$. Moreover, the kernel of ρ is the center of $\mathcal{E}$. □

THEOREM 2.2. *Suppose that a finite group G embeds in a finite Chevalley group E. Let $\mathcal{E}$ be the Lie algebra of E, and let U be a non-trivial absolutely irreducible E-module in natural characteristic. Let $\overline{k}$ be the algebraic closure of the field of definition of U. Let $\overline{\mathcal{E}}$ and $\overline{U}$ denote the Lie algebra and E-module obtained from $\mathcal{E}$ and U by extension of scalars to $\overline{k}$. Then there is a non-trivial G-invariant homomorphism of Lie algebras: $\overline{\mathcal{E}} \to \mathrm{Hom}_{\overline{k}}(\overline{U}, \overline{U})$.*

Proof. It is shown in [10, Lemma 2] that there is an action of E on $\mathcal{E}$ and a $\overline{k}\mathcal{E}$-module structure on $\overline{U}$ such that the action: $\overline{U} \times \mathcal{E} \to \overline{U}$ is E-invariant. (In the case where E is defined over a field that does not have prime order, there are inequivalent (twisted) actions of E on the Lie algebra $\mathcal{E}$. These are obtained from a preferred adjoint action by the application of a field automorphism. If the highest weight of U is not p-reduced, a twisted action of E on $\mathcal{E}$ is applied in [10, Lemma 2].) The E-invariant action gives an E-invariant Lie algebra homomorphism from $\overline{\mathcal{E}}$ to $\mathrm{Hom}_{\overline{k}}(\overline{U}, \overline{U})$. This homomorphism is perforce G-invariant. □

The following algorithm takes the action of a group on a Lie algebra as input. The action is specified by matrices over a finite field of scalars, but the algebra is considered to be defined over the algebraic closure of the input field. The algorithm determines all invariant subalgebras that meet specified representation-theoretic conditions. In the implementation, it is important to note that although we determine subalgebras defined over an algebraically closed field, we must work over one or more finite fields. Write k for a finite field and $\overline{k}$ for its algebraic closure. We add an overline to indicate that an algebra, module, or vector space defined over k should have its scalars extended to $\overline{k}$.

ALGORITHM 2.3. Suppose that G is a finite group and that $(\mathcal{L}, [\ ,\ ])$ is a G-invariant Lie algebra over k. Let V be a kG-module and let W be an irreducible submodule of V. Determine all G-invariant Chevalley Lie subalgebras of $\overline{\mathcal{L}}$ that are isomorphic (as $\overline{k}G$-modules) to $\overline{V}$ and are generated by a $\overline{k}G$-submodule isomorphic to $\overline{W}$.

METHOD.

1. Determine a basis $f_1, f_2, \ldots, f_\ell$ for $\mathrm{Hom}_{kG}(W, \mathcal{L})$.
2. Let X be an ℓ-dimensional k-space. (X parameterizes $\mathrm{Hom}_{kG}(W, \mathcal{L})$.)
3. Let M be a kG-submodule of $\mathcal{L}$ that contains all copies of V in $\mathcal{L}$.

4. Select a pair of random vectors w_1 and w_2 in W.
5. Let $f : X \otimes X \to \mathcal{L}$ be the linear transformation with

$$f((x_1, x_2, \ldots, x_\ell) \otimes (y_1, y_2, \ldots, y_\ell)) = \left[\sum_i x_i f_i(w_1)\,,\, \sum_j y_j f_j(w_2)\right].$$

6. Let $\overline{S}$ be the set of non-zero vectors in $\overline{X}$ for which the tensor square belongs to $\overline{f^{-1}(M)}$. Thus

$$\overline{S} = \left\{(x_1, x_2, \ldots, x_\ell) \mid f((x_1, x_2, \ldots, x_\ell) \otimes (x_1, x_2, \ldots, x_\ell)) \in \overline{M}\right\}.$$

The set $\overline{S}$ is closed under non-zero scalar multiplication. Let $\underline{\overline{S}}$ be a set of representatives of the 1-dimensional spaces spanned by elements of $\overline{S}$.

7. For each $(x_1, x_2, \ldots, x_\ell)$ in $\underline{\overline{S}}$:
   (a) let $\overline{L}$ be the subalgebra of $\overline{\mathcal{L}}$ generated by the image of W under $\sum_i x_i f_i$;
   (b) if $\overline{L}$ is isomorphic to $\overline{V}$ as a $\overline{k}G$-module, output $\overline{L}$. □

Note that the first six steps of this procedure serve to reduce the number of candidates that are considered in Step 7. It is obvious that all appropriate subalgebras would be obtained by consideration of every 1-dimensional subspace of $\overline{X}$ at Step 7. However, if $(x_1, x_2, \ldots, x_\ell)$ is an element of $\overline{X}$ that gives a successful outcome at Step 7(b), then $\sum_i x_i f_i(W)$ generates a subalgebra of $\overline{\mathcal{L}}$ that is contained in $\overline{M}$. In particular,

$$\left[\sum_i x_i f_i(w_1)\,,\, \sum_j x_j f_j(w_2)\right] \in \overline{M}.$$

Hence, $(x_1, x_2, \ldots, x_\ell) \in \overline{S}$. In other words, the reduction of candidates that is accomplished by Steps 1–6 does not lead to the omission of any invariant subalgebras.

Although there is no guarantee that the set $\overline{S}$ is finite, or even any smaller than $\overline{X}$ itself, in the two cases that we consider in this paper, the set $\underline{\overline{S}}$ that we obtain is in bijection with the set of subalgebras of $\overline{\mathcal{L}}$ that are isomorphic to $\overline{V}$ as G-modules. In other words, in our examples, $\underline{\overline{S}}$ is as small as is possible. In any case, the possibility of some reduction in work at Step 7 is worth the effort involved in the earlier steps. The running time of these earlier steps is comparable with that of a single iteration of Step 7(a). However, there may be instances where $\underline{\overline{S}}$ turns out to be infinite: in these instances our algorithm fails. (Formally, this possibility of failure means that our method is not an algorithm.)

Steps 1 and 3 are accomplished by application of the MeatAxe [8] to analyze the structure of the kG-module $\mathcal{L}$. In Step 3, it is desirable to use a module M that is as small as possible (since a larger module might result in an enlargement of the set $\overline{S}$). The algebra that is generated at Step 7 is computed with the spin routine of the MeatAxe. Steps 2, 4, and 5 present no problems. Note that the first five steps require only linear algebra over k.

The computation at Step 6 requires the solution over $\overline{k}$ of equations with coefficients in k. This is the only non-linear step in our method. In general, we seek a finite set of invariant subalgebras over $\overline{k}$ (see [14]). Therefore any method must involve a non-linear step. A strategy of Algorithm 2.3 is to delay this non-linear step for as long as possible.

Step 6 reduces to a relative eigenvector problem, as explained in [3]. A vector v is a relative eigenvector for a set of (not necessarily square) matrices $A_1, A_2, \ldots, A_r$ if the images $vA_1, vA_2, \ldots, vA_r$ are scalar multiples of each other. (In other words, these images lie in a 1-dimensional space.) In Step 6, elements of $X \otimes X$ may be represented by vectors in such a way that a tensor product

$$(x_1, x_2, \dots, x_\ell) \otimes (y_1, y_2, \dots, y_\ell)$$

is represented by the vector

$$(x_1y_1, x_1y_2, \dots, x_1y_\ell, x_2y_1, \dots, x_2y_\ell, \dots, x_\ell y_\ell).$$

In this way, a basis of $f^{-1}(M)$ is represented by an $m \times \ell^2$ matrix, B say. Partition the matrix B as a sequence $B_1, B_2, \dots B_\ell$ of ℓ blocks of size $m \times \ell$. Then $x = (x_1, x_2, \dots, x_\ell) \in \overline{S}$ if and only if there is a vector y with $yB_i = x_i x$, for $1 \leqslant i \leqslant \ell$. In this case, y is a relative eigenvector for the matrices $B_1, B_2, \dots, B_\ell$.

Other than an exhaustive search, we know of no general solution to a relative eigenvector problem. However, relative eigenvector problems often reduce to a sequence of ordinary eigenvector problems. This is the case in the example that arises in [**3**]. A similar decomposition occurs in the relative eigenvector problem of the first example considered in this paper.

Remark 2.4. Algorithm 2.3 can be adapted to work with higher tensor powers in place of $X \otimes X$.

For example, to work with the tensor cube of X, we modify Step 5 to select three vectors w_1, w_2, and w_3. We define a linear transformation $g\colon X \otimes X \otimes X \to \mathcal{L}$ such that:

$$g((x_1, x_2, \dots, x_\ell) \otimes (y_1, y_2, \dots, y_\ell) \otimes (z_1, z_2, \dots, z_\ell)) \\ = \big[\,\big[\,\textstyle\sum_i x_i f_i(w_1)\,,\, \sum_j y_j f_j(w_2)\,\big]\,,\, \sum_k z_k f_k(w_3)\,\big].$$

In this case, Step 6 would let $\overline{S}$ be the set of non-zero vectors in $\overline{X}$ for which the tensor cube belongs to $\overline{f^{-1}(M)}$. The other steps are exactly as in Algorithm 2.3.

In most situations, it is best to apply the original version of Algorithm 2.3. It leads to a simultaneous system of quadratic equations, whereas the modified algorithm that is described above leads to cubic equations. However, in the second example that we consider in this paper, it is advantageous to apply the modified algorithm. In Section 4, we explain why the modified algorithm is more useful in that instance.

3. *The embedding* ${}^2F_4(2) < E_8(3)$

Our strategy is to construct a natural action of ${}^2F_4(2)$ on the Lie algebra of $\mathrm{GO}^+_{248}(3)$. Then, Algorithm 2.3 is applied to obtain actions of ${}^2F_4(2)$ on the Lie algebra of $E_8(3)$ and embeddings ${}^2F_4(2) < E_8(3)$.

Write V for a 248-dimensional space over the finite field $k = \mathbb{Z}_3$. Let G be the group ${}^2F_4(2)$. Write χ for the irreducible 3-modular Brauer character of G that restricts to a sum of two conjugate irreducible characters (with degrees 124) of the Tits group G'. (The irreducible 3-modular characters of the Tits group are obtained in [**4**]. Exactly two of them have degree 124.)

Construct an embedding $G = {}^2F_4(2) < \mathrm{GL}(V) \cong \mathrm{GL}_{248}(3)$ that has character χ. (The 248-dimensional irreducible character of G has multiplicity 2 as a constituent of the skew square of the 52-dimensional irreducible character. The corresponding

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52-dimensional matrix representation of G is given in [16]; cf. also [15]. Its skew square is decomposed with the MeatAxe to yield an embedding $G < \mathrm{GL}(V)$.)

Let $(\ ,\ )$ be a G-invariant bilinear form on V. Irreducibility of V implies that this form is essentially unique. It can be constructed with the MeatAxe standard basis procedure [8] as follows. Let X be (a matrix that gives) a standard basis for the matrix representation ρ of G on V. Let Y be a standard basis for the image of ρ under the inverse transpose map. Then $X^{-1}Y$ represents an invariant bilinear form for ρ. This construction is due to R. A. Parker; see [3] for more details.

Let O be the subgroup of $\mathrm{GL}(V)$ that preserves the form $(\ ,\ )$. Then $G < O < \mathrm{GL}(V)$ and $O \cong \mathrm{GO}^+_{248}(3)$. The groups $\mathrm{GL}(V)$, O, and G act naturally on the Lie algebra $\mathrm{Hom}_k(V,V)$. Moreover, the groups O and G act naturally on the subalgebra $\mathcal{D} \subset \mathrm{Hom}_k(V,V)$ that consists of derivations of $(\ ,\ )$.

When considered as an O-module, V is self-dual. Hence, as O-modules, $\mathrm{Hom}(V,V) \cong V \otimes V$. The Lie product on $\mathrm{Hom}(V,V)$ is identified with an O-invariant Lie product on $V \otimes V$ that is given by $[v \otimes w, y \otimes z] = (w,y)v \otimes z - (v,z)w \otimes y$ (see [9]). The algebra $(V \otimes V, [\ ,\ ])$ has an O-invariant subalgebra $\bigwedge^2 V$. This subalgebra corresponds to the subalgebra $\mathcal{D}$ when $V \otimes V$ is identified with $\mathrm{Hom}_k(V,V)$ (see [9]).

Computational Theorem 3.1. *There are two G-invariant subalgebras of $(\bigwedge^2 V, [\ ,\ ])$ that have type E_8.*

Proof. Apply Algorithm 2.3 to compute all G-invariant subalgebras of $\bigwedge^2 V$ that give G-modules isomorphic to V. (Since V is irreducible, the submodule W of Algorithm 2.3 is taken as V itself.)

In the application of Algorithm 2.3, we find that $\ell = 3$, at Step 1. At Step 3, choose M to be the sum of the three copies of V in the socle of $\bigwedge^2 V$. The first random pair of vectors selected at Step 4 results in a set $\overline{S}$ that has size 2. At Step 7, both elements of $\overline{S}$ give 248-dimensional Lie subalgebras of $\bigwedge^2 V$. For each of these 248-dimensional subalgebras, apply the algorithm of [11] to compute a split Cartan subalgebra and its corresponding set of root spaces. In both cases, the root spaces form root systems of type E_8. $\square$

The two G-invariant E_8-subalgebras of $(\bigwedge^2 V, [\,,\,])$ give two G-invariant subalgebras of $\mathcal{D} \subset \mathrm{Hom}_k(V,V)$ under the O-invariant identification of $V \otimes V$ with $\mathrm{Hom}_k(V,V)$. Let $\mathcal{E}$ be one of these subalgebras of $\mathcal{D}$. The group $N_{\mathrm{GL}(V)}(\mathcal{E})$ has the form $Z(\mathrm{GL}(V)) \times E$, where $E \leqslant \mathrm{GL}(V)$ is isomorphic to the Chevalley group $E_8(3)$. (Note that $\mathrm{Aut}(\mathcal{E}) \cong E$ — see [13].) Hence, $G \leqslant 2 \times E_8(3)$, and we obtain an embedding $G \leqslant E_8(3)$ (since G has no center).

The module V decomposes as $124 \oplus 124'$ when it is restricted to the group G'. Hence, $C_{\mathrm{GL}(V)}(G') \cong 2 \times 2$. Let x be a non-scalar element of this centralizer. Then x acts by conjugation on $Z(\mathrm{GL}(V)) \times G$. (The element x negates elements of $G \setminus G'$.) It follows that for all $g \in G$, $\mathcal{E}^{xgx^{-1}} = \mathcal{E}^{\pm g} = \mathcal{E}$. Hence, $\mathcal{E}^{xg} = \mathcal{E}^x$, and therefore $\mathcal{E}^x$ is a G-invariant subalgebra of $\mathrm{Hom}_k(V,V)$. Moreover, x acts on $\mathcal{D}$. ($\mathcal{D}$ is the algebra of derivations of the (essentially) unique bilinear form invariant under $Z(\mathrm{GL}(V)) \times G$, and x normalizes this group.) Hence, $\mathcal{E}^x \subset \mathcal{D}$. However, $\mathcal{E}^x$ cannot be the algebra $\mathcal{E}$. (Otherwise $x \in Z(\mathrm{GL}(V)) \times E$, so that $[x, Z(\mathrm{GL}(V)) \times E] \subset E$. However, $-1 \in [x, G] \subset [x, Z(\mathrm{GL}(V)) \times E]$, and $-1 \notin E$: a contradiction.)

We have now shown that $\mathcal{E}$ and $\mathcal{E}^x$ are two G-invariant E_8-subalgebras of $\mathcal{D}$. According to Theorem 3.1, these are the only G-invariant E_8-subalgebras of $\mathcal{D}$.

Observe that only one of the groups G and G^x is a subgroup of E. (This is because $-1 \in \langle G, G^x \rangle$ and $-1 \notin E$.) Suppose that it is the group G that is contained in E. (If necessary, switch the algebras $\mathcal{E}$ and $\mathcal{E}^x$, which switches the groups E and E^x.)

THEOREM 3.2. *The group $E \cong E_8(3)$ has one conjugacy class of subgroups isomorphic to G that are irreducible on the adjoint module.*

Proof. Suppose that H is a subgroup of E that is isomorphic to ${}^2F_4(2)$ and is irreducible on the adjoint module V of E. Then there exists $\ell \in \mathrm{GL}(V)$ with $G = H^\ell$. (From [4], G has only one irreducible 3-modular representation with degree 248.) We must show that the conjugation is effected by an element of E.

The module V supports an E-invariant bilinear form. This is the (essentially) unique G-invariant form $(\ ,\ )$. The bilinear form is fixed by H. Therefore G fixes its transform $\langle\ ,\ \rangle$ under ℓ. (The transform is defined by $\langle u, v\rangle = (u\ell, v\ell)$.) However, since V is an irreducible G-module, the invariant bilinear forms are related by a scalar multiplication: $\langle\ ,\ \rangle = \pm(\ ,\ )$.

The subalgebra $\mathcal{D} \subset \mathrm{Hom}(V, V)$ that consists of derivations of $(\ ,\ )$ is invariant under the action of ℓ. (Let $\alpha \in \mathcal{D}$. Then

$$\begin{aligned}(v\alpha^\ell, w) + (v, w\alpha^\ell) &= (v\ell^{-1}\alpha\ell, w) + (v, w\ell^{-1}\alpha\ell)\\ &= \langle v\ell^{-1}\alpha, w\ell^{-1}\rangle + \langle v\ell^{-1}, w\ell^{-1}\alpha\ell\rangle\\ &= \pm\{(v\ell^{-1}\alpha, w\ell^{-1}) + (v\ell^{-1}, w\ell^{-1}\alpha)\}\\ &= \pm 0.)\end{aligned}$$

Now, $\mathcal{E}$ is an E-invariant subalgebra of $\mathcal{D}$. Hence, $\mathcal{E}^\ell$ is a G-invariant subalgebra of $\mathcal{D}$ (since $\mathcal{E}$ is invariant under H, and $G = H^\ell$). Hence, $\mathcal{E}^\ell \in \{\mathcal{E}, \mathcal{E}^x\}$. In the former case $\ell \in Z(\mathrm{GL}(V)) \times E$, and therefore $\pm\ell$ is an element of E that conjugates H to G. The latter case is impossible. (It gives $\mathcal{E}^{\ell x} = \mathcal{E}$. Hence, $\ell x \in Z(\mathrm{GL}(V)) \times E$. Now, one of the elements $\pm\ell x$ belongs to E. Thus $\langle G, H^{\pm\ell x}\rangle \leqslant E$. However, $-1 \in \langle G, G^x\rangle = \langle G, H^{\pm\ell x}\rangle$ and $-1 \notin E$: a contradiction.) $\square$

4. *The embedding $L_2(8).3 < E_6(\mathbb{C})$*

Our strategy is to construct a natural action of $L_2(8).3$ on the Lie algebra of $L_{27}(13)$. We then apply the modified algorithm described below Remark 2.4 to obtain embeddings of $L_2(8).3$ into an algebraic group of type E_6 in characteristic 13. Larsen's $(0, p)$-correspondence (see [3, Appendix 2]) provides embeddings $L_2(8).3 < E_6(\mathbb{C})$.

LEMMA 4.1. *Suppose that $f : L_2(8).3 \to E_6(\mathbb{C})$ is an embedding for which $f(L_2(8).3)$ acts irreducibly on the 27-dimensional projective representation of $E_6(\mathbb{C})$. Then the adjoint representation of $E_6(\mathbb{C})$ restricts to a module of $f(L_2(8).3)$ whose character is a sum of irreducibles of degrees 21, 27, 7, 7, 8, and 8. Moreover, in this sum of irreducibles, either all four characters of degrees 7 and 8 are rational, or the characters of degrees 7 and 8 form two pairs of conjugate, irrational characters.*

The first decomposition allowed by Lemma 4.1 is the one asserted by [1]. Here the characters of degree 7 and 8 are the two characters explicitly given in the ATLAS.

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These characters are written as 7^+ and 8^+ in [**1**]. (The decomposition of [**1**] includes a clearly spurious second copy of 21. However, after omission of this second copy of 21, the first decomposition allowed by Lemma 4.1 does remain.) Later we show that the restriction of the adjoint representation to $f(L_2(8).3)$ cannot have this decomposition.

In the second decomposition permitted by Lemma 4.1, the characters of degrees 7 and 8 are multiplied by cube roots of unity on outer elements of $L_2(8).3$. (In the notation of [**1**] that is applied to other groups with an automorphism of order 3:

$$78 = 21 + 27 + 7^{\omega} + 7^{\bar{\omega}} + 8^{\omega} + 8^{\bar{\omega}}.$$

Here ω and $\bar{\omega}$ denote cube roots of 1.)

Proof of Lemma 4.1. Write $\widetilde{E}$ for the triple cover of $E_6(\mathbb{C})$, and write $\widetilde{\psi}$ and $\widetilde{\chi}$ for irreducible characters of $\widetilde{E}$ that have degrees 27 and 78. Let ψ and χ be the restrictions of $\widetilde{\psi}$ and $\widetilde{\chi}$ to the preimage $\widetilde{L}$ of $f(L_2(8).3)$ in $\widetilde{E}$. The group $\widetilde{L}$ is an isocline of $3 \times L_2(8).3$ (see [**2**, p. xxiii]). In particular, $\widetilde{L}$ contains a copy of $L_2(8)$ on which ψ restricts to a sum of the three irreducible characters of degree 9. Let g_2, g_7, and g_9 denote elements of orders 2, 7, and 9 in this copy of $L_2(8)$.

The values of $\widetilde{\psi}$ and $\widetilde{\chi}$ at all elements of $\widetilde{E}$ that have order at most 7 are given in [**1**, Table 2]. Now, $\psi(g_2) = 3$. Hence g_2 belongs to the class $2A$ of [**1**] and therefore $\chi(g_2) = -2$. Similarly, $\psi(g_7) = -1$ so that g_7 belongs to the class $7N$ of [**1**] and $\chi(g_7) = 1$. Further, $\psi(g_9) = \psi(g_9^3) = 0$. A machine enumeration (using Kac theory; see [**7**]) of elements of order 9 in $\widehat{E}$ shows that the class of g_9 in $\widehat{E}$ is uniquely determined and has $\chi(g_9) = 0$ and $\chi(g_9^3) = -3$.

We have now obtained the value of χ at all elements in a copy of $L_2(8)$. Therefore the restriction of χ to this group is completely determined: it contains each irrational character with multiplicity 1 and each non-trivial rational character with multiplicity 2. In particular, the irreducible constituents of χ must have degrees 27, 21, 8, 8, 7, and 7.

Write g_3 for an element of $\widetilde{E}$ that maps to an outer element of order 3 in $f(L_2(8).3)$. The order of the element g_3 is either 3 or 9, but the order of its image in E is 3. Moreover, $\widetilde{\psi}(g_3) = 0$. Another machine enumeration (using Kac theory) shows that the only possibilities for such an element g_3 have order 3 and belong to one of the $\widetilde{E}$-classes called $3C$ and $3D$ in [**1**, Table 2]. Hence, $\widetilde{\chi}(g_3)$ is either -3 or 6 (see [**1**]). Together with the known degrees of the irreducible constituents of χ, each of these possibilities determines a single decomposition of the character χ. The two decompositions that arise are as stated above. $\square$

LEMMA 4.2. *There is no embedding of $G \cong L_2(8).3$ in $E_6(\mathbb{C})$ for which G is irreducible on the* 27*-dimensional module of* $3.E_6(\mathbb{C})$ *and G has adjoint character*

$$21 + 27 + 2 \times 7^+ + 2 \times 8^+.$$

Proof. We suppose that there is such an embedding, and obtain a contradiction. Let $(\mathcal{E}, [\ ,\ ])$ be the Lie algebra of $E_6(\mathbb{C})$. Let V be a copy of the $\mathbb{C}G$-module with character 7^+. Let f_1 and f_2 be two independent $\mathbb{C}G$-module homomorphisms from V to $\mathcal{E}$.

The space $\mathrm{Hom}_{\mathbb{C}G}(\Lambda^2 V, \mathcal{E})$ is 1-dimensional (because $\Lambda^2 V$ is irreducible). Let F be a non-zero element of $\mathrm{Hom}_{\mathbb{C}G}(\Lambda^2 V, \mathcal{E})$.

Three antisymmetric bilinear functions from $V \times V$ to $\mathcal{E}$ are given by mapping (u, v) to $[f_1(u), f_1(v)]$, $[f_2(u), f_2(v)]$, and $[f_1(u), f_2(v)] + [f_2(u), f_1(v)]$. The induced maps from $\Lambda^2 V$ to $\mathcal{E}$ may be written as αF, βF, and γF for some constants α, β, and γ.

Let (x, y) be a solution in $\mathbb{C}^2 \setminus (0, 0)$ to the equation $\alpha x^2 + \gamma xy + \beta y^2 = 0$. Write f for the homomorphism $xf_1 + yf_2 \colon V \to \mathcal{E}$. Then

$$[f(u), f(v)] = (\alpha x^2 + \gamma xy + \beta y^2) F(u \wedge v) = 0.$$

Therefore, $f(V)$ is a 7-dimensional abelian subalgebra of $(\mathcal{E}, [\ ,\ ])$.

The actions (on $\mathcal{E}$) of elements in the abelian subalgebra $f(V)$ commute. Therefore, the maps s and n that replace an element of $\mathcal{E}$ by the semisimple and nilpotent parts of its Jordan decomposition are linear on $f(V)$ (see [**5**, p. 18]). These maps are clearly G-invariant functions from $f(V)$. In other words, they are G-module homomorphisms.

We have: $\dim(s(f(V)) \leqslant 6 < \dim(f(V))$, since $s(f(V))$ is a toral subalgebra in a Lie algebra of rank 6. Hence, irreducibility of $f(V)$ gives $s(f(V)) = 0$. Therefore $f(V)$ acts as a set of commuting nilpotent endomorphisms of any $\mathcal{E}$-module ([**5**, p. 30]). In particular, if W is any $\mathcal{E}$-module, then $Wf(V)$ is a proper G-submodule. If W is 27-dimensional then $Wf(V) = 0$, since W is irreducible as a G-module. However the simple algebra $\mathcal{E}$ acts faithfully on W. Hence, $f(V) = 0$: a contradiction. $\square$

We now show that there exist embeddings with adjoint character $21 + 27 + 7^{\omega} + 7^{\bar{\omega}} + 8^{\omega} + 8^{\bar{\omega}}$. Our method is to classify embeddings in characteristic 13. This characteristic is convenient both because $13 \nmid |L_2(8){:}3|$ and because all irreducible representations of $L_2(8){:}3$ can be written over F_{13}. Larsen's $(0, p)$-correspondence shows that there is a bijection between conjugacy classes of embeddings in characteristics 0 and 13. Moreover, corresponding embeddings have the same adjoint character restrictions.

Write U for a 27-dimensional space over a finite field k of order 13. Let G be the group $L_2(8):3$. Write ψ for the character of the irreducible kG-module of degree 27. Construct an embedding $G = L_2(8):3 < \mathrm{GL}(U) \cong \mathrm{GL}_{27}(k)$ that has character ψ. The following steps implement such a construction. Make a 9-dimensional representation of $L_2(8)$ over k (for example by using [**3**, Recipe 2.2]). Decompose the skew square of the 9-dimensional representation and locate an irreducible constituent which gives the 7-dimensional representation of $L_2(8)$ that has rational character values. Find the images of generators of $L_2(8)$ under an automorphism of order 3. Apply the standard basis program in the usual way (see for example, [**12**, Section 4]) to extend the 7-dimensional representation of $L_2(8)$ to $L_2(8).3$. The 27-dimensional representation of $L_2(8).3$ is found as the non-trivial constituent of the symmetric square of the 7-dimensional representation.

The groups $\mathrm{GL}(U)$ and G act naturally on the Lie algebra $\mathrm{Hom}(U, U)$. Under the identification $\mathrm{Hom}(U, U) \cong U \otimes U^*$, the Lie product on $\mathrm{Hom}(U, U)$ is identified with an invariant Lie product on $U \otimes U^*$ that is given by $[u_1 \otimes v_1, u_2 \otimes v_2] = \langle u_1, v_2 \rangle u_2 \otimes v_1 - \langle u_2, v_1 \rangle u_1 \otimes v_2$ (see [**9**]). Let $\mathcal{L}$ be the Lie algebra $(U \otimes U^*, [\ ,\ ])$.

Write $\overline{k}$ for the algebraic closure of k and $\widehat{k}$ for the subfield of $\overline{k}$ that has order 13^2. We indicate that the scalars should be extended to either $\overline{k}$ or $\widehat{k}$ by applying

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an overline or a hat to the name of a k-space, k-module, or k-algebra. For example, the Lie algebras $\overline{\mathcal{L}}$ and $\widehat{\mathcal{L}}$ are obtained from $\mathcal{L}$ by extending scalars to these fields.

COMPUTATION 4.3. *There are six G-invariant subalgebras of $(\overline{\mathcal{L}}, [\ ,\ ])$ that have type E_6. Moreover, every G-invariant subalgebra with type E_6 has character*

$$27 + 21 + 7^{\omega} + 7^{\bar{\omega}} + 8^{\omega} + 8^{\bar{\omega}}.$$

This computation provides an alternative proof of Lemma 4.2.

Proof. The modified version of Algorithm 2.3 (see Remark 2.4) is applied twice to locate all G-invariant subalgebras of $\overline{\mathcal{L}}$ that have type E_6. In the first application the kG-modules called V and W in Algorithm 2.3 are taken to have characters $27 + 21 + 7^+ + 7^+ + 8^+ + 8^+$ and 7^+, respectively. In the second application the modules V and W are taken to have characters $27+21+7^{\omega}+7^{\bar{\omega}}+8^{\omega}+8^{\bar{\omega}}$ and 7^{ω}. The two applications correspond to the two possible characters that are allowed by Lemma 4.1.

In both applications, the space $\mathrm{Hom}_{kG}(W, \mathcal{L})$, that is computed at Step 1, is three-dimensional (because each of the characters 7^+, 7^{ω}, and $7^{\bar{\omega}}$ has multiplicity 3 in $U \otimes U^*$). Hence, at Step 2, X is a 3-dimensional k-space. At Step 3, M is taken to be the sum of all copies of irreducible constituents of V in $U \otimes U^*$. In the first application M has degree 620 and character $3 \times 7^+ + 4 \times 8^+ + 9 \times 21 + 14 \times 28$. In the second application M has degree 673 and character $3 \times 7^{\omega} + 3 \times 7^{\bar{\omega}} + 4 \times 8^{\omega} + 4 \times 8^{\bar{\omega}} + 9 \times 21 + 14 \times 28$.

At Step 4, three random vectors of W are chosen, since the modification of Remark 2.4 is being used. The map $g \colon X \otimes X \otimes X \to \mathcal{E}$ is defined by the formula given below Remark 2.4. The inverse image $g^{-1}(M)$ is calculated at Step 6. In the first application it has dimension 4 and in the second application it has dimension 14.

Let $\overline{S}$ be the set of non-zero vectors in $\overline{X}$ whose tensor cube belongs to $\overline{f^{-1}(M)}$. The set of perfect cubes (of the form $x \otimes x \otimes x$, for some $x \in X$) spans a copy of the symmetric cube of X, which we write as $S^3(X)$. Therefore, the tensor cubes of elements of $\overline{S}$ belong to $\overline{S^3(X) \cap f^{-1}(M)}$. In the two applications of our algorithm, the space $S^3(X) \cap f^{-1}(M)$ was computed and found to have dimension 0 and 7 in the respective cases. In particular, the first application terminates at this step: there are no invariant subalgebras of type E_6 with the first of the two characters under consideration. This verifies Lemma 4.2 computationally.

The second application proceeds with the computation of the set of representatives of the 1-dimensional spaces spanned by elements of $\overline{S}$. This set $\overline{S}$ is obtained as the set of solutions to a system of 3 homogeneous cubics in three variables. A series of substitutions reduces this system to a polynomial of degree 7 in one variable. The polynomial has six distinct roots, all of which belong to the field $\widehat{k}$. The six roots give rise to six elements of $\overline{S}$. The elements of $\overline{S}$ determine six particular copies of $\widehat{W}$ in $\widehat{\mathcal{L}}$. Each of these copies of $\widehat{W}$ does generate a 78-dimensional subalgebra of $\widehat{\mathcal{L}}$. In each case the algorithm of [11] locates a Cartan subalgebra (that splits over $\widehat{k}$) and a root system of type E_6. $\square$

In Computation 4.3 an application of the standard version of Algorithm 2.3 would be ineffective. This is because the kG-module $W \otimes W$ has structure $1^{\bar{\omega}} + 21 + 27$. Let M_1 be the unique submodule of $\mathcal{L}$ with character $1^{\bar{\omega}}$. At Step 5, the vectors of the form $[f_i(w_1), f_j(w_2)]$ all belong to the space $M + M_1$. Accordingly, at Step 6,

$\overline{S}$ would be computed from the weak condition that its elements have tensor squares in a particular subspace of co-dimension 1 in $\overline{S^2(X)}$. The modified algorithm leads to a much more stringent restriction that the tensor cube of a vector should belong to a subspace of co-dimension 21 in $\overline{S^3(X)}$.

Write $\widehat{\mathcal{E}}_1$, $\widehat{\mathcal{E}}_2$, $\widehat{\mathcal{E}}_3$, $\widehat{\mathcal{E}}_4$, $\widehat{\mathcal{E}}_5$, and $\widehat{\mathcal{E}}_6$ for the G-invariant subalgebras of $\widehat{\mathcal{L}}$ of type E_6. For each i, the $\overline{k}$-algebra $\overline{\mathcal{E}_i}$ is obtained from $\widehat{\mathcal{E}}_i$ by extension of scalars.

The next theorem is our main goal. The remaining rather technical arguments serve merely to count the number of conjugacy classes of embeddings.

THEOREM 4.4. *For each algebra $\overline{\mathcal{E}_i}$, the action of G gives an embedding of G into the algebraic group $E_6(\overline{k})$.*

Proof. Computation 4.3 gives an embedding $G \leqslant \mathrm{Aut}(\overline{\mathcal{E}}_i)$. Now, $\mathrm{Aut}(\overline{\mathcal{E}}_i)$ has structure $E_6(\overline{k}).2$ (see [**13**]). However, $G \cong L_2(8).3$ has no homomorphic image of order 2. Therefore the embedding into $\mathrm{Aut}(\overline{\mathcal{E}}_i)$ gives an embedding into $E_6(\overline{k})$. □

The action of G on $\widehat{\mathcal{E}}_i$ does give an embedding of G into the Chevalley group $E_6(\widehat{k})$. However, in this case the proof is more tricky because the automorphism group of the Lie algebra includes diagonal automorphisms of order 3. One approach is to show that the action of G gives an embedding into ${}^2E_6(k)$; this is an easy corollary of the following computation.

COMPUTATION 4.5. *Each algebra $\overline{\mathcal{E}_i}$ has a G-invariant k-form. Moreover, the six k-forms obtained from these algebras are isomorphic kG-algebras.*

Proof. For each algebra $\overline{\mathcal{E}_i}$ we compute a 'standard basis'. We then verify that the k-span of this basis is closed under both the action of G and the Lie product.

Observe that if a scalar multiple cv of some vector v belongs to a kG-form of $\overline{\mathcal{E}_i}$ and $\sigma_1 = vG$, then $c\sigma_1$ is a set of vectors in the k-form. Moreover, if further sets of vectors are defined inductively by $\sigma_n = [\sigma_1, \sigma_{n-1}]$, then $c^n\sigma_n$ is also a set of vectors in the k-form. Our strategy is to find a 'seed vector' with the properties of v, and compute sets $\sigma_1, \sigma_2, \ldots$ until we reach a spanning set for $\overline{\mathcal{E}_i}$. A lexicographically earliest maximal independent subset of vectors provides a standard basis for any available k-form.

We locate an appropriate seed vector v by an application of the MEATAXE to output a vector that spans the nullspace of an element of kG and lies in the 21-dimensional G-submodule of $\overline{\mathcal{E}_i}$. With this choice of seed, it turns out that σ_2 spans $\overline{\mathcal{E}_i}$. (It is not surprising that the Lie square of a 21-dimensional constituent should span a 78-dimensional algebra). We select a maximal independent subset $\beta \subset \sigma_2$. We check that the Lie product can be rescaled so that all elements of $[\beta, \beta]$ are k-linear combinations of elements of β. This verifies the existence of an appropriate scale factor c and gives a kG-form of $\overline{\mathcal{E}_i}$.

The six k-forms that we obtain are observed to have identical structure constants, and to support identical actions of G with respect to the bases constructed as above. □

The k-form of $\overline{\mathcal{E}_i}$ must be the Lie algebra of ${}^2E_6(k)$ (since according to Computation 4.3 there is no G-invariant action on the Lie algebra of $E_6(k)$).

COMPUTATIONAL THEOREM 4.6. *There are two conjugacy classes of embeddings of G in $E_6(\overline{k})$ that are irreducible on a 27-dimensional module of $3.E_6(\overline{k})$.*

Proof. Let $\mathcal{E}$ be a G-invariant Lie algebra over $\overline{k}$ such that G is irreducible on a 27-dimensional $\mathcal{E}$-module. Consider the embedding $G \leqslant E \leqslant \mathrm{Aut}(\mathcal{E}) \leqslant \mathrm{GL}(\mathcal{E})$, where E is a Chevalley group isomorphic to $E_6(\overline{k})$. Computation 4.5 shows that there is just one $\mathrm{Aut}(\mathcal{E})$ class of embeddings of G in E. However, there are two cosets of E in $\mathrm{Aut}(\mathcal{E})$ — see [**13**]. We complete the proof by showing that if $\alpha \in \mathrm{Aut}(\mathcal{E}) \setminus E$, then G^α is not an E-conjugate of G. Hence the conjugates of G by each of the two cosets of E in $\mathrm{Aut}(\mathcal{E})$ give two E-classes of embeddings of G in E.

For suppose that $\alpha \in \mathrm{Aut}(\mathcal{E})$, $e \in E$, and $G^\alpha = G^e$. Then $e\alpha^{-1}$ acts on G. Therefore it acts on $G' \cong L_2(8)$. However, G is the full group of automorphisms of $L_2(8)$. Hence, there exists $g \in G$ such that $x = ge\alpha^{-1}$ centralizes G'. View x as a linear transformation of $\mathcal{E}$. By Schur's lemma, x acts as a scalar c on U, an absolutely irreducible 9-dimensional G'-submodule of $\mathcal{E}$. (As a G'-module, $\mathcal{E}$ has 3 non-isomorphic 9-dimensional constituents. The element x must act on each of these constituents so as to commute with the action of G'.) However, we compute that $[U,U]$ is 36-dimensional and

$$[[U,U],U] = [[[U,U],U],U] = \mathcal{E}.$$

It follows that x acts on $\mathcal{E}$ as the scalar c^3 and also as the scalar c^4. In particular, $c = 1$. Therefore x acts as the identity on $\mathcal{E}$. We deduce that $\alpha = ge \in GE \subset E$. This completes the proof, as we observed above. $\square$

As a corollary, Larsen's $(0,p)$-correspondence gives the following theorem.

THEOREM 4.7. *There are two conjugacy classes of embeddings of $L_2(8).3$ in $E_6(\mathbb{C})$ that are irreducible on a 27-dimensional module.*

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