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ON THE SIGNATURE OF GENERALISED SEIFERT FIBRATIONS

PETER Y. PANG

In this note, we prove a signature product formula for generalised Seifert fibrations. We also discuss how this result can be viewed using the theory of minimal models.

1. INTRODUCTION

In this short note, we deduce a signature product formula for generalised Seifert fibrations based on results of Haefliger [6] and Chern-Hirzebruch-Serre [1]. Then we note that Haefliger's result has a natural interpretation from the point of view of the theory of minimal models, and the signature formula obtained can be viewed as a generalisation of Chern-Hirzebruch-Serre. Further, we demonstrate that the well-known result: that a compact, connected, orientable Riemannian manifold with a non-singular Killing vector-field has signature zero, can be interpreted in the same vein.

2. GENERALISED SEIFERT FIBRATIONS

DEFINITION 2.1: A foliation $\mathcal{F}$ is called a generalised Seifert fibration if (i) all its leaves are compact, and (ii) it is locally stable; equivalently, each leaf has a finite holonomy group, and $\mathcal{F}$ is Riemannian and taut [8]. See for example [2].

REMARK 2.2: It follows that the leaf space $M/\mathcal{F}$ of a generalised Seifert fibration on a manifold M is a Satake manifold, and its cohomology (as a Satake manifold) coincides with the cohomology of the complex of basic differential forms of the foliation, $H_B(\mathcal{F})$. Also, the leaves of $\mathcal{F}$ have a comon holonomy covering, known as the universal leaf, which is a compact, connected manifold.

THEOREM 2.3. *Let $\mathcal{F}$ be a generalised Seifert fibration on a connected, compact, simply-connected manifold M. Let $\mathcal{L}$ denote the universal leaf of $\mathcal{F}$. Then, there is a signature product formula as follows:*

$$\mathit{sign}\,(M) = \mathit{sign}\,(\mathcal{L})\,.\ \mathit{sign}\,(H_B(\mathcal{F})).$$

PROOF: By [6], there is a locally trivial fibration

$$\mathcal{L} \longrightarrow M \longrightarrow B\Gamma$$

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where $B\Gamma$ is the classifying space of the transverse holonomy groupoid of $\mathcal{F}$. Furthermore, there is a map

$$B\Gamma \longrightarrow M/\mathcal{F}$$

which induces an isomorphism in rational cohomology. Thus, we have

$$H^{\cdot}(B\Gamma) \cong H^{\cdot}(M/\mathcal{F}) \cong H_B{}^{\cdot}(\mathcal{F}).$$

By the duality theorem of Kamber-Tondeur [**8**], $H_B(\mathcal{F})$ satisfies Poincaré duality, hence by [**1**], we have

$$\begin{aligned} \text{sign}\,(M) &= \text{sign}\,(\mathcal{L})\,.\ \text{sign}\,(B\Gamma) \\ &= \text{sign}\,(\mathcal{L})\,.\ \text{sign}\,(H_B(\mathcal{F})). \end{aligned}$$

∎

Corollary 2.4. *Let M be a connected, compact, simply-connected manifold with non-zero signature. Let $\mathcal{F}$ be a Riemannian foliation on M. Then the codimension of $\mathcal{F}$ is divisible by 4.*

Proof: By a theorem of Ghys [**3**], $\mathcal{F}$ can be approximated by a generalised Seifert fibration of the same codimension. Thus the corollary follows from the above theorem. ∎

3. Minimal models

Definition 3.1: Let $\mathcal{F}$ be a foliation on a connected manifold M. Then, by the minimal model (Λ-minimal Λ-extension) of $\mathcal{F}$, we mean the minimal model (Λ-minimal Λ-extension) of the differential graded algebra map given by the inclusion of the basic differential forms into the de Rham algebra of M, $\Omega_B(\mathcal{F}) \longrightarrow \Omega_{DR}(M)$ [**7**, **9**].

Theorem 3.2. *The Λ-minimal Λ-extension of a generalised Seifert fibration $\mathcal{F}$ on a connected, compact, simply-connected manifold M coincides with that of the fibration $\mathcal{L} \longrightarrow M \longrightarrow B\Gamma$. In particular, the relative minimal model* [**9**] *is that of the universal leaf $\mathcal{L}$.*

Proof: This follows because we have

$$\Omega_B(\mathcal{F}) \longrightarrow \Omega(B\Gamma) \longrightarrow \Omega_{DR}(M)$$

where the map on the left induces an isomorphism in rational cohomology. The last assertion follows from a theorem of Grivel [**5**]. ∎

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THEOREM 3.3. *Let $\mathcal{M} \longrightarrow \mathcal{A} \longrightarrow \mathcal{B}$ be a Λ-minimal Λ-extension, where $H^1(\mathcal{B}) = 0$. Then, if $\mathcal{M}$, $\mathcal{A}$ and $\mathcal{B}$ satisfy Poincaré duality, there is a signature product formula as follows:*

$$sign\,(\mathcal{A}) = sign\,(\mathcal{M})\,.\,sign\,(\mathcal{B}).$$

PROOF: As a graded algebra, $\mathcal{A}$ is the tensor product of $\mathcal{M}$ and $\mathcal{B}$. We can put a filtration on $\mathcal{A}$ by

$$F^r(\mathcal{A}) = \mathcal{B}^{\geqslant r} \otimes \mathcal{M}.$$

Then F is a descending filtration, and it is preserved by the differential. Thus, F is a canonically cobounded filtration which gives rise to a convergent third quadrant spectral sequence with

$$E_2 = H(\mathcal{B}) \otimes H(\mathcal{M}) \Longrightarrow H(\mathcal{A}).$$

The signature product formula is then obtained similarly to [**1**], with the above spectral sequence replacing the Leray-Serre. □

4. Taut Riemannian flows

It is well-known that a taut Riemannian flow $\mathcal{F}$ is given by a non-singular Killing vector-field $\mathcal{X}$, and if the ambient manifold M is connected and compact, there is a decomposition of the de Rham algebra of M, up to cohomology, as follows [**4**]:

$$\Omega_{DR}(M) \simeq \Omega_B(\mathcal{F}) \otimes \Lambda X,$$

where ΛX denotes the exterior algebra with one generator of degree one with the trivial differential. This describes the minimal model of $\mathcal{F}$. Thus, it follows from Theorem 3.3 that the signature of M is zero.

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Department of Mathematics
National University of Singapore
Kent Ridge 0511
Republic of Singapore