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TREE MAPS WITH NON DIVISIBLE PERIODIC ORBITS

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Let End(T) be the number of ends of a tree T and $f : T \longrightarrow T$ be continuous. We show that f has a non divisible periodic orbit if and only if there are some $x \in T$ and $n > 1$ with $(n, m) = 1$ for each $2 \leqslant m \leqslant$ End(T) such that $x \in (f(x), f^n(x))$. Consequently the property of a tree map with a non divisible periodic orbit is preserved under small perturbation.

1. INTRODUCTION

In the recent years there has been a growing interest in studying the dynamics of continuous maps of a *graph*, that is, a one-dimensional connected compact branched manifold, see for instance [**1, 2, 3, 5, 8, 11**], as this kind of research is closely related to the study of disk homeomorphisms [**5**] and the topological structure of one-dimensional continua through the inverse limit process [**12**]. In the study one of the important problems is to study periodic orbits and related problems such as to estimate the topological entropy (see [**10**] for a definition) since a periodic orbit plays an important role in determining the dynamics and the entropy is a good tool for measuring the chaoticity of the system.

In this paper we shall deal with the above problem in the particular case of a *tree*, that is, a graph without a cycle. For simplicity, a continuous map from a tree into itself will be called a *tree map*. The notion of no division first appeared in [**6, 7**] for interval maps and then was developed for tree maps in [**1, 2**]. We remark that if a tree map has a non divisible periodic orbit then the dynamics of f is complicated, for instance, the set of periods of f is cofinite and the topological entropy of f is positive [**2**].

It is known that for an interval map f, if there are some odd $n > 1$ and some x with $x \in (f(x), f^n(x))$ then there is a periodic orbit of odd period $1 < q \leqslant n$ [**6**]. The aim of the paper is to try to generalise the above result from interval maps to tree maps and to show the converse. The result will be useful in dealing with the ω-limit set of f, see for instance [**4**]. To be precise we need some notion.

We shall consider non-degenerate trees. A *subtree* of T is a subset of T, which is a tree itself. For $x \in T$ the number of connected components of $T \setminus \{x\}$ is called the

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valence of x in T. A point of T of valence 1 is called an *end* of T, and a point of valence different from 2 is called a *vertex* of T. The set of ends of T, the set of the vertices of T and the number of ends of T will be denoted by $E(T)$, $V(T)$ and $\text{End}(T)$ respectively. The closure of each connected component of $T \setminus V(T)$ is called an *edge* of T.

Let $A \subset T$ have more than one point. We shall use $[A]$ to denote the smallest subtree containing A. If $A = \{a, b\}$ then we use $[a, b]$ to denote $[A]$. We define $(a, b) = [a, b] \setminus \{a, b\}$ and we similarly define $(a, b]$ and $[a, b)$. For a subtree S of T, we shall use $r_S : T \longrightarrow S$ to denote the natural retraction from T to S.

Lastly we need the definition of no division introduced in [**2**]. Let $f : T \longrightarrow T$ be a continuous map of T and P a periodic orbit of f with period larger than 1. Assume $y \in [P]$ is a fixed point of $r_{[P]} \circ f \mid [P]$, Z is the connected component of $[P] \setminus P$ containing y and $Z_1, \ldots, Z_k$ are the connected components of $[P] \setminus Z$.

Definition 1.1. We say that P has a *division* if there is a partition $\{M_1, \ldots, M_m\}$ with $m \geqslant 2$ of $\{Z_1, \ldots, Z_k\}$ such that

$$f(M_i \cap P) = M_{i+1(\text{mod } m)} \cap P, \quad 1 \leqslant \text{i} \leqslant \text{m}.$$

Otherwise we say that P has *no division* or P is a *non divisible periodic orbit.*

The main results of the paper are:

Theorem A. *Let $f : T \longrightarrow T$ be a continuous map of a tree T. Then f has a non divisible periodic orbit if and only if there are some $x \in T$ and $n > 1$ with $(n, m) = 1$ for each $2 \leqslant m \leqslant \text{End}(T)$ such that $x \in (f(x), f^n(x))$.*

Corollary B. *Let T be a tree and $f : T \longrightarrow T$ be continuous. Then the set $\mathcal{A}$ of all continuous maps of T with a non divisible periodic orbit is open in $C(T, T)$ and the set $\mathcal{B}$ of all continuous maps with vanishing topological entropy is closed in $C(T, T)$, where $C(T, T)$ is the set of all continuous maps of T.*

2. The proofs

In this section we shall give the proofs of Theorem A and Corollary B stated in the Introduction. To do this we need

Lemma 2.1. [6] *Let I be a closed interval and $f : I \longrightarrow I$ continuous. If there are some $x \in T$ and some odd $n > 1$ such that $f^n(x) \leqslant x < f(x)$ or $f(x) < x \leqslant f^n(x)$ then f has a periodic orbit of period q, for some odd q satisfying $1 < q \leqslant n$.*

Lemma 2.2. [**11**] *Let T be a tree and $f : T \longrightarrow T$ be continuous. If there are x, y in the same edge of T such that $[x, y] \subset [f(x), f(y)]$, or $x \notin [f(x), y]$ and $y \notin [x, f(y)]$ then f has a fixed point in $[x, y]$.*

Lemma 2.3. [3] *Let f be a continuous map of a tree T and S be a subtree of T. Then $Per(r_S \circ f \mid S) \subset Per(f)$.*

Now we are ready to prove Theorem A.

PROOF OF THEOREM A (SUFFICIENCY). Let $C_{f^n(x)}$ be the closure of the connected component of $T \setminus \{x\}$ containing $f^n(x)$. We use induction on $\operatorname{End}\left(C_{f^n(x)}\right)$.

Assume that $\operatorname{End}\left(C_{f^n(x)}\right) = 2$, that is $C_{f^n(x)} = [e, x]$. Give an orientation of $[e, x]$ such that $e < x$. Define $y > x$ if $y \in T \setminus [e, x]$. As $f^n(e) \geqslant e$ by Lemma 2.2 there is a $t \in [e, x)$ with $f^n(t) = t$. Let $t_0 = \sup\{t \in [e, x) : f^n(t) = t\}$. If t_0 is not a fixed point of f the period l of t_0 is larger than 1 and $l \mid n$. Thus f has a periodic orbit with period $1 < l \leqslant n$ with no division. Hence we may assume that $f(t_0) = t_0$. By the definition of t_0 we have that $f(t) > t$ and $f^n(t) < t$ for each $t \in (t_0, x]$. Take $t \in (t_0, x]$ closed to t_0 such that $\{t, f(t), \dots, f^n(t)\} \subset [e, x]$. By Lemma 2.1 and Sharkovskii's Theorem [**9**] $r_{[e,x]} \circ f \mid [e, x]$ has a periodic orbit of period n, consequently f has a periodic orbit with period n according to Lemma 2.3. This orbit has no division.

Now assume that for each $x \in T$ with $\operatorname{End}\left(C_{f^n(x)}\right) \leqslant i < \operatorname{End}(T)$, Theorem A holds. Suppose now $\operatorname{End}\left(C_{f^n(x)}\right) = i + 1$.

Let $v \neq x$ be a vertex of $C_{f^n(x)}$ with $(v, x) \cap V(T) = \emptyset$. If $f^n(v) \in C_{f^n(x)} \setminus (v, x]$ and $v \in (f^n(v), f(v))$ then f has a periodic orbit of period $1 < l \leqslant n$ with no division by the induction assumption (with x replaced by v). Hence we assume that there are (Case 1:) $f^n(v) \notin C_{f^n(x)} \setminus (v, x)$ or (Case 2:) $f^n(v) \in C_{f^n(x)} \setminus (v, x)$ but $v \notin (f^n(v), f(v))$.

Give an orientation on $[v, x]$ such that $v < x$. Define $y < v$ if $y \in C_{f^n(x)} \setminus [v, x]$ and $x < y$ if $y \in T \setminus C_{f^n(x)}$. In Case 1 $f^n(v) \geqslant v$, and in Case 2 $f(v) \leqslant v$. Hence by Lemma 2.2 there exists $y \in [v, x)$ such that $f^n(y) = y$. Let $t_0 = \sup\{y \in [v, x) : f^n(y) = y\}$. If t_0 is not a fixed point of f then f has a periodic orbit with period $1 < l \leqslant n$ with no division. Hence we assume that t_0 is a fixed point of f. Then for each $t \in (t_0, x]$ we have $f(t) > t$ and $f^n(t) < t$. Take $t \in (t_0, x]$ close to t_0 such that $\{t, f(t), \dots, f^n(t)\} \subset S(v)$, where $S(v)$ is the union of all edges of $C_{f^n(x)}$ containing v.

Let $g = r_{S(v)} \circ f \mid S(v)$. Then $g^i(t) = f^i(t)$, $0 \leqslant i \leqslant n$. Assume that $t, g(t) \subset [e, v]$, where e is an end point of $S(v)$. Give an orientation of $[e, v]$ with $e < v$. As $g(e) \geqslant e$, by Lemma 2.2 $\{t_1 \in [e, x) : g(t_1) = t_1\} \neq \emptyset$. Let $t_0 = \sup\{t_1 \in [e, t) : g^n(t_1) = t_1\}$. If t_0 is not a fixed point of g then g, hence f, has a periodic orbit with period $q > 1$ such that $q \mid n$. Hence we assume that t_0 is a fixed point of g. Thus for $s \in (t_0, t]$ close to t_0 we have $\{s, g(s), \dots, g^n(s)\} \subset [e, v]$ and $g(s) < s < g^n(s)$. Then by Lemma 2.1 and Sharkovskii's theorem [**9**] $r_{[e,v]} \circ g \mid [e, v]$ has a periodic orbit of period n. It follows by Lemma 2.3 that g, and hence f, has a periodic orbit of period n. This orbit has no division and the proof of sufficiency is completed.

NECESSITY. As f has a non divisible periodic orbit f has a periodic orbit P of period $p > \operatorname{End}(T)$ by [**2**, Theorem A], where p is a prime number. Let $y \in [P]$ be a fixed point of $r_{[P]} \circ f \mid [P]$. Then y has the property that $f([y, x]) \supset [y, f(x)]$ for each $x \in [P]$ with $f(x) \in [P]$.

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Let $\{z_1, z_2, \dots, z_k\}$ be the end points of the closure of the connected components of $[P] \setminus P$. Define $\phi : \{z_1, z_2, \dots, z_k\} \longrightarrow \{z_1, z_2, \dots, z_k\}$ as follows: $\phi(z_i) = z_j$ if $f(z_i) \in Z_j$. Without loss of generality we assume that $\{z_1, z_2, \dots, z_l\}$ is a periodic orbit of ϕ with $\phi(z_i) = z_{i+1(\text{mod } l)}$, $1 \leqslant i \leqslant l$. It is clear that $l \leqslant \text{End}(T)$.

As $f([y, z_1]) \supset [y, z_2]$, $f([y, z_2]) \supset [y, z_3]$, $\dots$, $f([y, z_l]) \supset [y, z_1]$ there is $x_1 \in (y, z_1)$ such that $f^l(x_1) = z_1$. Inductively if we have define x_i for $i \in \mathbb{N}$ then we define $x_{i+1} \in (y, x_i)$ with $f^l(x_{i+1}) = x_i$.

If there is x_{i_0} such that $x_{i_0} \in (y, f(x_{i_0}))$ then we define y_j with $f(y_{j+1}) = y_j$ and $y_{j+1} \in (y, y_j)$, where $y_1 = x_{i_0}$. In this case let $z \in P$ with $y \in (z, z_1)$. Assume $z = f^{i_1}(z_1)$. Take j_1 with $n = j_1 - 1 + li_0 + i_1 > \text{End}(T)$ being prime and let $x = y_{j_1}$. Then we have $x = y_{j_1} \in (f(y_{j_1}), z) = (f(x), f^n(x))$ as $z = f^{i_1}(z_1) = f^{i_1 + li_0 + j_1 - 1}(y_{j_1}) = f^n(x)$.

Hence we may assume that $x_i \in (f(x_i), z_1)$ for each $i \geqslant 1$. By Dirichlet's Theorem there is $j_0 \in \mathbb{N}$ such that $j_0 l + p > \text{End}(T)$ is a prime number as $(p, l) = 1$. Let $x = x_{j_0}$ and $n = j_0 l + p$. Then we have $x \in (f(x), f^n(x))$ as $f^n(x) = f^{j_0 l + p}(x_{j_0}) = f^p(z_1) = z_1$. This ends the proof of Theorem A. ∎

Let d be a metric on a tree T. Then the topology of $C(T, T)$ is induced by the metric d_1 defined by $d_1(f, g) = \sup\{d(f(x), g(x)) : x \in T\}$ for each $f, g \in C(T, T)$.

PROOF OF COROLLARY B. By Theorem A we know that there are some $x \in T$ and prime $n > \text{End}(T)$ such that $x \in (f(x), f^n(x))$. We may assume that x is not a vertex of T as if x is a vertex of T then we may take y close to x with $y \in (f(y), f^n(y))$ and y not a vertex of T. Hence there is a neighbourhood U of f in $C(T, T)$ such that for each $g \in U$ we have $x \in (g(x), g^n(x))$. Then by Theorem A, g has a non divisible periodic orbit, that is, $\mathcal{A}$ is an open subset of $C(T, T)$.

Let $\mathcal{A}'$ be the set of all continuous maps of T with property that there is $n \in \mathbb{N}$ such that f^n has a non divisible periodic orbit. It is easy to see that $\mathcal{A}'$ is also an open subset of $C(T, T)$. The complement of $\mathcal{A}'$ is $\mathcal{B}$, by [**2**, Corollary C], hence $\mathcal{B}$ is a closed subset of $C(T, T)$. We remark that the fact $\mathcal{B}$ is closed also follows by [**8**], as the entropy function is lower semi-continuous. ∎

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