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L^1-CONVERGENCE OF FOURIER SERIES

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Abstract

For an integrable function f on T, we introduce a modified partial sum $S_n^\Delta(f, t)$ and establish its L^1-convergence property. The relation between the sum and L^1-convergence classes is also established. As a corollary, a new L^1-convergence class is obtained. It is shown that this class covers all known L^1-convergence classes.

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1. Introduction

For an integrable function f defined on the circle group $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $S_n(f, t)$ and $\sigma_n(f, t)$ denote the nth partial sum and the nth Cesàro sum of its Fourier series $\sum_{|n|<\infty} \hat{f}(n)e^{int}$, respectively. Define $\Delta\hat{f}(n)$ as follows: for $n > 0$, $\Delta\hat{f}(n) = \hat{f}(n) - \hat{f}(n+1)$ and $\Delta\hat{f}(-n) = \hat{f}(-n) - \hat{f}(-n-1)$. It is well known that there exists an integrable function on $\mathbb{T}$ whose Fourier series does not converge to itself in L^1-norm. Hence, many authors have defined L^1-convergence classes in terms of conditions on sequences of Fourier coefficients. An *L^1-convergence class* is a class of Fourier coefficients $\{\hat{f}(n)\}$ for which

$$\text{(Y)} \qquad \|S_n(f) - f\|_1 = o(1) \qquad (n \to \infty)$$

$$\text{if and only if } \hat{f}(n)\log|n| = o(1) \qquad (|n| \to \infty).$$

This development was influenced by [14]. In that paper, A. N. Kolmogorov proved that even quasi-convex sequences form an L^1-convergence class. After

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that, many L^1-convergence classes have been found (see [1, 2, 3, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17, 18], and these are subclasses of the following three classes:

$$\text{(i)} \qquad \lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta \hat{f}(k)|^p = 0 \quad \text{for } 1 < p \leqslant 2;$$

$$\text{(ii)} \qquad \mathbf{C} \cap \mathbf{BV};$$

$$\text{(iii)} \qquad \mathbf{QM}.$$

Condition (i) is a Tauberian condition of Hardy-Karamata kind (see [12]). Notations in (ii) and (iii) are defined as follows.

Definition. We say that an even sequence $\{\hat{f}(n)\}$ belongs to the class $\mathbf{C}$ if for every $\varepsilon > 0$, there exists $\delta > 0$, independent of n, such that

$$C_n(\delta) = \frac{1}{\pi} \int_0^{\delta} \left| \sum_{k=n}^{\infty} \Delta \hat{f}(k) D_k(t) \right| dt < \varepsilon \quad \text{for all } n,$$

where D_k is the Dirichlet kernel.

Definition. We say that $\{\hat{f}(n)\}$ belongs to $\mathbf{BV}$, the *class of bounded variation*, if $\sum_{|n|=1}^{\infty} |\Delta \hat{f}(n)| < \infty$.

Definition. We say that an even sequence $\{\hat{f}(n)\}$ belongs to $\mathbf{QM}$, the *class of quasi-monotone sequences*, if, for some $\alpha \geqslant 0$, $\hat{f}(n)/n^{\alpha}$ is monotone decreasing as n varies from 1 to ∞.

In this paper, we shall introduce in Section 2 a new modified partial sum $S_n^{\Delta}(f, t)$ and establish a result about its L^1-convergence property which says that the L^1-convergence problem of $S_n^{\Delta}(f, t)$ is closely related to the behavior of a new series $\sum_{n \neq 0} \Delta \hat{f}(n) E_n^*(t)$. The series has the form $\sum_{n=1}^{\infty} \Delta \hat{f}(n) D_n(t)$ in the case that f is even, or equivalently, the Fourier series of f is a cosine series. In Section 3, it will be shown that $S_n^{\Delta}(f, t)$ controls the truth of the statement (Y). As a corollary, we shall establish a new L^1-convergence class which covers all known L^1-convergence classes mentioned before.

2. A modified partial sum $S_n^{\Delta}(f, t)$

For $n > 0$, define $E_n^*(t)$ as follows:

$$E_n^*(t) = \frac{e^{i(n+1)t}}{2ie^{it/2} \sin t/2} = \sum_{k=0}^{n} e^{ikt} + \frac{1}{2ie^{it/2} \sin t/2},$$

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and $E^*_{-n}(t) = E^*_n(-t)$. Clearly, for any positive integer n, we have the following identities:

(i)
$$E^*_n(t) - E^*_{n-1}(t) = e^{int},$$

(ii)
$$E^*_n(t) + E^*_{-n}(t) = D_n(t) = \frac{\sin\left(n+\frac{1}{2}\right)t}{\sin\frac{1}{2}t}.$$

(iii)
$$E^*_n(t) - E^*_{-n}(t) = \frac{\cos\left(n+\frac{1}{2}\right)t}{i\sin\frac{1}{2}t}.$$

We introduce a new modified partial sum $S^\Delta_n(f,t)$ as follows.

$$S^\Delta_n(f) = S^\Delta_n(f,t) = S_n(f,t) - \left(\hat{f}(n)E^*_n(t) + \hat{f}(-n)E^*_{-n}(t)\right).$$

In order to establish the L^1-convergence property of $S^\Delta_n(f,t)$, we need the following two lemmas.

Lemma 2.1. *Let $\{\lambda_n\}$ be a sequence of integers with $\lambda_n > n$ for all n. Then for any $0 < |t| \leqslant \pi$, we have*

$$S^\Delta_n(f,t) - \sigma_n(f,t) = \frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n}(f,t) - \sigma_n(f,t)\right)$$
$$- \sum_{|k|=n}^{\lambda_n-1} \frac{\lambda_n-|k|}{\lambda_n-n}\Delta\hat{f}(k)E^*_k(t) + R_n(t),$$

where

$$\begin{aligned}R_n(t) = &-\frac{1}{\lambda_n-n}\sum_{|k|=n}^{\lambda_n-1}\Delta\hat{f}(k)E^*_k(t)\\ &-\frac{1}{\lambda_n-n}\sum_{k=n}^{\lambda_n-1}\left(\hat{f}(k+1)E^*_k(t) + \hat{f}(-k-1)E^*_{-k}(t)\right)\\ &+\frac{1}{\lambda_n-n}\left(\hat{f}(n)E^*_n(t) + \hat{f}(-n)E^*_{-n}(t)\right)\\ &-\frac{1}{\lambda_n-n}\left(\hat{f}(\lambda_n)E^*_{\lambda_n}(t) + \hat{f}(-\lambda_n)E^*_{-\lambda_n}(t)\right).\end{aligned}$$

PROOF. It is well known that

$$S_n(f,t)-\sigma_n(f,t)=\frac{\lambda_n+1}{\lambda_n-n}\left(\sigma_{\lambda_n}(f,t)-\sigma_n(f,t)\right)$$
$$-\sum_{|k|=n+1}^{\lambda_n}\frac{\lambda_n+1-|k|}{\lambda_n-n}\hat{f}(k)e^{ikt}.$$

By using summation by parts, we get

$$\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n+1-k}{\lambda_n-n}\hat{f}(k)e^{ikt}=\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n+1-k}{\lambda_n-n}\hat{f}(k)\left(E_k^*(t)-E_{k-1}^*(t)\right)$$
$$=\sum_{k=n}^{\lambda_n-1}\frac{\lambda_n+1-k}{\lambda_n-n}\Delta\hat{f}(k)E_k^*(t)-\hat{f}(n)E_n^*(t)$$
$$+\frac{1}{\lambda_n-n}\left(\sum_{k=n}^{\lambda_n-1}\hat{f}(k+1)E_k^*(t)-\hat{f}(n)E_n^*(t)+\hat{f}(\lambda_n)E_{\lambda_n}^*(t)\right).$$

Similarly, we have

$$\sum_{k=n+1}^{\lambda_n}\frac{\lambda_n+1-k}{\lambda_n-n}\hat{f}(-k)e^{-ikt}$$
$$=\sum_{k=n}^{\lambda_n-1}\frac{\lambda_n+1-k}{\lambda_n-n}\Delta\hat{f}(-k)E_{-k}^*(t)-\hat{f}(-n)E_{-n}^*(t)$$
$$+\frac{1}{\lambda_n-n}\left(\sum_{k=n}^{\lambda_n-1}\hat{f}(-k-1)E_{-k}^*(t)-\hat{f}(-n)E_{-n}^*(t)+\hat{f}(-\lambda_n)E_{-\lambda_n}^*(t)\right).$$

From these identities we get the desired result.

LEMMA 2.2. *Let* $\{\lambda_n\}$ *and* $R_n(t)$ *be as in Lemma* 2.1. *If a sequence* $\{\rho_n\}$ *satisfies* $\rho_n=0(\lambda_n-n)$ *and* $\rho_n\geqslant 1$ *for all* n, *then*

$$\lim_{n\to\infty}\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}|R_n(t)|\,dt=0.$$

PROOF. From the fact that

$$\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}|E_k^*(t)|\,dt\leqslant\pi\log\rho_n\quad\text{for all integers } k,$$

we find that

$$\lim_{n\to\infty}\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}\left|\frac{1}{\lambda_n-n}\left(\hat{f}(n)E_n^*(t)+\hat{f}(-n)E_{-n}^*(t)\right)\right|dt=0$$

and

$$\lim_{n\to\infty}\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}\left|\frac{1}{\lambda_n-n}\left(\hat f(\lambda_n)E^*_{\lambda_n}(t)+\hat f(-\lambda_n)E^*_{-\lambda_n}(t)\right)\right|dt=0.$$

Therefore, it suffices to show that if $\{c_n\}$ is a sequence tending to 0 as $|n|\to\infty$, then

$$\lim_{n\to\infty}\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}\left|\frac{1}{\lambda_n-n}\sum_{|k|=n}^{\lambda_n-1}c_kE^*_k(t)\right|dt=0.$$

If we apply the Hölder inequality and the Parseval formula, then we get

$$\begin{aligned}\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}\left|\sum_{k=n}^{\lambda_n-1}c_kE^*_k(t)\right|dt&=\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}\left|\frac{1}{2\sin t/2}\sum_{k=n}^{\lambda_n-1}c_ke^{i(k+1)t}\right|dt\\&\leqslant\left\{\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}\frac{dt}{|2\sin t/2|^2}\right\}^{1/2}\cdot\left\{\int_0^{2\pi}\left|\sum_{k=n}^{\lambda_n-1}c_ke^{i(k+1)t}\right|^2dt\right\}^{1/2}\\&\leqslant\pi\rho_n^{1/2}\left(\sum_{k=n}^{\lambda_n-1}|c_k|^2\right)^{1/2}.\end{aligned}$$

Similarly, we have

$$\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}\left|\sum_{k=n}^{\lambda_n-1}c_{-k}E^*_{-k}(t)\right|dt\leqslant\pi\rho_n^{1/2}\left(\sum_{k=n}^{\lambda_n-1}|c_{-k}|^2\right)^{1/2}.$$

From the last two results, we get

$$\begin{aligned}&\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}\left|\frac{1}{\lambda_n-n}\sum_{|k|=n}^{\lambda_n-1}c_kE^*_k(t)\right|dt\\&\quad\leqslant\pi\left(\frac{\rho_n}{\lambda_n-n}\right)^{1/2}\cdot\left\{\left(\frac{1}{\lambda_n-n}\sum_{k=n}^{\lambda_n-1}|c_k|^2\right)^{1/2}+\left(\frac{1}{\lambda_n-n}\sum_{k=n}^{\lambda_n-1}|c_{-k}|^2\right)^{1/2}\right\}\\&\quad\leqslant\pi\left(\frac{\rho_n}{\lambda_n-n}\right)^{1/2}\cdot\left\{\sup_{k\geqslant n}|c_k|+\sup_{k\geqslant n}|c_{-k}|\right\}.\end{aligned}$$

It follows that

$$\lim_{n\to\infty}\int_{\pi\geqslant|t|\geqslant\pi/\rho_n}\left|\frac{1}{\lambda_n-n}\sum_{|k|=n}^{\lambda_n-1}c_kE^*_k(t)\right|dt=0.$$

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It is well-known that for any integrable function f on $\mathbb{T}$, $\sigma_n(f, t)$ converges to f in L^1-norm. If we apply Lemmas 2.1 and 2.2 to the case $\lambda_n = [\lambda n]$ and $\rho_n = n$, then we get the following L^1-convergence property of $S_n^{\Delta}(f, t)$.

THEOREM 2.1. *Let $f \in L^1(\mathbb{T})$. Then the following are equivalent*

(i) $$\lim_{n\to\infty} \int_{\pi \geqslant |t| \geqslant \pi/n} |S_n^{\Delta}(f, t) - f(t)|\, dt = 0,$$

(ii) $$\lim_{\lambda \downarrow 1} \varlimsup_{n\to\infty} \int_{\pi \geqslant |t| \geqslant \pi/n} \left| \sum_{|k|=n}^{[\lambda n]-1} \frac{[\lambda n] - |k|}{[\lambda n] - n} \Delta \hat{f}(k) E_k^*(t) \right| dt = 0.$$

REMARK. The sum inside the second integral of the above theorem can be written as

$$\sum_{|k|=n}^{[\lambda n]-1} \frac{[\lambda n] - |k|}{[\lambda n] - n} \Delta \hat{f}(k) E_k^* = \frac{1}{[\lambda n] - n} \sum_{m=n}^{[\lambda n]-1} \left(\sum_{|k|=n}^{m} \Delta \hat{f}(k) E_k^* \right),$$

which is the average of the sums

$$\sum_{|k|=n}^{m} \Delta \hat{f}(k) E_k^* \qquad (n \leqslant m < [\lambda n]).$$

Therefore, the above theorem tells us that the L^1-convergence problem of $S_n^{\Delta}(f, t)$ is closely related to the behavior of the new series $\sum_{n\neq 0} \Delta \hat{f}(n) E_n^*(t)$, which is $\sum_{n=1}^{\infty} \Delta \hat{f}(n) D_n(t)$ for the case that f is even, or equivalently, the Fourier series of f is a cosine series. In Section 3, we shall describe the relation between $S_n^{\Delta}(f, t)$ and the statement (Y).

In the rest of this section, we shall discuss the relation between $S_n^{\Delta}(f, t)$ and the modified partial sum g_n defined in [7, 8]. Therefore, throughout this part, we shall assume that f is even, or equivalently, that $\sum_{|n|<\infty} \hat{f}(n) e^{int}$ is a cosine series. We have

$$S_n^{\Delta}(f, t) = S_n(f, t) - \hat{f}(n) D_n(t).$$

From the following estimate

$$\begin{aligned} \int_{|t| \leqslant \pi/n} |S_n^{\Delta}(f, t)|\, dt &= \int_{|t| \leqslant \pi/n} |S_n(f, t) - \hat{f}(n) D_n(t)|\, dt \\ &\leqslant \frac{2\pi}{n} \sum_{|k| \leqslant n} |\hat{f}(k)| + \frac{(4n+2)\pi}{n} |\hat{f}(n)|, \end{aligned}$$

we get

$$\lim_{n\to\infty} \int_{|t| \leqslant \pi/n} |S_n^{\Delta}(f, t) - f(t)|\, dt = 0.$$

Therefore, in this case, Theorem 2.1 can be transformed into the following form.

COROLLARY 2.1. *Let f be an even integrable function on $\mathbb{T}$. Then $S_n^\Delta(f)$ converges to f in L^1-norm if and only if*

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \int_{\pi \geqslant |t| \geqslant \pi/n} \left| \sum_{k=n}^{[\lambda n]-1} \frac{[\lambda n]-k}{[\lambda n]-n} \Delta \hat{f}(k) D_k(t) \right| dt = 0.$$

We have the following relation

$$S_n^\Delta(f,t) = g_n(t) - \Delta\hat{f}(n) D_n(t).$$

For any $\delta > 0$ and $m \geqslant n$, we have

$$\int_\delta^\pi \left| \sum_{k=n}^{m} \Delta\hat{f}(k) D_k(t) \right| dt \leqslant \sum_{k=n}^{m} |\Delta\hat{f}(k)| \cdot \frac{\pi}{\sin \delta/2},$$

and

$$\int_0^\delta \left| \sum_{k=n}^{m} \Delta\hat{f}(k) D_k(t) \right| dt$$
$$\leqslant \int_0^\delta \left| \sum_{k=n}^{\infty} \Delta\hat{f}(k) D_k(t) \right| dt + \int_0^\delta \left| \sum_{k=m+1}^{\infty} \Delta\hat{f}(k) D_k(t) \right| dt.$$

It follows that if $\{\hat{f}(n)\}$ belongs to $\mathbf{C}$, then $\|\Delta\hat{f}(n)D_n(t)\|_1 = o(1)$ $(n \to \infty)$, and so

$$\left\| S_n^\Delta(f) - g_n \right\|_1 = o(1) \qquad (n \to \infty).$$

If in addition, $\{\hat{f}(n)\}$ belongs to $\mathbf{BV}$, then for any $\lambda > 1$, we have

$$\lim_{n\to\infty} \left\| \sum_{k=n}^{[\lambda n]-1} \frac{[\lambda n]-k}{[\lambda n]-n} \Delta\hat{f}(k) D_k \right\|_1 = 0.$$

This follows from the observation that the last sum is the average of the sums

$$\sum_{k=n}^{m} \Delta\hat{f}(k) D_k(t) \qquad (n \leqslant m < [\lambda n]).$$

By Corollary 2.1, g_n converges to f in L^1-norm. Therefore, $S_n^\Delta(f,t)$ can be regarded as a generalization of $g_n(t)$.

3. L^1-convergence classes

In this section, we first establish the relation between $S_n^\Delta(f,t)$ and the statement (Y). As a corollary, we establish a new L^1-convergence class, and then show that this class covers all known L^1-convergence classes mentioned before. To establish these, we need the following lemma.

LEMMA 3.1. *Suppose that* $\rho_n = O(n^\alpha)$ *for some* $\alpha > 0$ *and that* $\rho_n \geqslant 1$ *for all* n. *Then*

$$\int_{\pi \geqslant |t| \geqslant \pi/\rho_n} |\hat{f}(n)E_n^*(t) + \hat{f}(-n)E_{-n}^*(t)|\, dt = o(1) \qquad (n \to \infty)$$

if and only if $\hat{f}(n) \log \rho_{|n|} = o(1)$ $(|n| \to \infty)$.

PROOF. The "if" part follows from the inequality

$$\int_{\pi \geqslant |t| \geqslant \pi/\rho_n} |E_k^*(t)|\, dt \leqslant \pi \log \rho_n \qquad \text{for all integers } k.$$

To establish the "only if" part, it suffices to show that for some $M > 0$, dependent on α,

$$\int_{\pi/\rho}^{\pi} |aE_n^*(t) + bE_{-n}^*(t)|\, dt \geqslant M(|a| + |b|) \log \rho$$

for all complex numbers a, b, all $n \geqslant 3$, and all ρ of the form $3 \leqslant \rho \leqslant n^\alpha$. The last inequality can be proved as follows:

$$\int_{\pi/\rho}^{\pi} |aE_n^*(t) + bE_{-n}^*(t)|\, dt \geqslant \int_{\pi/\rho}^{\pi} \frac{|ae^{i(n+1/2)t} - be^{-i(n+1/2)t}|}{t}\, dt$$
$$= \int_{(n+1/2)\pi/\rho}^{(n+1/2)\pi} \frac{|ae^{it} - be^{-it}|}{t}\, dt.$$

If we can get a favorable estimate for the last integral, we shall be done. The last integral can be estimated by using the following inequality:

$$|ae^{it} - be^{-it}| \geqslant \frac{|a| + |b|}{2}$$

for all complex numbers a and b, and any real number t of the form

$$2t = -\arg(a\bar{b}) + \tau + 2k\pi,$$

where k is an integer and $\pi/3 \leqslant |\tau| \leqslant \pi$. To see this, first divide the interval $[(n + \frac{1}{2})\pi/\rho, (n + \frac{1}{2})\pi]$ into intervals of the form

$$\left[-\tfrac{1}{2}\arg(a\bar{b}) - \pi/2, -\tfrac{1}{2}\arg(a\bar{b}) + \pi/2\right] + k\pi,$$

with something left over. Second, apply the mentioned inequality to the last integral and then do a calculation similar to that used in the estimate of Lebesgue's constants (cf. [4, Vol. 1, p. 80] or [19, p. 172]). Note that the condition $\log \rho \leqslant \alpha(\log n)$ will be used at the last step. After that, the desired result will follow.

It is clear from the Riemann-Lebesgue lemma and from the integrability of f that

$$\lim_{n \to \infty} \int_{|t| \leqslant \pi/n} |S_n(f, t) - f(t)|\, dt = 0.$$

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If we apply Lemma 3.1 to the case $\rho_n = n$, then we get the following theorem, which says that $S_n^{\Delta}(f, t)$ controls the truth of the statement (Y).

THEOREM 3.1. *Let $f \in L^1(\mathbb{T})$. If*

$$\lim_{n\to\infty} \int_{\pi \geqslant |t| \geqslant \pi/n} |S_n^{\Delta}(f,t) - f(t)|\, dt = 0,$$

then $\|S_n(f) - f\|_1 = o(1)$ $(n \to \infty)$ if and only if $\hat{f}(n)\log|n| = o(1)$ $(|n| \to \infty)$.

As a corollary of Theorems 2.1 and 3.1, we get the following L^1-convergence class.

THEOREM 3.2. *Let $f \in L^1(\mathbb{T})$. If*

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \int_{\pi \geqslant |t| \geqslant \pi/n} \left| \sum_{|k|=n}^{[\lambda n]-1} \frac{[\lambda n] - |k|}{[\lambda n] - n} \Delta \hat{f}(k) E_k^*(t) \right| dt = 0,$$

then $\|S_n(f) - f\|_1 = o(1)$ $(n \to \infty)$ if and only if $\hat{f}(n)\log|n| = o(1)$ $(|n| \to \infty)$.

REMARK. If both conditions in the statement (Y) are true, then it is easy to see from Lemma 3.1 that the condition (i) in Theorem 2.1 holds, and so the condition (ii) in Theorem 2.1 holds, i.e., the hypothesis of Theorem 3.2 holds. This shows that the above L^1-convergence class is best possible.

In the rest of this section, we shall show that the above L^1-convergence class covers all known L^1-convergence classes mentioned before. The first case we want to investigate is the following Tauberian condition of Hardy-Karamata kind (I was informed that Professor W. O. Bray and Professor Č. V. Stanojević also got this result by using the Hölder-Hausdorff-Young technique).

COROLLARY 3.1. *Let $f \in L^1(\mathbb{T})$. If for some $1 < p \leqslant 2$, we have*

$$\text{(HK)} \qquad \lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta \hat{f}(k)|^p = 0,$$

then (Y) holds.

PROOF. Let Σ denote the integral

$$\int_{\pi \geqslant |t| \geqslant \pi/n} \left| \sum_{|k|=n}^{[\lambda n]-1} \frac{[\lambda n] - |k|}{[\lambda n] - n} \Delta \hat{f}(k) E_k^*(t) \right| dt.$$

Then we have

$$\Sigma = \int_{\pi \geqslant |t| \geqslant \pi/n} \left| \frac{\phi(t)}{2 \sin t/2} \right| dt,$$

where

$$\phi(t) = \sum_{k=n}^{[\lambda n]-1} \frac{[\lambda n]-k}{[\lambda n]-n} \Delta\hat{f}(k) e^{i(k+1)t} - \sum_{k=n}^{[\lambda n]-1} \frac{[\lambda n]-k}{[\lambda n]-n} \Delta\hat{f}(-k) e^{-ikt}.$$

Applying first the Hölder inequality and then the Hausdorff-Young inequality, we get

$$\begin{aligned} \Sigma &\leqslant \left(\int_{\pi \geqslant |t| \geqslant \pi/n} \frac{dt}{|2\sin t/2|^p} \right)^{1/p} \left(\int_0^{2\pi} |\phi(t)|^q \, dt \right)^{1/q} \\ &\leqslant A_p n^{1/q} \left(\sum_{|k|=n}^{[\lambda n]-1} \left(\frac{[\lambda n]-|k|}{[\lambda n]-n} \right)^p |\Delta\hat{f}(k)|^p \right)^{1/p} \\ &\leqslant A_p \left(\sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta\hat{f}(k)|^p \right)^{1/p}, \end{aligned}$$

where $1/p + 1/q = 1$, and where A_p is a constant dependent on p only. From the last estimate, we find that this corollary follows from Theorem 3.2.

From the following set of implications

$$\begin{aligned} \sum_{|k|=1}^{\infty} |k|^{p-1} |\Delta\hat{f}(k)|^p < \infty \Rightarrow n \left(\frac{\sum_{|k|=n}^{\infty} |\Delta\hat{f}(k)|^p}{n} \right)^{1/p} &= o(1) \\ \Rightarrow \frac{1}{n} \sum_{|k|=n}^{2n} |k|^p |\Delta\hat{f}(k)|^p &= o(1) \\ \Rightarrow \lim_{\lambda \downarrow 1} \overline{\lim_{n\to\infty}} \sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta\hat{f}(k)|^p &= 0, \end{aligned}$$

we find that the following three classes are L^1-convergence classes:

(i) $\mathbf{C}_p^* \cap \mathbf{BV}$ (cf. [15]),

(ii) $\mathbf{C}_p \cap \mathbf{BV}$ (cf. [15]),

(iii) $\mathbf{V}_p \cap \mathbf{F}$ (cf. [1]).

Moreover, Corollary 3.1 generalizes those corresponding results in [1, 2, 3, 10, 15, 16]. It is easy to see that the condition $n\Delta\hat{f}(n) = O(1)$ implies

$$\sum_{|k|=n}^{[\lambda n]} |k|^{p-1} |\Delta\hat{f}(k)|^p \leqslant M \log \lambda$$

for some absolute constant M. Therefore, the above corollary has the following consequence.

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COROLLARY 3.2. *Let* $f \in L^1(\mathbb{T})$. *If* $n\Delta\hat{f}(n) = O(1)$ $(|n| \to \infty)$, *then* (Y) *holds.*

Recall the proof of Corollary 3.1. We used the Hölder-Hausdorff-Young technique to get an estimate of

$$\Sigma = \int_{\pi \geqslant |t| \geqslant \pi/n} \left| \sum_{|k|=n}^{[\lambda n]-1} \frac{[\lambda n] - |k|}{[\lambda n] - n} \Delta\hat{f}(k) E_k^*(t) \right| dt$$

and then got the desired result. In fact, we can directly get the following estimate of Σ without using the above famous inequalities

$$\Sigma \leqslant \left(\sum_{|k|=n}^{[\lambda n]-1} |\Delta\hat{f}(k)| \right) \int_{\pi \geqslant |t| \geqslant \pi/n} \frac{dt}{|2\sin t/2|}$$

$$\leqslant \pi(\log n) \sum_{|k|=n}^{[\lambda n]-1} |\Delta\hat{f}(k)|.$$

From this point of view, we get the following L^1-convergence class, which corresponds to the limit of condition (HK) as $p \to 1$.

COROLLARY 3.3. *Let* $f \in L^1(\mathbb{T})$. *If*

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} (\log n) \sum_{|k|=n}^{[\lambda n]} |\Delta\hat{f}(k)| = 0,$$

then (Y) *holds.*

The second case we want to investigate is the L^1-convergence class $\mathbf{C} \cap \mathbf{BV}$, which was obtained in [7, 8]. From the discussion at the end of Section 2, we find that the following is a consequence of Theorem 3.2.

COROLLARY 3.4. *Let* $f \in L^1(\mathbb{T})$. *If* $\{\hat{f}(n)\} \in \mathbf{C} \cap \mathbf{BV}$, *then* (Y) *holds.*

Let $\mathbf{S}$ and $\mathbf{F}_p$ be as in [15]. As shown in [15], the following relations hold

$$\{\text{even quasi-convex null-sequences}\} \subset \mathbf{S} \subset \mathbf{F}_p \subset \mathbf{C} \cap \mathbf{BV}.$$

From these relations, we find that all the above classes are L^1-convergence classes, and Theorem 3.2 generalizes those corresponding results in [6, 7, 8, 14, 17].

The third case we want to investigate is the class $\mathbf{QM}$. Suppose that $\{\hat{f}(n)\}$ is even and belongs to $\mathbf{QM}$. The quasi-monotonicity of $\{\hat{f}(n)\}$ yields

$$|\Delta\hat{f}(n)| \leqslant \Delta\hat{f}(n) + c\alpha\frac{\hat{f}(n)}{|n|},$$

where $c = \max(2, 2^\alpha)$. This implies that

$$\int_{\pi \geqslant |t| \geqslant \pi/n} \left| \sum_{|k|=n}^{[\lambda n]-1} \frac{[\lambda n] - |k|}{[\lambda n] - n} \Delta \hat{f}(k) E_k^*(t) \right| dt \leqslant (\pi \log n) \sum_{|k|=n}^{[\lambda n]-1} |\Delta \hat{f}(k)|$$

$$\leqslant (\pi \log n) \sum_{|k|=n}^{[\lambda n]-1} \left(\Delta \hat{f}(k) + c\alpha \frac{\hat{f}(k)}{|k|} \right)$$

$$\leqslant (\pi \log n) \Big\{ \hat{f}(n) + \hat{f}(-n) - \hat{f}([\lambda n]) - \hat{f}(-[\lambda n])$$

$$+ \Big(\sup_{n \leqslant |k| < [\lambda n]} \hat{f}(k) \Big) M \log \lambda \Big\}$$

for some absolute constant M. Therefore, if $\hat{f}(n) \log |n| = o(1)$ ($|n| \to \infty$), then

$$\lim_{\lambda \downarrow 1} \overline{\lim_{n \to \infty}} \int_{\pi \geqslant |t| \geqslant \pi/n} \left| \sum_{|k|=n}^{[\lambda n]-1} \frac{[\lambda n] - |k|}{[\lambda n] - n} \Delta \hat{f}(k) E_k^*(t) \right| dt = 0.$$

By Theorem 3.2, we get the following corollary.

COROLLARY 3.5. *Let $f \in L^1(\mathbb{T})$. If $\{\hat{f}(n)\} \in$* **QM** *and if $\hat{f}(n) \log |n| = o(1)$ as $|n|$ tends to ∞, then $\|S_n(f) - f\|_1 = o(1)$ $(n \to \infty)$.*

Under a suitable definition of **QM** for a Fourier sine series, such as that given in [9] and [18], we can obtain the same result for sine series as Corollary 3.5 provides for cosine series. We know that the concept of quasi-monotonicity is a generalization of the concept of monotonicity, so the above corollary generalizes the corresponding result in [5]. It is easy to see that if $\{\hat{f}(n)\}_{n=1}^\infty$ is decreasing and $\sum_{n=1}^\infty \hat{f}(n)/n < \infty$, then $\hat{f}(n) \log n = o(1)$ $(n \to \infty)$. This shows that the above corollary also generalizes the corresponding result in [11].

4. A modified approach

It was seen in Sections 2 and 3 that the L^1-convergence problem of the Fourier series of f is closely related to the L^1-convergence problem of the modified partial sum $S_n^\Delta(f, t)$, and, moreover, that the L^1-convergence problem of $S_n^\Delta(f, t)$ is completely determined by the following two factors.

(i) $$\frac{\lambda_n + 1}{\lambda_n - n} \left\| \sigma_{\lambda_n}(f) - \sigma_n(f) \right\|_1 \quad \text{and}$$

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(ii)
$$\int_{\pi \geqslant |t| \geqslant \pi/\rho_n} \left| \sum_{|k|=n}^{\lambda_n - 1} \frac{\lambda_n - |k|}{\lambda_n - n} \Delta \hat{f}(k) E_k^*(t) \right| dt.$$

Therefore, it becomes a crucial issue as to how to control the above two quantities. We have seen in Sections 2 and 3 how the choice $\lambda_n = [\lambda n]$ and $\rho_n = n$ works. In this section, we shall see how the new choice $\lambda_n = n + [n/l_n]$ and $\rho_n = [n/l_n]$ also works, where $\{l_n\}$ is defined below. Assume that for an integrable function f on $\mathbb{T}$, the following better estimate of $\|\sigma_{\lambda_n}(f) - \sigma_n(f)\|_1$ holds:

$$l_n \left\| \sigma_{n+[n/l_n]}(f) - \sigma_n(f) \right\|_1 = o(1) \qquad (n \to \infty)$$

for some sequence $\{l_n\}$ satisfying $1 \leqslant l_n \leqslant n$ for all n. From the first identity in the proof of Lemma 2.1 and the following estimate

$$\int_{|t| \leqslant \pi/\rho_n} \left| \sum_{|k|=n+1}^{\lambda_n} \frac{\lambda_n + 1 - |k|}{\lambda_n - n} \hat{f}(k) e^{ikt} \right| dt$$
$$\leqslant \frac{2\pi}{\rho_n} \sum_{|k|=n+1}^{\lambda_n} |\hat{f}(k)| \leqslant 2\pi \cdot \sup_{k>n} \left(|\hat{f}(k)| + |\hat{f}(-k)| \right),$$

we get

$$\int_{|t| \leqslant \pi/\rho_n} |S_n(f,t) - f(t)|\, dt = o(1) \qquad (n \to \infty).$$

If we apply the argument in Sections 2 and 3 to the choice $\lambda_n = n + [n/l_n]$ and $\rho_n = [n/l_n]$, then the results corresponding to Theorems 2.1, 3.1, 3.2 and their consequences will follow, especially the following results, which correspond to Corollaries 3.1 and 3.3.

RESULT 1. *Let $f \in L^1(\mathbb{T})$. Assume the existence of the sequence $\{l_n\}$ defined above. If for some $1 < p \leqslant 2$ $(1/p + 1/q = 1)$, we have*

$$l_n^{-1/q} \left(\sum_{|k|=n}^{n+[n/l_n]} |k|^{p-1} |\Delta \hat{f}(k)|^p \right)^{1/p} = o(1) \qquad (n \to \infty),$$

then $\|S_n(f) - f\|_1 = o(1)$ $(n \to \infty)$ if and only if $\hat{f}(n) \log(|n|/l_{|n|}) = o(1)$ $(|n| \to \infty)$.

RESULT 2. *Let $f \in L^1(\mathbb{T})$. Assume the existence of the sequence $\{l_n\}$ defined above. If*

$$(\log [n/l_n]) \sum_{|k|=n}^{n+[n/l_n]} |\Delta \hat{f}(k)| = o(1) \qquad (n \to \infty),$$

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then $\|S_n(f) - f\|_1 = o(1)$ $(n \to \infty)$ *if and only if* $\hat{f}(n) \log(|n|/l_{|n|}) = o(1)$ $(|n| \to \infty)$.

It is obvious that the above two results are better than Corollaries 3.1 and 3.3 for the case that the sequence $\{l_n\}$ exists. From the fact that

$$\|\sigma_n(f) - f\|_1 = O(n^{-\alpha}) \quad \text{for } f \in \mathrm{Lip}_\alpha(\mathbb{T}),\ 0 < \alpha < 1,$$

and

$$\|\sigma_n(f) - f\|_1 = O(\log n/n) \quad \text{for } f \in \mathrm{Lip}_1(\mathbb{T}),$$

we know that the sequence $\{l_n\}$ exists at least for functions satisfying a Lipschitz condition. This shows that the above two results make sense at least for Lipschitz classes. From the definition of $\{l_n\}$, we know that the existence problem of the sequence $\{l_n\}$ is completely dependent on the estimate of the quantity $\|\sigma_{\lambda_n}(f) - \sigma_n(f)\|_1$. Therefore, how to obtain a better estimate of the quantity $\|\sigma_{\lambda_n}(f) - \sigma_n(f)\|_1$ is a problem of special significance.

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