MAP-COLOUR THEOREMS

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Introduction. A map wil be called *k-chromatic* or said to have *chromatic number* k if k is the least positive integer having the property that the countries of the map can be divided into k mutually disjoint (colour) classes in such a way that no two countries which have a common frontier line are in the same (colour) class. Heawood [4] proved that for $h > 1$ the chromatic number of a map on a surface of connectivity h is at most n_h, where

$$n_h = [\tfrac{1}{2}(7 + \sqrt{24h - 23})].$$

($[x]$ denotes the integral part of x.) It is known also [5] that for $2 \leqslant h \leqslant 15$, n_h different colours are sometimes needed, because maps consisting of n_h mutually adjacent countries can be drawn on the surfaces concerned.

The main purpose of this paper is to establish the following:

THEOREM I. *For $h = 3$ and for $h \geqslant 5$ a map on a surface of connectivity h with chromatic number n_h always contains n_h mutually adjacent countries.*

It follows from this theorem that for $h = 3$ and for $h \geqslant 5$ every n_h-chromatic map on a surface of connectivity h consists essentially (in a sense which will become clear later) of n_h mutually adjacent countries; all maps drawn on such a surface which do not contain n_h mutually adjacent countries can be coloured with less than the full n_h colours.

On the other hand, it is possible that for some values of h the surfaces of connectivity h are such that all maps on them can be coloured using less than n_h colours. The theorem furnishes a procedure for deciding this. For any given surface of connectivity $h > 1$, n_h colours are needed only if a map consisting of n_h mutually adjacent countries can be drawn on the surface. (For $h = 2$ and 4 this follows not from Theorem I but from Theorems II and III.) For $h = 1$ ($n_h = 4$) the theorem is clearly not true, and it has of course never been proved, or disproved, that four colours are always sufficient to colour a map on the sphere or in the plane.

For the cases $h = 2$ and $h = 4$, Theorems II and III respectively will be proved; they are somewhat weaker than Theorem I.

The following table shows the values of n_h corresponding to values of h up to 16.

$$h\colon\ 1,2,3,4,5,6,7,\ 8,\ 9,10,11,12,13,14,15,16$$

$$n_h\colon\ 4,6,7,7,8,9,9,10,10,10,11,11,12,12,12,13.$$

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1. Combinatorial basis. It is convenient to consider graphs rather than maps, the graphs being the duals of maps and therefore such that no node is joined to itself by an edge, two nodes are joined by at most one edge, two edges meet in a node or not at all, and there is no crossing of edges. A graph will be called k-chromatic, or said to have chromatic number k, if k colours are just sufficient to colour the nodes in such a way that two nodes which are joined by an edge are never coloured the same. Theorem I, which concerns maps, is equivalent to the following theorem concerning graphs:

THEOREM I′. *For $h = 3$ and for $h \geqslant 5$, a graph of chromatic number n_h on a surface of connectivity h always contains n_h mutually adjacent nodes.*

To prove this theorem some combinatorial notations and results are necessary.

If Γ is a graph, its chromatic number will be denoted by $K(\Gamma)$. A graph will be called *critical* if it has no subgraph of smaller order with the same chromatic number. Clearly, *if Γ is critical the degree of every node of Γ is at least $K(\Gamma) - 1$.* De Bruijn proved [**2**] that an infinite k-chromatic graph always contains a finite k-chromatic subgraph. It follows that *a k-chromatic graph always contains a critical k-chromatic subgraph, and a critical graph is finite and connected.* If N denotes the number of nodes and E denotes the number of edges of a critical k-chromatic graph, since the degree of every node is at least k-1, the following simple inequality holds:

1.1 $$\frac{2E}{N} \geqslant k - 1.$$

This inequality was improved by Brooks [**1**] who proved that *if $k \geqslant 4$ a k-chromatic graph either contains k nodes such that every pair are joined by an edge, or it contains a node of degree $\geqslant k$.* Hence

1.2 $$\text{if } k \geqslant 4 \text{ and } N > k \text{ then } \frac{2E}{N} > k - 1.$$

(A graph which consists of n nodes, every pair of distinct nodes being joined by an edge, is usually called *a complete n-graph.* This terminology will be adopted.)

It is necessary for the purpose of this paper to strengthen 1.2 considerably for $N \leqslant k + 3$, and the following will be now established:

1.3 *A k-chromatic graph which does not contain a complete k-graph as a subgraph contains at least $k + 2$ nodes.*

1.4 *In the notation of* 1.2, *if $k \geqslant 5$ and $N = k + 2$ for a critical graph, then*

$$\frac{2E}{N} > k + 1 - \frac{12}{k+2}.$$

1.5 *In the same notation, if $k \geqslant 5$ and $N = k + 3$ for a critical graph, then*

$$\frac{2E}{N} > k + 2 - \frac{24}{k+3}.$$

In a previous paper [**3**] I proved the following result:

1.6 *If* $0 \leqslant n \leqslant k-1$, *a k-chromatic graph either contains a complete* $(k-n)$*-graph as a subgraph or it has at least* $k+n+2$ *nodes.*

1.3 follows from this on substituting $n = 0$.

Proof of 1.4. It also follows that a critical k-chromatic graph of order $k+2$ contains a complete $(k-1)$-graph as a subgraph. Let $k \geqslant 5$ and let the nodes of such a graph be denoted by $a_1, a_2, \ldots, a_{k-1}, b_1, b_2, b_3$; where every pair of $a_1, a_2, \ldots, a_{k-1}$, is joined by an edge. Because the graph is critical, b_1, b_2, and b_3 are each joined to at least $k-1$ nodes. The number of edges in the graph, consistent with this requirement, is least (i.e., the most economical distribution of edges is obtained) if each of b_1, b_2, and b_3 is joined to the other two and to $k-3$ of the nodes $a_1, a_2, \ldots, a_{k-1}$. In this case the graph contains

$$\tfrac{1}{2}(k-1)(k-2) + 3(k-3) + 3$$

edges; with any other distribution of edges it contains more. Hence for such a graph,

$$\frac{2E}{N} \geqslant k + 1 - \frac{12}{k+2}.$$

But with the distribution described above, it is easy to see that unless b_1, b_2, and b_3 are all joined to the same $k-3$ nodes from among $a_1, a_2, \ldots, a_{k-1}$, the chromatic number of the graph is $k-1$ ($k \geqslant 5$ was assumed). If b_1, b_2, and b_3 are all joined to the same $k-3$ nodes from among $a_1, a_2, \ldots, a_{k-1}$, then these nodes together with b_1, b_2, and b_3 form a set of k nodes of which each pair is joined by an edge, so that the graph contains a complete k-graph as a subgraph and is therefore not critical. This most economical distribution is therefore not permissible, and so for a critical k-chromatic graph of order $k+2$,

$$\frac{2E}{N} > k + 1 - \frac{12}{k+2},$$

and this proves 1.4.

Proof of 1.5. By 1.6 a critical k-chromatic graph of order $k+3$ contains a complete $(k-2)$-graph as a subgraph. It is easiest to prove 1.5 by considering two alternatives: the graph contains a complete $(k-1)$-graph as a subgraph or it does not.

(i) *The graph contains a complete* $(k-1)$*-graph as a subgraph.* Let the nodes of the graph be denoted by $a_1, a_2, \ldots, a_{k-1}, b_1, b_2, b_3, b_4$; where every pair of $a_1, a_2, \ldots, a_{k-1}$ is joined by an edge. Because the graph is critical, b_1, b_2, b_3, and b_4 are each joined to at least $k-1$ nodes. The number of edges in the graph, consistent with this requirement, is least (i.e., the most economical distribution of edges is obtained) if each of b_1, b_2, b_3, and b_4 is joined to the other three and to $k-4$ of the nodes $a_1, a_2, \ldots, a_{k-1}$. In this case the graph contains

$$\tfrac{1}{2}(k-1)(k-2) + 4(k-4) + 6$$

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edges; with any other distribution of edges it contains more. Hence for such a graph,

$$\frac{2E}{N} \geqslant k + 2 - \frac{24}{k+3}.$$

But with the distribution described above it is easy to see that unless b_1, b_2, b_3, and b_4 are all joined to the same $k - 4$ nodes from among $a_1, a_2, \ldots, a_{k-1}$, the chromatic number of the graph is $k - 1$ ($k \geqslant 5$ was assumed). If b_1, b_2, b_3, and b_4 are all joined to the same $k - 4$ nodes from among $a_1, a_2, \ldots, a_{k-1}$ then these nodes together with b_1, b_2, b_3, and b_4 form a set of k nodes of which each pair is joined by an edge, so that the graph contains a complete k-graph as a subgraph and is therefore not critical. This most economical distribution is therefore not permissible, and so, for a critical k-chromatic graph of order $k + 3$ containing a complete $(k - 1)$-graph as a subgraph,

$$\frac{2E}{N} > k + 2 - \frac{24}{k+3}.$$

(ii) *The graph does not contain a complete $(k - 1)$-graph as a subgraph.* Let Γ denote the graph. It contains a complete $(k - 2)$-graph as a subgraph and is critical (by 1.6 with $n = 2$). Let a be any node of Γ which does not belong to this complete $(k - 2)$-graph and let Γ' denote the graph obtained from Γ by deleting a and all edges incident in a. Γ' is $(k - 1)$-chromatic and it is not critical. For if Γ' were critical, then a would have to be joined to every node of Γ', since Γ is k-chromatic; and as Γ' contains a complete $(k - 2)$-graph as a subgraph, Γ would contain a complete $(k - 1)$-graph as a subgraph, contrary to hypothesis. Therefore, Γ' contains a node b such that the graph obtained from Γ' by deleting b and all edges incident in b, say Γ'', is $(k - 1)$-chromatic.

Now Γ'' is $(k - 1)$-chromatic and of order $k + 1$ and does not contain a complete $(k - 1)$-graph as a subgraph. It is therefore critical: if it were not, it would contain a $(k - 1)$-chromatic subgraph of order k without a complete $(k - 1)$-graph, whereas, by 1.6 with $k - 1$ in place of k, and $n = 0$, a $(k - 1)$-chromatic graph either contains a complete $(k - 1)$-graph as a subgraph or it has at least $k + 1$ nodes. Since Γ'' is critical, by 1.4 with $k - 1$ in place of k, the number of its edges is at least $\frac{1}{2}k(k + 1) - 5$.

The nodes a and b in Γ are each of degree $\geqslant k - 1$, since Γ is critical, and so contribute at least $2(k - 1) - 1$ edges to Γ. The number of edges in Γ is therefore at least

$$\tfrac{1}{2}k(k + 1) - 5 + 2(k - 1) - 1 = \tfrac{1}{2}k^2 + \tfrac{5}{2}k - 8.$$

Hence, for Γ:

$$\frac{2E}{N} \geqslant k + 2 - \frac{22}{k+3}.$$

Under assumption (i) it was shown that

$$\frac{2E}{N} > k + 2 - \frac{24}{k+3},$$

so that 1.5 is now proved.

Theorems 1.2, 1.3, 1.4, and 1.5 form the combinatorial basis for Theorem I.

2. Topological basis and the proof of Heawood's formula. Let a graph be drawn on a surface of connectivity h which divides the surface into polygonal regions. If N denotes the number of nodes, E the number of edges, and F the number of polygonal regions into which the surface is divided, then Euler's Theorem states that

$$N + F - E = 3 - h.$$

If there are regions on the surface which are bounded by more than three edges, it is possible to add new edges until a graph is obtained which divides the surface into polygons bounded by three edges, i.e., triangles. (It is to be understood of course that we speak of polygons and triangles drawn on the surface in question, whose vertices are nodes and whose sides are edges of the graph.) The number of nodes of the new graph is still N. Let the number of edges be E' and the number of triangular regions be F', then $E' \geqslant E$ and $F' \geqslant F$. Now every triangle is bounded by three edges and every edge separates two triangles, hence

$$3F' = 2E'.$$

By Euler's Theorem,

$$3N + 3F' - 3E' = 9 - 3h;$$

hence

$$3N - E' = 9 - 3h,$$

and so

$$\frac{2E'}{N} = 6\left(1 + \frac{h-3}{N}\right).$$

Since $E' \geqslant E$, for the original graph,

2.1 $$\frac{2E}{N} \leqslant 6\left(1 + \frac{h-3}{N}\right).$$

A graph drawn on the surface which does not divide it into polygonal regions can be drawn on a surface with smaller connectivity, or can be made to divide the surface into polygonal regions by the addition of edges. *Thus* 2.1 *holds for all graphs drawn on a surface of connectivity* h.

Let k be the chromatic number of a graph drawn on a surface of connectivity h. Such a graph contains a critical k-chromatic subgraph. Let this subgraph have N nodes and E edges. Then clearly $N \geqslant k$, and the degree of each node is at least $k - 1$, so that $2E/N \geqslant k - 1$. Hence, from 2.1, if $h \geqslant 3$,

$$k - 1 \leqslant 6\left(1 + \frac{h-3}{k}\right).$$

It follows that:

If $h \geqslant 3$, *every graph drawn on a surface of connectivity* h *can be coloured using* *at most n_h colours, where n_h is the greatest integer satisfying*

2.2 $$n_h - 1 \leqslant 6\left(1 + \frac{h-3}{n_h}\right).$$

If $h = 2$ we have from 2.1 that $k - 1 < 6$, that is, $k \leqslant 6$.

The value of n_h from 2.2 explicitly is

$$[\tfrac{1}{2}(7 + \sqrt{24h - 23})]$$

and, when $h = 2$, gives the correct value 6. Thus we have a very simple proof of the well-known result quoted in the beginning of this paper.

3. Proof of Theorem I. To prove Theorem I it is to be shown that for $h = 3$ and $h \geqslant 5$ the only critical n_h-chromatic graph which can be drawn on a surface of connectivity h is the complete n_h-graph. To do this will first be proved that for $h = 3$ and $h \geqslant 5$ no critical n_h-chromatic graph of order $\geqslant n_h + 4$ can be drawn on a surface of connectivity h. Then it will be proved that no critical n_h-chromatic graph of order $n_h + 2$ or $n_h + 3$ can be drawn on such a surface. Theorem I will then follow by 1.3.

Suppose a critical n_h-chromatic graph of order $\geqslant n_h + 4$ is drawn on a surface of connectivity h. If E denotes the number of edges and N the number of nodes then by 1.2,

$$\frac{2E}{N} > n_h - 1;$$

and by 2.1, since $h \geqslant 3$,

$$\frac{2E}{N} \leqslant 6\left(1 + \frac{h-3}{n_h + 4}\right).$$

Hence

3.1 $$n_h - 1 < 6\left(1 + \frac{h-3}{n_h + 4}\right),$$

while n_h satisfies the inequalities:

3.2 $$n_h - 1 \leqslant 6\left(1 + \frac{h-3}{n_h}\right),$$

3.3 $$n_h > 6\left(1 + \frac{h-3}{n_h + 1}\right).$$

From 3.1,

$$n_h^2 - 3n_h \leqslant 6h + 9,$$

and from 3.3,

$$n_h^2 - 5n_h \geqslant 6h - 11;$$

hence $2n_h \leqslant 20$, that is, $n_h \leqslant 10$. It remains to examine those cases where $n_h \leqslant 10$.

Case $n_h = 7$. By the table on page 480, the case to be examined is $h = 3$. (The

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case $h = 4$ is excluded from Theorem I.) By 1.2 with $k = 7$,

$$\frac{2E}{N} > 6;$$

and by 2.1 with $h = 3$,

$$\frac{2E}{N} \leqslant 6.$$

This is a contradiction. Actually, 1.2 with $k = 7$ applies to all critical 7-chromatic graphs of order exceeding 7, and so we have completed the proof of Theorem I for the case $h = 3$. (The theorem with $h = 3$, was first established by P. Ungár. His proof is different from this one.)

Case $n_h = 8$. From the table, if $n_h = 8$ then $h = 5$. By 1.2 with $k = 8$,

$$\frac{2E}{N} > 7.$$

By 2.1 with $h = 5$ and $N \geqslant n_h + 4 = 12$,

$$\frac{2E}{N} \leqslant 6\left(1 + \frac{2}{12}\right) = 7.$$

This is a contradiction.

Case $n_h = 9$. From the table, if $n_h = 9$ then $h = 6$ or $h = 7$. Consider first the case $h = 7$. By 1.2 with $k = 9$,

$$\frac{2E}{N} > 8.$$

By 2.1 with $h = 7$ and $N \geqslant n_h + 4 = 13$,

$$\frac{2E}{N} \leqslant 6\left(1 + \frac{4}{13}\right) = 7\tfrac{11}{13}.$$

This is a contradiction. *A fortiori* the case $h = 6$ would lead to a contradiction.

Case $n_h = 10$. From the table, if $n_h = 10$ then $h = 8$ or $h = 9$ or $h = 10$. Consider first the case $h = 10$. By 1.2 with $k = 10$,

$$\frac{2E}{N} > 9.$$

By 2.1 with $h = 10$ and $N \geqslant n_h + 4 = 14$,

$$\frac{2E}{N} \leqslant 6\left(1 + \frac{7}{14}\right) = 9.$$

This is a contradiction. *A fortiori* the cases $h = 8$ and $h = 9$ lead to a contradiction.

These contradictions prove that for $h \geqslant 5$ no critical n_h-chromatic graph of order $\geqslant n_h + 4$ can be drawn on a surface of connectivity h, and that for $h = 3$ the only critical n_h-chromatic graphs which can be drawn on a surface of connectivity h are the complete n_h-graphs.

It remains to see whether an n_h-chromatic critical graph of order $n_h + 2$ or $n_h + 3$ can be drawn on a surface of connectivity h. These cases will be considered in turn:

Graphs of order $n_h + 2$. Suppose a critical n_h-chromatic graph of order $n_h + 2$ is drawn on a surface of connectivity h. By 1.4 for such a graph,

$$\frac{2E}{N} > n_h + 1 - \frac{12}{n_h + 2}.$$

By 2.1,

$$\frac{2E}{N} \leqslant 6\left(1 + \frac{h-3}{n_h + 2}\right);$$

hence

$$n_h + 1 - \frac{12}{n_h + 2} < 6\left(1 + \frac{h-3}{n_h + 2}\right),$$

that is, $n_h^2 - 3n_h \leqslant 6h + 3$. But from the definition of n_h, for $h \geqslant 3$,

$$n_h > 6\left(1 + \frac{h-3}{n_h + 1}\right),$$

that is, $n_h^2 - 5n_h \geqslant 6h - 11$, and so $2n_h \leqslant 14$, or $n_h \leqslant 7$. But we have already disposed of the case $n_h = 7$ ($h = 3$).

Graphs of order $n_h + 3$. Suppose a critical n_h-chromatic graph of order $n_h + 3$ is drawn on a surface of connectivity h. By 1.5, for such a graph,

$$\frac{2E}{N} > n_h + 2 - \frac{24}{n_h + 3}.$$

By 2.1,

$$\frac{2E}{N} \leqslant 6\left(1 + \frac{h-3}{n_h + 3}\right);$$

hence

$$n_h + 2 - \frac{24}{n_h + 3} < 6\left(1 + \frac{h-3}{n_h + 3}\right),$$

that is, $n_h^2 - n_h \leqslant 6h + 17$. But by the definition of n_h,

$$n_h > 6\left(1 + \frac{h-3}{n_h + 1}\right),$$

that is, $n_h^2 - 5\,n_h \geqslant 6h - 11$, and so $4\,n_h \leqslant 28$, or $n_h \leqslant 7$. But we have already disposed of this case. This completes the proof of Theorem I.

The cases $h = 2$ and $h = 4$ have not been included. If $h = 2$ then $n_h = 6$, and the graphs drawn on these surfaces must be such that

$$\frac{2E}{N} \leqslant 6\left(1 - \frac{1}{N}\right).$$

The theorem of Brooks (1.2 above) states only that for critical 6-chromatic

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graphs of order > 6, $2E/N > 5$; it is not strong enough to settle the question. It is of course well known that a map consisting of six mutually adjacent countries can be drawn on the projective plane [5] (for which $h = 2$); but I do not know whether it is possible to draw a map on a surface of connectivity 2 which does not contain six mutually adjacent countries and is nevertheless 6-chromatic. If $h = 4$ then $n_h = 7$ and the graphs drawn on these surfaces must be such that

$$\frac{2E}{N} \leqslant 6\left(1 + \frac{1}{N}\right).$$

Theorem 1.2 states only that for critical 7-chromatic graphs of order exceeding 7, $2E/N > 6$; so that it fails to deal with this case also. I think that it is very unlikely that a 7-chromatic map which does not contain seven mutually adjacent countries can be drawn on a surface of connectivity 4.

4. Weaker theorems for $h = 2$ and for $h = 4$. It is curious, but not unusual in this subject, that the surfaces of greater connectivity should be more easily amenable to treatment than the simpler surfaces with low connectivity. But instead in the cases $h = 2$ and $h = 4$, I will prove the following:

THEOREM II. *A 6-chromatic map on a surface of connectivity 2 either contains 6 mutually adjacent countries, or a map containing 6 mutually adjacent countries can be obtained from it by deleting suitably chosen frontier lines and uniting those countries which they separate.*

THEOREM III. *A 7-chromatic map on a surface of connectivity 4 either contains 7 mutually adjacent countries, or a map containing 7 mutually adjacent countries can be obtained from it by deleting suitably chosen frontier lines and uniting those countries which they separate.*

The proof is based on the following simple

LEMMA. *If $h \geqslant 2$, a map on a surface of connectivity h contains a country with fewer than n_h neighbours.*

A graph in which the degree of every node is at least n_h has at least $n_h + 1$ nodes. If N denotes the number of nodes and E the number of edges of such a graph then

$$\frac{2E}{N} \geqslant n_h.$$

On the other hand, if the graph can be drawn on a surface of connectivity h, by 2.1,

$$\frac{2E}{N} \leqslant 6\left(1 + \frac{h-3}{N}\right).$$

Hence

$$n_h \leqslant 6\left(1 + \frac{h-3}{n_h + 1}\right) \quad \text{if } h > 2 \text{ and } \quad n_h < 6 \quad \text{if } h = 2.$$

But from the definition of n_h,

$$n_h > 6\left(1 + \frac{h-3}{n_h+1}\right) \quad \text{if } h > 2 \text{ and} \quad n_h = 6 \quad \text{if } h = 2.$$

So a graph on a surface of connectivity h contains a node with fewer than n_h neighbours. The Lemma follows.

Proof of Theorem II. The theorem is certainly true for maps containing 6 countries. We shall assume it to be true for maps containing not more than $C - 1$ countries ($C \geqslant 7$) and deduce that it is true for maps containing C countries. The truth of the theorem will then follow by the induction principle.

Let M be a 6-chromatic map containing C countries, B frontier lines, and A frontier points common to three or more countries, and not having 6 mutually adjacent countries. If, on deleting a country from M, there remains a 6-chromatic map, then by the induction hypothesis the theorem is true for M. We may therefore suppose that on deleting any country from M there remains a 5-chromatic map, and in this case *every country has at least* 5 *neighbours having a common frontier line with it.*

By the Lemma, M *therefore contains a country with just* 5 *neighbours.*

Let X be such a country and let its neighbours be Y, Z, U, V, W. If each pair of Y, Z, U, V, W were neighbours then M would contain the 6 mutually adjacent countries X, Y, Z, U, V, W, contrary to hypothesis. So among Y, Z, U, V, W there are two countries which are not neighbours, say Y and Z. Let M′ denote the map obtained from M by deleting the frontier line separating X and Y and the frontier line separating X and Z and uniting the three countries X, Y and Z into one country X′.

Then M′ is 6-chromatic; for the chromatic number of M′ is at most 6. Suppose it could be coloured using five colours. Then the map M − X, obtained from M by deleting X, could be coloured with five colours in such a way that Y and Z receive the same colour. In this colouring the countries Y, Z, U, V, W between them receive at most four colours. If X is now re-introduced into M − X, it can be given the fifth colour, and this gives a colouring of M using five colours, which contradicts the datum that M is 6-chromatic.

So M′ is 6-chromatic and contains fewer countries than M, and therefore by the induction hypothesis it either contains 6 mutually adjacent countries, or a map containing 6 mutually adjacent countries can be obtained from it by deleting suitably chosen frontier lines, and uniting those countries which they separate. *A fortiori* the same is true of M, and so the theorem is proved.

Proof of Theorem III. Similar to the proof of Theorem II, with $n_h = 7$ instead of 6.

It follows that a 7-chromatic map can be drawn on a surface of connectivity 4 if and only if there is room for a complete 7-graph on the surface.

NOTE. By a very similar method a short proof of the five-colour theorem of

Heawood [4] can be obtained. For a plane or spherical graph ($h = 1$), $2E/N < 6$; hence a map on the plane or the sphere contains a country with not more than five neighbours, of which two have no common frontier line. If the map obtained by uniting the country and two non-adjacent neighbours can be coloured using five colours, so can the original map. The five-colour theorem now follows by induction.

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For $h = 8$ see R. C. Bose, *On the construction of balanced incomplete block designs*, Annals of Eugenics, vol. 9 (1939), 353.

The cases $h = 10, 12, 14$: The connectivities of the surfaces obtained from the sphere and from the projective plane by attaching n handles are $2n + 1$ and $2n + 2$ respectively. Any map drawn on the surface of a sphere with n handles attached can also be drawn on a projective plane with n handles attached. It follows that $k_{2n+2} \geqslant k_{2n+1}$. Hence, by Heffter's results quoted above and by the table on p. 480, Heawood's result is best possible also for $h = 10, 12$, and 14.

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