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A WEAKLY ANALYTIC LOCALLY CONVEX SPACE WHICH IS NOT K-ANALYTIC

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Abstract

It is shown that the dual of the space $C_p(I)$ of all real-valued continuous functions on the closed unit interval with the pointwise topology, when equipped with the Mackey topology, is a non K-analytic but weakly analytic locally convex space.

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1. Introduction

A question raised in [2] of whether every weakly analytic locally convex space is analytic is answered in the negative. If $C_p(I)$ denotes the linear space of all real-valued continuous functions defined on the closed unit interval I of the real line, provided with the pointwise convergence topology, it is shown that the dual E of $C_p(I)$ equipped with the Mackey topology is a weakly analytic locally convex space which is not K-analytic. This solution provides an interesting link between descriptive set topology, C_p-theory and locally convex space theory.

2. Preliminaries

If X is a completely regular Hausdorff space the linear space $C(X)$ of the real-valued continuous functions on X equipped with the topology of pointwise convergence is denoted by $C_p(X)$. The topological dual of $C_p(X)$ is denoted by $L(X)$, whereas $L_p(X)$ designs the weak* dual of $C_p(X)$. By $C_c(X)$ we represent the linear space $C(X)$ equipped with the compact-open topology, whose dual is denoted by $C_c(X)^*$.

Let us recall that $L(X)$ consists of the linear span of the vectors of the standard copy $\delta(X)$ of X in $C_p(C_p(X))$, that is, each $x \in X$ is depicted in $L(X)$ by the evaluation map

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δ_x at x, defined by $\delta_x(f) = f(x)$ for each $f \in C(X)$. This forces X to be represented in $L(X)$ as an algebraic basis. Indeed, if $\{x_1, \ldots, x_n\}$ is a finite subset of X with $\sum_{i=1}^n \zeta_i \delta_{x_i} = \mathbf{0}$, where $\{\zeta_1, \ldots, \zeta_n\}$ are real numbers and $\mathbf{0}$ stands for the null linear form on $C(X)$, by choosing $f_i \in C(X)$ such that $f_i(x_i) = 1$ and $f_i(x_j) = 0$ for $i \neq j$, $1 \leq i, j \leq n$, the equation $(\sum_{i=1}^n \zeta_i \delta_{x_i})(f_i) = 0$ implies that $\zeta_i = 0$. Hence $L(X)$ consists of the linear span of the basic vectors $\{\delta_x \mid x \in X\}$ of the standard copy of X in $C_p(C_p(X))$. The mapping $\delta : X \to L_p(X)$ defined by $\delta(x) = \delta_x$ is a homeomorphism from X onto the (closed) subset $\delta(X)$ of $L_p(X)$.

A Hausdorff topological space Y is said to be analytic if it is a continuous image of the universal Polish space $\mathbb{N}^{\mathbb{N}}$. A Hausdorff topological space Y is said to be K-analytic if there exists an upper semicontinuous map T from $\mathbb{N}^{\mathbb{N}}$ into the family $\mathcal{K}(Y)$ of all compact subsets of Y, such that $\bigcup\{T(\alpha) \mid \alpha \in \mathbb{N}^{\mathbb{N}}\} = Y$. Different definitions of K-analytic spaces have been shown to be equivalent in the completely regular case [4]. Analytic and K-analytic spaces have been studied in [6] under the names of Suslin and K-Suslin spaces, respectively. Every analytic space is K-analytic and a nonseparable compact space is an example of a K-analytic space which is not analytic. It can be easily shown that a compact set K is metrizable if and only if the space $C_p(K)$ is analytic. A compact Hausdorff space K is said to be Talagrand compact if $C_p(K)$ is K-analytic. Weakly analytic and K-analytic spaces have been extensively investigated in Banach space theory since Talagrand's seminal paper [5]. For instance, it is well known that a Banach space is weakly K-analytic (analytic) if and only if its dual unit ball with the relative weak* topology is Talagrand compact (respectively metrizable).

3. Main theorem

In what follows X will be the closed interval $I = [0, 1]$ equipped with the relative topology of the real line and the subset $\delta(I)$ of $L(I)$ will be represented by Δ. The Mackey topology on $L(I)$ of the dual pair $\langle L(I), C(I)\rangle$ will be denoted as usual by $\mu(L(I), C(I))$, and the corresponding weak topology, namely the weak* topology of $L(I)$, by $\sigma(L(I), C(I))$. The topology on Δ induced by $\mu(L(I), C(I))$ will be denoted by μ, whereas σ will design the relative topology of $\sigma(L(I), C(I))$ on Δ.

LEMMA 3.1. *Each nontrivial convergent sequence of* (Δ, σ) *does not converge in* (Δ, μ).

PROOF. Let $\{u_n \mid n \in \mathbb{N}\}$ be a nontrivial convergent sequence of (Δ, σ) and let u be its σ-limit. Then put $a_n = \delta^{-1}(u_n) \in I$ for each $n \in \mathbb{N}$. Since $\{a_n\}$ is a nontrivial convergent sequence of I, working with a subsequence if necessary we may assume without loss of generality that $\{a_n\}$ is strictly monotone.

We may suppose for instance (the other case is totally analogous) that $\{a_n\}$ is strictly decreasing with $a_1 < a_0 = 1$ and $a_n \to a = \delta^{-1}(u)$. Then let us consider a sequence $\{f_n \mid n \in \mathbb{N}\}$ of functions in $C(I)$ satisfying the following conditions:

(1) supp $f_n \subseteq [(3a_n + a_{n+1})/4, (3a_n + a_{n-1})/4]$, where supp f_n means the support of f_n;
(2) $0 \le f_n \le 1$;
(3) $f_n(a_n) = 1$ for each $n \in \mathbb{N}$.

For example f_n can be the function on I taking the value zero in $[0, (3a_n + a_{n+1})/4] \cup [(3a_n + a_{n-1})/4, 1]$ and whose graph in the band $[(3a_n + a_{n+1})/4, (3a_n + a_{n-1})/4] \times \mathbb{R}$ of $\mathbb{R}^2$ is a triangle with vertices at the points $A_n((3a_n + a_{n+1})/4, 0)$, $B_n(a_n, 1)$ and $C_n((3a_n + a_{n-1})/4, 0)$ of the plane. Since supp $f_i \cap$ supp $f_j = \emptyset$ if $i \neq j$, the functions f_n are disjointly supported. Let us write $A_n := [(3a_n + a_{n+1})/4, (3a_n + a_{n-1})/4]$ for each $n \in \mathbb{N}$.

As is well known, the topological dual $C_c(I)^*$ of $C_c(I)$ can be identified with the space $rca(\mathcal{B})$ of regular (countably additive) Borel measures defined on the σ-algebra $\mathcal{B}$ of all Borel subsets of I. Note that if $\mu \in rca(\mathcal{B})$, since $f_n(x) \le \chi_{A_n}(x)$ for $0 \le x \le 1$, then, with respect to the dual pair $\langle C(I), C_c(I)^*\rangle$, one has

$$|\langle f_n, \mu\rangle| = \left|\int_0^1 f_n \, d\mu \le \int_0^1 \chi_{A_n}\right| d|\mu| = |\mu|(A_n)$$

for every $n \in \mathbb{N}$. Given that $A_i \cap A_j = \emptyset$ for $i \neq j$ and μ is countably additive, then $\mu(A_n) \to 0$ in $\mathbb{R}$. Hence $\langle f_n, \mu\rangle \to 0$, which shows that $f_n \to 0$ in $C(I)$ under the weak topology of the Banach space $C_c(I)$.

If P stands for the $\sigma(C(I), C_c(I)^*)$-closure of abx$\{0, f_n : n \in \mathbb{N}\}$, the absolutely convex cover of the weakly compact subset $\{0, f_n : n \in \mathbb{N}\}$ of the Banach space $C_c(I)$, Krein's theorem ensures that P is an absolutely convex weakly compact set in $C_c(I)$. Since $C_c(I)^* \supseteq L(I)$, it follows that P is an absolutely convex compact set in $C_p(I)$.

On the one hand, the fact that $f_n \in P$ for all $n \in \mathbb{N}$ implies that, with respect to the dual pair $\langle C(I), L(I)\rangle$,

$$\sup\{|\langle f, u_n\rangle| : f \in P\} \ge |\langle u_n, f_n\rangle| = f_n(a_n) = 1 \tag{3.1}$$

holds for every $n \in \mathbb{N}$. On the other hand, one has

$$\sup\{|\langle f, u\rangle| : f \in P\} = \sup\{|f(a)| : f \in P\} = 0. \tag{3.2}$$

Indeed, since $a \notin \bigcup_{n=1}^{\infty} A_n$ then $\langle f_n, \delta(a)\rangle = \langle f_n, \delta_a\rangle = f_n(a) = 0$ for each $n \in \mathbb{N}$, which means that $f(a) = 0$ for each $f \in$ abx$\{0, f_n : n \in \mathbb{N}\}$ and hence for every $f \in P$. From (3.1) and (3.2) it follows that

$$\sup\{|\langle f, u_n - u\rangle| : f \in P\} \ge 1$$

for every $n \in \mathbb{N}$. Consequently, $u_n \nrightarrow u$ in Δ under $\mu(L(I), C(I))$ whilst $u_n \to u$ in Δ under $\sigma(L(I), C(I))$. □

THEOREM 3.2. *The locally convex space $E = (L(I), \mu(L(I), C(I)))$ is weakly analytic but not K-analytic.*

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PROOF. Since I and $\mathbb{R}$ are analytic sets and the class of analytic spaces is closed, among other properties, under continuous images, countable products and countable unions of subspaces, then $L_p(I)$ is an analytic space as a consequence of [1, Proposition 0.5.13]. So E is weakly analytic. Proceeding by contradiction we assume that E is a K-analytic space. Note that since Δ is closed in $L_p(I)$, it is closed in E and consequently Δ is a K-analytic set under the relative topology of E. □

CLAIM 3.3. *There is a completely regular K-analytic topology τ_2 on I stronger than the relativization τ_1 of the usual topology of $\mathbb{R}$ to I such that δ is an homeomorphism from (I, τ_2) onto Δ, when considered as a subspace of E.*

PROOF. Since $\mu(L(I), C(I))$ is the strongest locally convex topology on $L(I)$ of the dual pair $\langle L(I), C(I)\rangle$, the topology μ on Δ induced by $\mu(L(I), C(I))$ is stronger than the Souslin topology σ on Δ induced by $\sigma(L(I), C(I))$. Now let us consider the homeomorphism $\delta : (I, \tau_1) \to (\Delta, \sigma)$ and let τ_2 be the topology on I consisting of the family $\{\delta^{-1}(U) : U \in \mu\}$ of subsets of I. Since μ is stronger than σ, if W is a τ_1-open subset of I then $\delta(W)$ is a σ-open subset of Δ and hence a μ-open set. Hence $W = \delta^{-1}(\delta(W))$ is τ_2-open and $\tau_1 \leq \tau_2$.

Clearly, $\delta : (I, \tau_2) \to (\Delta, \mu)$ is continuous. On the other hand, if V is a τ_2-open set of I there is a μ-open set U in Δ with $V = \delta^{-1}(U)$, so that $\delta(V) = U$. This shows that $\delta : (I, \tau_2) \to (\Delta, \mu)$ is open. Thus δ is an homeomorphism from (I, τ_2) onto (Δ, μ). □

CLAIM 3.4. *There exists a nonempty perfect set J in I where τ_1 coincides with τ_2.*

PROOF. Since (I, τ_1) is a Baire space and (I, τ_2) is a regular K-analytic space, Nakamura's closed graph theorem [3, Theorem] applied to the identity map $\varphi : (I, \tau_1) \to (I, \tau_2)$ yields a subset D of I with $I \setminus D$ of the first category in (I, τ_1) such that φ is continuous on D; hence $\tau_1|_D = \tau_2|_D$. If $\{N_n\}$ is a sequence of nowhere dense subsets of (I, τ_1) such that $I \setminus D = \bigcup_{n=1}^{\infty} N_n$ and F_n denotes the τ_1-closure of N_n in I, then $G = I \setminus \bigcup_{n=1}^{\infty} F_n$ is a τ_1-dense G_δ in I of the second category (hence uncountable) in (I, τ_1) with $G \subseteq D$. Since G is an uncountable analytic set, it must contain a nonempty perfect set J. □

If J is the nonempty perfect subset of (I, τ_1) determined by Claim 3.4, let us denote by K the compact subset $\delta(J)$ of $L_p(I)$. Given that J is a nonempty perfect subset of the compact space (I, τ_1), then $|J| = 2^{\aleph_0}$ and there exists a nontrivial (injective) τ_1-convergent sequence $\{\zeta_n : n \in \mathbb{N}\}$ in I contained (together with its limit a) in J. So, if we put $v_n := \delta(\zeta_n)$ for each $n \in \mathbb{N}$, then $\{v_n\}$ is a nontrivial σ-convergent sequence of K, hence of (Δ, σ). But since $\tau_1|_J = \tau_2|_J$ by Claim 3.4, the second statement of Claim 3.3 implies that $\sigma|_K = \mu|_K$. So $\{v_n\}$ is a nontrivial convergent sequence in (Δ, σ) which also converges in (Δ, μ), contradicting Lemma 3.1. □

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