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COMMUTING PROPERTIES OF EXT

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Abstract

We characterize the abelian groups G for which the functors $\mathrm{Ext}(G, -)$ or $\mathrm{Ext}(-, G)$ commute with or invert certain direct sums or direct products.

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1. Introduction

Let $\mathcal{F}$ be a functor acting on the category of abelian groups and let C be a class of abelian groups. We address the problem of characterizing those $\mathcal{F}$ which preserve or invert direct sums or products from C, according to the following definitions, in which the indices run over arbitrary sets and the A_i are groups in C:

(1) $\mathcal{F}$ preserves direct sums from C if $\mathcal{F}(\bigoplus_i A_i) \cong \bigoplus_i(\mathcal{F}(A_i))$;
(2) $\mathcal{F}$ preserves direct products from C if $\mathcal{F}(\prod_i A_i) \cong \prod_i(\mathcal{F}(A_i))$;
(3) $\mathcal{F}$ inverts direct products from C if $\mathcal{F}(\prod_i A_i) \cong \bigoplus_i(\mathcal{F}(A_i))$;
(4) $\mathcal{F}$ inverts direct sums from C if $\mathcal{F}(\bigoplus_i A_i) \cong \prod_i(\mathcal{F}(A_i))$.

This problem is a generalization of many well-known results in the theory of abelian groups and modules. For example, the classic text of Fuchs [9] contains proofs that the covariant functors $\mathrm{Hom}(G, -)$ and $\mathrm{Ext}(G, -)$ preserve direct products, whereas the contravariant functors $\mathrm{Hom}(-, G)$ and $\mathrm{Ext}(-, G)$ invert direct sums from the class $\mathcal{A}$ of all abelian groups.

The simplest case for the preserving/inverting properties of Hom functors is that of small abelian groups, that is, abelian groups G for which the functor $\mathrm{Hom}(G, -)$ preserves direct sums from $\mathcal{A}$. These are precisely the finitely generated abelian groups. However, the characterization of the other preserving/inverting properties is more complicated. For instance, in [10], the class of slender groups is characterized, where slender groups are those abelian groups G for which the functor $\mathrm{Hom}(-, G)$ inverts direct products of non-measurable index from $\mathcal{A}$. Each of the four properties for the two cases $\mathcal{F} = \mathrm{Hom}(-, G)$ and $\mathcal{F} = \mathrm{Hom}(G, -)$ are considered in the recent papers [5, 15]. A related problem concerns the 'self' specialization: describe the class of groups G for which $\mathrm{Hom}(G, -)$ or $\mathrm{Hom}(-, G)$ preserve or invert sums or products of copies of G. For example, G is self-small [4] if for all cardinals κ, $\mathrm{Hom}(G, \bigoplus_\kappa G)$ is naturally isomorphic to $\bigoplus_\kappa \mathrm{Hom}(G, G)$ and self-slender if $\mathrm{Hom}(\prod_\kappa G, G) \cong \bigoplus_\kappa \mathrm{Hom}(G, G)$. These properties are considered in [3, 8, 11, 15].

Concerning corresponding properties of the extension functors, Göbel and Trlifaj proved in [14, Example 3.1.8], using a result of Salce [18], that despite the results in [9] quoted above, there are important differences between the cases of Hom and Ext. In [12] Göbel and Prelle showed that $\mathrm{Ext}(-, G)$ inverts direct products and preserves direct products from the class of abelian groups if and only if G is divisible, and from the class of torsion-free abelian groups if and only if G is cotorsion; that is, $\mathrm{Ext}(J, G) = 0$ for all torsion-free groups J.

In this paper, we complete the characterization of the groups G for which the functors $\mathrm{Ext}(-, G)$ or $\mathrm{Ext}(G, -)$ preserve or invert direct sums and products from various classes of abelian groups.

If $\mathrm{Ext}(-, G)$ or $\mathrm{Ext}(G, -)$ preserve or invert arbitrary sums or products of copies of G, we say that they preserve or invert *self-sums* or *self-products*.

Just like the Hom functors, it turns out that the most interesting cases are:

- *C-Ext-small groups* $= \{G : \mathrm{Ext}(G, -)$ preserves sums from $C\}$,
- *C-Ext-slender groups* $= \{G : \mathrm{Ext}(-, G)$ inverts products from $C\}$,

together with the corresponding 'self' versions. Note that in the papers [1, 7] these concepts are called C-extendible and C-coextendible, respectively.

Henceforth, to simplify the notation, 'group' means abelian group, 'sum' means direct sum and 'product' means direct product. For a group A and a cardinal κ, let $A^{(\kappa)} = \bigoplus_\kappa A$ and $A^\kappa = \prod_\kappa A$. For any group A we denote by $T(A)$ the torsion subgroup of A. For a prime p, we denote by A_p the p-component of $T(A)$ and by $A[p]$ the socle of A_p. Except where explicitly stated, we adopt the notation of the standard reference [9]. In particular, $\mathbb{Z}(p^n)$ is the cyclic group of order p^n for some prime p, J_p is the group of p-adic integers and $\mathbb{Z}_p$ the localization of the integers $\mathbb{Z}$ at the prime p.

The various classes C of groups from which we take direct sums or products are as follows:

- $\mathcal{A}$, all abelian groups;
- $\mathcal{TF}$, torsion-free groups;
- $\mathcal{T}$, torsion groups;
- $\mathcal{D}$, divisible groups;
- $p^\infty\mathcal{A}$, p-divisible groups;
- $\mathcal{A}[p^\infty]$, p-groups;
- C_p, cyclic p-groups;
- $\{B\}$, a singleton group B.

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We use repeatedly the trivial but useful result that if $\mathcal{C} \subseteq \mathcal{C}'$ are classes of groups and $\mathcal{F}$ is a functor which preserves (inverts) sums (products) from $\mathcal{C}'$, then $\mathcal{F}$ preserves (inverts) sums (products) from $\mathcal{C}$.

In Section 2 we survey the known results and characterize the $\mathcal{C}$-Ext-slender groups for various classes $\mathcal{C}$, and the self-Ext-slender groups.

In Section 3 we characterize the groups G for which $\mathrm{Ext}(G, -)$ inverts products from some classes $\mathcal{C}$, and in Section 4 those for which $\mathrm{Ext}(G, -)$ inverts sums from $\mathcal{C}$. For example, we prove in Theorem 4.5 that $\mathrm{Ext}(G, -)$ inverts sums from $\mathcal{TF}$, and hence from $\mathcal{A}$, if and only if G is free.

Section 5 contains a characterization of the $\mathcal{C}$-Ext-small groups for some important classes $\mathcal{C}$. This complements the results from [7], in which we proved that a group is $\mathcal{A}$-Ext-small if and only if it is $\mathcal{TF}$-Ext-small, and the class of these groups coincides with the class of groups which are direct sums of a finitely generated group and a free group.

Remark 1.1. (1) For any family $(A_i)_{i\in I}$ of groups from a class $\mathcal{C}$, there are canonical injections $\iota_i : A_i \to \bigoplus_i A_i$ and projections $\pi_i : \prod_i A_i \to A_i$. The functors $\mathrm{Ext}(G, -)$ and $\mathrm{Ext}(-, G)$ induce natural homomorphisms from $\bigoplus_i \mathrm{Ext}(G, A_i)$ to $\mathrm{Ext}(G, \bigoplus_i A_i)$ and from $\bigoplus_i \mathrm{Ext}(A_i, G)$ to $\mathrm{Ext}(\prod_i A_i, G)$. Moreover, if we start with the canonical projections $\bigoplus_i A_i \to A_i$, we obtain a natural homomorphism from $\mathrm{Ext}(G, \bigoplus_i A_i)$ to $\prod_i \mathrm{Ext}(G, A_i)$; see [1].

We might require these natural homomorphisms to be isomorphisms, but we do not impose this condition since in all situations considered here, the existence of an isomorphism implies that the natural homomorphism is an isomorphism. Solutions for the 'natural' case were obtained in [6] for modules over right hereditary rings, and in [2] for self-Ext-small abelian groups.

(2) In many cases, it turns out that sums or products are preserved or inverted only because both sides of an isomorphism evaluate to zero. In this situation, we say that the functor preserves or inverts products or sums *trivially*. For instance, an abelian group G is free if and only if $\mathrm{Ext}(G, -)$ preserves direct sums from $\mathcal{A}$ trivially. Therefore many techniques used here are closely related to those used in the study of cotorsion theories in [19, 22, 23].

2. The functor $\mathrm{Ext}(-, G)$

Some of the properties of $\mathrm{Ext}(-, G)$ described in the introduction are well known but are listed here for completeness.

Proposition 2.1 [9, Theorem 52.2 1.]. *For all groups G, $\mathrm{Ext}(-, G)$ inverts sums from $\mathcal{A}$.*

In [12, Section 5], Göbel and Prelle studied the properties of the functor $\mathrm{Ext}(-, G)$. By reducing to the cases of elementary p-groups and free groups and using the basic properties of Ext described in [9, Section 52], they showed that $\mathrm{Ext}(-, G)$ preserves (inverts) products in various classes $\mathcal{C}$ if and only if it does so trivially.

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Hence we have the following results.

Proposition 2.2. *The following statements are equivalent for a group G and a prime p.*

(1) *G is p-divisible.*
(2) $\operatorname{Ext}(-, G)$ *preserves products from* $p^\infty\mathcal{A}$.
(3) $\operatorname{Ext}(-, G)$ *inverts products from* $p^\infty\mathcal{A}$.

Corollary 2.3 [12, Corollary 5.6(a)]. *The following statements are equivalent for a group G.*

(1) *G is divisible.*
(2) $\operatorname{Ext}(-, G)$ *preserves products from* $\mathcal{A}$.
(3) $\operatorname{Ext}(-, G)$ *inverts products from* $\mathcal{A}$.

Proposition 2.4 [12, Theorem 3.3, Corollary 5.6(b)]. *The following statements are equivalent for a group G.*

(1) *G is cotorsion; that is,* $\operatorname{Ext}(J, G) = 0$ *for every torsion-free group J.*
(2) $\operatorname{Ext}(-, G)$ *preserves products from* $\mathcal{TF}$.
(3) $\operatorname{Ext}(-, G)$ *preserves products from* $\{\mathbb{Q}\}$.

Similarly, it is easy to show that $\operatorname{Ext}(-, G)$ preserves sums from a class C if and only if it does so trivially.

Lemma 2.5. $\operatorname{Ext}(-, G)$ *preserves sums from any class C if and only* $\operatorname{Ext}(H, G) = 0$ *for all $H \in C$.*

Proof. It is clear that if $\operatorname{Ext}(H, G) = 0$ for all $H \in C$, then sums are preserved trivially.

Conversely, assume that $\operatorname{Ext}(-, G)$ preserves sums from C and let $|\operatorname{Ext}(H, G)| = \lambda > 1$ for some $H \in C$. Let $\kappa \geq \lambda$ be an infinite cardinal. Then

$$|\operatorname{Ext}(H^{(\kappa)}, G)| = |\operatorname{Ext}(H, G)^\kappa| = \lambda^\kappa \quad \text{but } |\operatorname{Ext}(H, G)^{(\kappa)}| = \lambda\kappa = \kappa.$$

Hence $\operatorname{Ext}(H^{(\kappa)}, G) \ncong \operatorname{Ext}(H, G)^{(\kappa)}$. □

Proposition 2.6.

(1) $\operatorname{Ext}(-, G)$ *preserves sums from* $\mathcal{A}$ *if and only if G is divisible.*
(2) $\operatorname{Ext}(-, G)$ *preserves sums from* $\mathcal{TF}$ *if and only if G is cotorsion.*
(3) $\operatorname{Ext}(-, G)$ *preserves sums from* $\mathcal{A}[p^\infty]$ *if and only if G is p-divisible.*
(4) $\operatorname{Ext}(-, G)$ *preserves self-sums if and only if* $\operatorname{Ext}(G, G) = 0$.

Proof. In each of the four parts, it is well known that the structure condition is trivially sufficient. To show that it is also necessary, it is enough to apply Lemma 2.5. □

Remark 2.7. Groups satisfying $\operatorname{Ext}(G, G) = 0$ are called *splitters*. We introduced these groups in [20], pointing out the obvious facts that free groups and cotorsion groups are splitters, and posed the question whether these classes are the only examples. This question was answered immediately in the negative by Göbel and Shelah [13],

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who by an elaborate construction using generators and relations found large classes of counterexamples.

It remains to characterize the C-Ext-slender and self-Ext-slender groups.

Lemma 2.8. *Let H be a group. A group G is $\{H\}$-Ext-slender if and only if* $\mathrm{Ext}(H^\kappa, G) = 0$ *for all cardinals κ.*

Proof. Suppose that G is $\{H\}$-Ext-slender. For every infinite cardinal κ, we have an epimorphism

$$\mathrm{Ext}(H,G)^{(\kappa)} \cong \mathrm{Ext}(H^\kappa, G) \twoheadrightarrow \mathrm{Ext}(H^{(\kappa)}, G) \cong \mathrm{Ext}(H,G)^\kappa.$$

This is possible only if $\mathrm{Ext}(H,G) = 0$. Therefore, for every cardinal κ, $\mathrm{Ext}(H^\kappa, G) \cong \mathrm{Ext}(H,G)^{(\kappa)} = 0$. □

Conversely, if $\mathrm{Ext}(H^\kappa, G) = 0$ for all κ, it follows that $\mathrm{Ext}(H,G) = 0$, hence $\mathrm{Ext}(H,G)^{(\kappa)} = 0 = \mathrm{Ext}(H^\kappa, G)$.

Göbel and Prelle proved the following result in [12, Theorem 3.3 and Corollary 5.6(b)]. We present some details for the reader's convenience.

Theorem 2.9. *The following statements are equivalent for a group G.*

(1) *G is $\mathcal{TF}$-Ext-slender.*
(2) *G is $\{\mathbb{Q}\}$-Ext-slender.*
(3) *G is $\{\mathbb{Z}\}$-Ext-slender.*
(4) *If $0 \neq H \in \mathcal{TF}$, then G is $\{H\}$-Ext-slender;*
(5) *G is cotorsion.*

Proof. It is obvious that (1) implies (2), (3) and (4) and that (5) implies (1).

That (2) implies (5) is a consequence of Lemma 2.8.

To prove that (4) implies (3), let $H \neq 0$ be a torsion-free group such that G is $\{H\}$-Ext-slender. By Lemma 2.8, it follows that $\mathrm{Ext}(H^\kappa, G) = 0$ for all κ. Since $\mathbb{Z}^\kappa$ embeds in H^κ, we obtain $\mathrm{Ext}(\mathbb{Z}^\kappa, G) = 0$ for all κ, so G is $\{\mathbb{Z}\}$-Ext-slender.

Finally, we prove that (3) implies (5). Göbel and Prelle proved in [12, Theorem 3.3] that if a group G is not cotorsion, then there is a cardinal κ such that $\mathrm{Ext}(\mathbb{Z}^\kappa, G) \neq 0$. But the $\{\mathbb{Z}\}$-Ext-slender condition shows that $\mathrm{Ext}(\mathbb{Z}^\kappa, G) = 0$ for all κ, so G must be cotorsion. □

Theorem 2.10. *The following statements are equivalent for a group G and a prime p.*

(1) *G is $\{\mathbb{Z}(p)\}$-Ext-slender.*
(2) *If $B \neq 0$ is a bounded p-group then G is $\{B\}$-Ext-slender.*
(3) *G is p-divisible.*

Proof. It is well known that if G is p-divisible, then $\mathrm{Ext}(A,G) = 0$ for any p-group A. Moreover, if $B \neq 0$ is a bounded p-group such that $\mathrm{Ext}(B,G) = 0$ then there is a positive integer n such that $\mathrm{Ext}(\mathbb{Z}(p^n), G) = 0$. Since $\mathbb{Z}(p)$ can be embedded in $\mathbb{Z}(p^n)$, it follows that $G/pG \cong \mathrm{Ext}(\mathbb{Z}(p), G) = 0$, so G is p-divisible. □

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Theorem 2.11. *The following are equivalent for a group G and a prime p.*

(1) *G is $\mathcal{A}[p^\infty]$-Ext-slender.*
(2) *G is $\{\mathbb{Z}(p^\infty)\}$-Ext-slender.*
(3) *If $H \neq 0$ is an unbounded p-group then G is $\{H\}$-Ext-slender.*
(4) *G is a cotorsion p-divisible group.*

Proof. Since $\{\mathbb{Z}(p^\infty)\} \subseteq \mathcal{A}[p^\infty]$, (1) implies (2)

By Lemma 2.7, $\mathrm{Ext}(\mathbb{Z}(p^\infty)^\kappa, G) = 0$ for all cardinals κ. For all cardinals λ, there is a cardinal κ such that H^λ embeds in $\mathbb{Z}(p^\infty)^\kappa$. Since $\mathrm{Ext}(H^\lambda, G)$ is a homomorphic image of $\mathrm{Ext}(\mathbb{Z}(p^\infty)^\kappa, G)$, it follows that G is $\{H\}$-Ext-slender, thus (2) implies (3).

By Lemma 2.7, $\mathrm{Ext}(H^\kappa, G) = 0$ for all cardinals κ. Since H is unbounded, the group H^ω is not a torsion group, so for every cardinal λ there is a cardinal κ such that $\mathbb{Z}^\lambda$ embeds in H^κ. Hence $\mathrm{Ext}(\mathbb{Z}^\lambda, G) = 0$ for all λ. By [12, Theorem 3.3], it follows that G is a cotorsion group.

If H is not divisible then it has a nonzero cyclic direct summand, and from $\mathrm{Ext}(H, G) = 0$ it follows that G is p-divisible. If H is divisible, we obtain the condition $\mathrm{Ext}(\mathbb{Z}(p^\infty), G) = 0$, and from the structure of reduced cotorsion groups, [9, 54(I)], it follows that G must be p-divisible. Thus (3) implies (4).

If G is p-divisible cotorsion group then $\mathrm{Ext}(-, G)$ annihilates all p-groups and all torsion-free groups, hence it annihilates all direct products of p-groups. Thus (4) implies (1). □

Corollary 2.12. *The following statements are equivalent for a group G.*

(1) *G is $\mathcal{A}$-Ext-slender.*
(2) *G is $\mathcal{T}$-Ext-slender.*
(3) *G is $\mathcal{D}$-Ext-slender.*
(4) *G is divisible.*

For the characterization of self-Ext-slender groups, recall from [20] that a group G is a Π-splitter if for all cardinals κ, $\mathrm{Ext}(G^\kappa, G) = 0$. Using [20, Corollary 4.3] we obtain the following result.

Lemma 2.13. *Let G be a torsion-free group. The following statementsare equivalent.*

(1) *G is a Π-splitter.*
(2) *G is self-Ext-slender.*
(3) *G is cotorsion.*

Proof. Let κ be any cardinal. Then $\mathrm{Ext}(G^\kappa, G) = 0 = \mathrm{Ext}(G, G)^{(\kappa)}$. Thus (1) implies (2).

That (2) implies (3) is a consequence of Theorem 2.9.

It follows from [20, Corollary 4.3] that (3) implies (1). □

It remains to characterize the self-Ext-slender groups that are not necessarily torsion-free.

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Theorem 2.14. *A group G is self-Ext-slender if and only if $G = D \oplus H$ where:*

(1) *H is a cotorsion torsion-free group.*
(2) *D is divisible torsion.*
(3) *If $D_p \neq 0$ then H is p-divisible.*

Proof. Let G be self-Ext-slender. By Lemma 2.8, $\mathrm{Ext}(G, G) = 0$ so that $G = D \oplus H$, where D is torsion divisible and H is torsion-free such that H is p-divisible whenever $D_p \neq 0$.

Conversely, let $G = D \oplus H$ satisfying (1), (2) and (3). For every cardinal λ we have the isomorphism $0 = \mathrm{Ext}(G^\lambda, G) \cong \mathrm{Ext}(H^\lambda, H)$. Hence H is a Π-splitter so G is self-Ext-slender. □

3. When $\mathrm{Ext}(G, -)$ inverts products

As in Section 2, some of the properties of $\mathrm{Ext}(G, -)$ are well known, and others can be derived by straightforward cardinality arguments.

Proposition 3.1 [9, Theorem 52.2(2)]. *For all groups G, $\mathrm{Ext}(G, -)$ preserves products from $\mathcal{A}$.*

Corresponding to Lemma 2.5, we show that $\mathrm{Ext}(G, -)$ inverts products from a class C if and only if it does so trivially.

Proposition 3.2. $\mathrm{Ext}(G, -)$ *inverts products from a class C if and only if* $\mathrm{Ext}(G, A) = 0$ *for all $A \in C$.*

Proof. Let $A \in C$ satisfy $|\mathrm{Ext}(G, A)| = \kappa$. Then for any infinite cardinal $\lambda > \kappa$, $\kappa^\lambda = |\mathrm{Ext}(G, A)^\lambda| = |\mathrm{Ext}(G, A^\lambda)| = |\mathrm{Ext}(G, A)^{(\lambda)}| = \kappa\lambda$. Hence $\kappa = 0$.

The converse is evident. □

Lemma 3.3. *If G is a group and H is a p-group then* $\mathrm{Ext}(G \otimes \mathbb{Z}_p, H) \cong \mathrm{Ext}(G, H)$.

Proof. If G is a group and $G^p = G/\bigoplus_{q \neq p} G_q$ then, for every p-group H, $\mathrm{Ext}(G, H) \cong \mathrm{Ext}(G^p, H)$ and $G^p \otimes \mathbb{Z}_p \cong G \otimes \mathbb{Z}_p$. Therefore, we can suppose that $G_q = 0$ for all $q \neq p$.

Hence there is an embedding $G \rightarrowtail G \otimes \mathbb{Z}_p$ whose cokernel is a torsion group with trivial p-component. Consequently, for every p-group H, there is an isomorphism $\mathrm{Ext}(G \otimes \mathbb{Z}_p, H) \cong \mathrm{Ext}(G, H)$. □

Lemma 3.4. *The following statements are equivalent for a group G and a prime p.*

(1) $\mathrm{Ext}(G, H) = 0$ *for all p-groups H.*
(2) *$G \otimes \mathbb{Z}_p$ is free over $\mathbb{Z}_p$.*

Proof. If $\mathrm{Ext}(G, H) = 0$ for all p-groups H, then $\mathrm{Ext}(G \otimes \mathbb{Z}_p, H) = 0$ for all p-groups H. Since all p-groups are naturally $\mathbb{Z}_p$-modules, we can look at this as an equality in the category of $\mathbb{Z}_p$-modules. Using Griffith's solution of the Baer splitting problem [16], we obtain that $G \otimes \mathbb{Z}_p$ is free as a $\mathbb{Z}_p$-module. Thus (1) implies (2).

If $G \otimes \mathbb{Z}_p$ is free over $\mathbb{Z}_p$, it follows that $\mathrm{Ext}(G, H) \cong \mathrm{Ext}(G \otimes \mathbb{Z}_p, H) = 0$ for all p-groups H since all p-groups are naturally $\mathbb{Z}_p$-modules. Thus (2) implies (1). □

Corollary 3.5.

(1) $\mathrm{Ext}(G, -)$ *inverts products from* $\mathcal{A}$ *if and only if* G *is free.*
(2) $\mathrm{Ext}(G, -)$ *inverts products from* $\mathcal{TF}$ *if and only if* G *is free.*
(3) $\mathrm{Ext}(G, -)$ *inverts products from* $\{B\}$, *where* B *is a bounded* p-*group, if and only if* $G_p = 0$.
(4) $\mathrm{Ext}(G, -)$ *inverts products from* $\mathcal{A}[p^\infty]$ *if and only if* $G \otimes \mathbb{Z}_p$ *is free over* $\mathbb{Z}_p$.
(5) $\mathrm{Ext}(G, -)$ *inverts products from* $\{\mathbb{Z}\}$ *if and only if* G *is a Whitehead group.*
(6) $\mathrm{Ext}(G, -)$ *inverts self-products if and only if* G *is a splitter.*

Proof. (1) By Proposition 3.2, it suffices to show that if $\mathrm{Ext}(G, A) = 0$ for all $A \in \mathcal{A}$, then A is free. But this is well known; see [9, p. 222(A)].

(2) Once again, it suffices to show that if $\mathrm{Ext}(G, A) = 0$ for all $A \in \mathcal{TF}$, then G is free. But any abelian group H is a homomorphic image of a free group F, so $\mathrm{Ext}(G, H)$ is a homomorphic image of $\mathrm{Ext}(G, F) = 0$. By (1), G is free.

(3) and (4) follow similarly from Proposition 3.2 and Lemma 3.4.

(5) and (6) follow similarly from Proposition 3.2 and the definitions of Whitehead groups (A is Whitehead if $\mathrm{Ext}(\mathbb{Z}, A) = 0$) and splitters ($A$ is a splitter if $\mathrm{Ext}(A, A) = 0$). □

4. When $\mathrm{Ext}(G, -)$ inverts sums

We shall use the following cardinal property.

Lemma 4.1 [21, pp. 153, 154], [16, Lemma 3.1]. *For every cardinal* κ *there is a cardinal* $\lambda \geq \kappa$ *such that* $\lambda^{\aleph_0} = 2^\lambda$.

Theorem 4.2. *Let* G *be a group. Then:*

(1) $\mathrm{Ext}(G, -)$ *inverts sums from* $\{\mathbb{Z}(p)\}$ *if and only if* $G_p = 0$;
(2) $\mathrm{Ext}(G, -)$ *inverts sums from* $\mathcal{A}[p^\infty]$ *if and only if* $G \otimes \mathbb{Z}_p$ *is free over* $\mathbb{Z}_p$.

Proof. (1) Suppose that $G_p \neq 0$, so that $\mathrm{Ext}(G, \mathbb{Z}(p)) \neq 0$. Denote by ν the cardinality of $\mathrm{Ext}(G, \mathbb{Z}(p))$, and, using Lemma 4.1, fix an infinite cardinal λ such that $\nu \leq \lambda$ and $\lambda^{\aleph_0} = 2^\lambda$.

Since $\mathrm{Ext}(G, -)$ inverts sums of p-groups, $\mathrm{Ext}(G, \mathbb{Z}(p)^{(2^\lambda)}) \cong \mathrm{Ext}(G, \mathbb{Z}(p))^{2^\lambda}$ has cardinality ν^{2^λ}. On the other side, $\mathbb{Z}(p)^{(2^\lambda)} \cong \mathbb{Z}(p)^\lambda$, hence $\mathrm{Ext}(G, \mathbb{Z}(p)^{(2^\lambda)}) \cong \mathrm{Ext}(G, \mathbb{Z}(p))^\lambda$, and its cardinality is ν^λ. However, $\nu^\lambda \leq \lambda^{\aleph_0 \lambda} = (2^\lambda)^\lambda = 2^\lambda < 2^{2^\lambda} \leq \nu^{2^\lambda}$. Therefore, $G_p = 0$.

The converse is clear.

(2) If $G \otimes \mathbb{Z}_p$ is free over $\mathbb{Z}_p$, it follows using Lemma 3.3 that $\mathrm{Ext}(G, H) = 0$ for all p-groups H, so that $\mathrm{Ext}(G, -)$ inverts sums from $\mathcal{A}[p^\infty]$.

Conversely, suppose $\mathrm{Ext}(G, -)$ inverts sums of p-groups. We have proved that $G_p = 0$. By Lemma 3.3, we observe that $\mathrm{Ext}(G \otimes \mathbb{Z}_p, H) \cong \mathrm{Ext}(G, H)$ for all

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p-groups H. Therefore $\mathrm{Ext}(G \otimes \mathbb{Z}_p, -)$ inverts sums from $\mathcal{A}[p^\infty]$. It follows that for every family $\mathcal{E}$ of cyclic p-groups,

$$\mathrm{Ext}\left(G \otimes \mathbb{Z}_p, \bigoplus_{C \in \mathcal{E}} C\right) \cong \prod_{C \in \mathcal{E}} \mathrm{Ext}(G \otimes \mathbb{Z}_p, C) = 0.$$

This property is also valid in the category of $\mathbb{Z}_p$-modules, hence $G \otimes \mathbb{Z}_p$ is a Baer module. Therefore it is free. □

From Theorem 4.2, the following useful result can be deduced.

Lemma 4.3. *Let G and H be groups such that $\mathrm{Ext}(G, -)$ inverts sums from $\{H\}$. If p is a prime such that $pH \neq H$ then $G_p = 0$.*

Proof. First observe that if H is an elementary p-group, then in the proof of Theorem 4.2(1) λ can be chosen such that $H^{(2^\lambda)} \cong H^\lambda$, and it follows that $\mathrm{Ext}(G, H) = 0$, so that $G_p = 0$.

Now suppose that H is any group satisfying $pH \neq H$. Let κ be a cardinal. Starting with an exact sequence $H^{(\kappa)} \xrightarrow{p} H^{(\kappa)} \twoheadrightarrow (H/pH)^{(\kappa)}$, where the first arrow is multiplication by p, apply the functor $\mathrm{Ext}(G, -)$ to obtain the exact sequence

$$\mathrm{Ext}(G, H)^\kappa \cong \mathrm{Ext}(G, H^{(\kappa)}) \xrightarrow{p} \mathrm{Ext}(G, H^{(\kappa)}) \cong \mathrm{Ext}(G, H)^\kappa \twoheadrightarrow \mathrm{Ext}(G, (H/pH)^{(\kappa)}).$$

Using [9, Lemma 52.1], observe that the first arrow is again multiplication by p. Therefore $\mathrm{Ext}(G, (H/pH)^{(\kappa)}) \cong \mathrm{Ext}(G, (H/pH))^\kappa$, so G inverts sums from $\{H/pH\}$. By Theorem 4.2(1), it follows that the p-component of G is trivial. □

Corollary 4.4. $\mathrm{Ext}(G, -)$ *inverts sums from $\{B\}$, where $B \neq 0$ is a bounded p-group, if and only if $G_p = 0$.*

Theorem 4.5. *The following are equivalent statements for a group G.*

(1) *The functor* $\mathrm{Ext}(G, -)$ *inverts sums from $\mathcal{A}$.*
(2) $\mathrm{Ext}(G, -)$ *inverts sums from $\mathcal{TF}$.*
(3) *G is free.*

Proof. It is clear that (1) implies (2) and (3) implies (1).

We prove that (2) implies (3). Using Lemma 4.3, we see that G is torsion-free. Let L be a direct sum of finite cyclic groups. Then L is an epimorphic image of a direct sum H of p-adic integers where p runs over some set of primes. Hence $\mathrm{Ext}(G, L)$ is an epimorphic image of $\mathrm{Ext}(G, H)$. But since $\mathrm{Ext}(G, -)$ inverts sums of torsion-free groups, $\mathrm{Ext}(G, H)$ is a product of terms $\mathrm{Ext}(G, J_p)$, each of which is zero since G is torsion-free. Thus $\mathrm{Ext}(G, L) = 0$ for all direct sums L of finite cyclic groups, so by [17, Proposition 2.5], G is free. □

5. $\mathcal{C}$-Ext-small groups

We can now characterize the $\mathcal{C}$-Ext-small groups for various classes $\mathcal{C}$.

PROPOSITION 5.1.

(1) *Every finitely generated group is $\mathcal{A}$-Ext-small.*
(2) *If B is a bounded p-group then $\mathbb{Z}(p^\infty)$ is $\{B\}$-Ext-small, but it is not $\mathcal{A}[p^\infty]$-Ext-small.*

PROOF. (1) This is a consequence of [14, Lemma 3.1.6].

(2) Let κ be a cardinal and B a bounded p-group. From the short exact sequence $\mathbb{Z}_p \rightarrowtail \mathbb{Q} \twoheadrightarrow \mathbb{Z}(p^\infty)$ we obtain the exact sequence

$$\mathrm{Hom}(\mathbb{Q}, B^{(\kappa)}) \to \mathrm{Hom}(\mathbb{Z}_p, B^{(\kappa)}) \to \mathrm{Ext}(\mathbb{Z}(p^\infty), B^{(\kappa)}) \to \mathrm{Ext}(\mathbb{Q}, B^{(\kappa)}),$$

where the first and the last groups are zero so that $\mathrm{Ext}(\mathbb{Z}(p^\infty), B^{(\kappa)}) \cong B^{(\kappa)}$. Therefore $\mathbb{Z}(p^\infty)$ is $\{B\}$-Ext-small.

On the other hand, $\mathrm{Ext}(\mathbb{Z}(p^\infty), \bigoplus_{n>0} \mathbb{Z}(p^n))$ is not a torsion group since we have an exact sequence

$$\mathrm{Ext}\Big(\mathbb{Z}(p^\infty), \bigoplus_{n>0} \mathbb{Z}(p^n)\Big) \to \mathrm{Ext}\Big(\mathbb{Q}, \bigoplus_{n>0} \mathbb{Z}(p^n)\Big) \to \mathrm{Ext}\Big(\mathbb{Z}_p, \bigoplus_{n>0} \mathbb{Z}(p^n)\Big) = 0,$$

and the nonzero group $\mathrm{Ext}(\mathbb{Q}, \bigoplus_{n>0} \mathbb{Z}(p^n))$ is torsion-free divisible. □

We now characterize the $\mathcal{C}$-Ext-small groups for certain classes of p-groups.

LEMMA 5.2. *Let G be an infinite bounded p-group. For every group B which is not p-divisible and for every cardinal μ, there is a cardinal $\lambda > \mu$ such that* $|\mathrm{Ext}(G, B^{(\lambda)})| > |\mathrm{Ext}(G, B)^{(\lambda)}|$.

PROOF. Given the group B and the cardinal μ, we claim first that the property is valid for $G = (\mathbb{Z}(p^n))^{(\nu)}$ for any positive integer n and any infinite cardinal ν. Moreover, the inequality holds for all cardinals $\lambda \geq \max\{|(B/p^nB)|^\nu, \mu\}$ such that $\lambda^{\aleph_0} = 2^\lambda$. In that case,

$$\mathrm{Ext}(G, B) \cong \mathrm{Ext}(\mathbb{Z}(p^n), B)^\nu \cong (B/p^nB)^\nu \neq 0.$$

Let $|(B/p^nB)^\nu| = \kappa$. Then for any cardinal λ, $|\mathrm{Ext}(G, B^{(\lambda)})| = \kappa\lambda^\nu$, whereas $|\mathrm{Ext}(G, B)^{(\lambda)}| = \lambda\kappa$.

If $\lambda \geq \max\{\kappa, \mu\}$ such that $\lambda^{\aleph_0} = 2^\lambda$ then $\lambda\kappa = \lambda < 2^\lambda = \lambda^{\aleph_0} \leq \lambda^\nu$, and the proof of our claim is complete. Moreover, such a cardinal λ exists by Lemma 4.1.

Now let G be an arbitrary bounded p-group, so G is a finite direct sum of its homogeneous components. Since G is infinite we can write $G = F \oplus G_1 \oplus \cdots \oplus G_m$, where F is a finite group and $G_1, \ldots, G_m$ are infinite homogeneous bounded p-groups. Let n_i be the exponent for the group G_i and let λ_i be a cardinal satisfying our claim for the group G_i.

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By Lemma 4.1 again, there is a cardinal $\lambda \geq \max\{\lambda_1, \ldots, \lambda_m, \mu\}$ such that $\lambda^{\aleph_0} = 2^\lambda$. Then by our claim

$$|\mathrm{Ext}(G_i, B^{(\lambda)})| > |\mathrm{Ext}(G_i, B)^{(\lambda)}| \quad \text{for all } i \in \{1, \ldots, n\},$$

so that

$$|\mathrm{Ext}(G, B^{(\lambda)})| > |\mathrm{Ext}(G, B)^{(\lambda)}|. \qquad \square$$

Theorem 5.3. *Let $B \neq 0$ be a bounded p-group. A group G is $\{B\}$-Ext-small if and only if the p-component of G has finite rank.*

Proof. Let κ be an infinite cardinal and B a bounded p-group bounded by p^n. Using the proof of [9, 52(F)], there is an isomorphism

$$\mathrm{Ext}(G, B^{(\kappa)}) \cong \mathrm{Ext}(G[p^n], B^{(\kappa)}). \tag{$\sharp$}$$

Suppose that G is $\{B\}$-Ext-small. Then $G[p^n]$ is $\{B\}$-Ext-small, and it follows by Lemma 5.2 that it is finite. Therefore G_p has finite rank.

Conversely, if G_p has finite rank then G is $\{B\}$-Ext-small as a consequence of the isomorphism ($\sharp$) and Proposition 5.1. $\square$

Our final result is the following theorem.

Theorem 5.4. *The following statements are equivalent for a group G.*

(1) *G is $\mathcal{A}[p^\infty]$-Ext-small.*
(2) *G is C_p-Ext-small, where C_p is the class of cyclic p-groups.*
(3) *$G = B \oplus K$, where B is a finite p-group and K is a group with trivial p-component such that $K \otimes \mathbb{Z}_p$ is free over $\mathbb{Z}_p$.*

Proof. That (1) implies (2) and (3) implies (1) is obvious.

We prove that (2) implies (3). Let G be C_p-Ext-small. It follows from Theorem 5.3 that the p-component of G is of finite rank, so it is a direct sum of a finite p-group and finitely many copies of $\mathbb{Z}(p^\infty)$.

Moreover,

$$\mathrm{Ext}\left(G, \bigoplus_{k>0} \mathbb{Z}(p^k)\right) \cong \bigoplus_{k>0} \mathrm{Ext}(G, \mathbb{Z}(p^k)) \cong \bigoplus_{k>0} G[p^k]$$

is a torsion group.

But we have seen in the proof of Proposition 5.1 that $\mathrm{Ext}(\mathbb{Z}(p^\infty), \bigoplus_{k>0} \mathbb{Z}(p^k))$ is not a torsion group. It follows that the p-component of G is finite. Hence $G = B \oplus K$ where B is a finite p-group and the p-component of K is 0. Then for every family $\mathcal{E}$ of cyclic p-groups $C = \mathbb{Z}(p^{n_C})$,

$$\mathrm{Ext}\left(G, \bigoplus_{C \in \mathcal{E}} C\right) \cong \bigoplus_{C \in \mathcal{E}} \mathrm{Ext}(G, C) \cong \bigoplus_{C \in \mathcal{E}} B[p^{n_C}].$$

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Hence if p^n is a positive integer such that $p^n B = 0$, then $p^n\,\mathrm{Ext}(K, -)$ annihilates all direct sums of cyclic p-groups. Let L be a direct sum of cyclic p-groups. The bounded group $\mathrm{Ext}(K, L)$ is p-divisible, since the p-component of K is zero. It follows that $\mathrm{Ext}(K, L) = 0$ for all direct sums of cyclic p-groups L.

If H is a p-group with basic subgroup L, then there is an epimorphism $\mathrm{Ext}(K, L) \twoheadrightarrow \mathrm{Ext}(K, H)$ and it follows that $\mathrm{Ext}(K, H) = 0$ for all p-groups H. Then $K \otimes \mathbb{Z}_p$ is free over $\mathbb{Z}_p$ as a consequence of Lemma 3.4. □

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