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A NOTE OF THE UNION-CLOSED SETS CONJECTURE

R. M. NORTON and D. G. SARVATE

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Abstract

Let $\mathscr{A} = \{A_1, \dots, A_n\}$ be a union-closed set. This note establishes a property which must be possessed by any smallest counterexample to the Union-Closed Sets Conjecture. Specifically, a counterexample to the conjecture with minimal n has at least three distinct elements, each of which appears in exactly $(n-1)/2$ of the A_j's.

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1. Introduction

A union-closed set is defined as a nonempty finite collection of distinct nonempty finite sets closed under union. The following conjecture is referred to as the Union-Closed Sets Conjecture:

CONJECTURE. *Let $\mathscr{A} = \{A_1, \dots, A_n\}$ be a union-closed set. Then there exists an element which belongs to at least $\lceil n/2 \rceil$ of the A_j's, where*

$$\lceil n/2 \rceil = \begin{cases} n/2 & \text{if } n \text{ is even} \\ (n+1)/2 & \text{if } n \text{ is odd.} \end{cases}$$

For some historical remarks see [6]. The origin of the conjecture should be attributed to a problem of Peter Frankl (see [2, p. 525] and [5, pp. 161 and 186]).

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Frankl's problem was stated so as to include the empty set in $\mathscr{A}$. In a letter to Fred Galvin, Frankl traces his posing of the problem to December, 1979, and mentions that in early 1980 he relayed the problem to Ron Graham, whom he credits with giving the conjecture some publicity. Sarvate and Renaud [3, 4] have given some bounds on the size of te smallest set in a counterexample to the conjecture, proved that the conjecture is true when $m \leq 18$, and given some other results. The purpose of this note is to give a necessary property for a counterexample of minimal n.

2. A necessary property of a smallest counterexample

Let $\mathscr{A} = \{A_1, \dots, A_n\}$ be a union-closed set. Assume for convenience that $|A_1| \leq |A_j|$ for all j. Let N be the minimum value of n taken over all counterexamples to the union-closed sets conjecture. If a counterexample to the conjecture exists then N is odd (see [3, Theorem 1] and the following theorem holds.

THEOREM. *Let $\mathscr{A} = \{A_1, \dots, A_n\}$ be a counterexample to the conjecture with n minimal. Then there are at least three distinct elements, each of which appears in exactly $(n-1)/2$ of the A_j's.*

PROOF. Let $n = 2t + 1$ and $\mathscr{B} = \{A_2, A_3, \dots, A_n\}$. Now $\mathscr{B}$ is union-closed. Therefore there exists an element x in t sets of $\mathscr{B}$. Let A_x be a smallest set containing x in $\mathscr{B}$. Note that x is not in A_1. Consider $\mathscr{C} = \mathscr{B} - \{A_x\}$ Now $\mathscr{C}$ is union-closed, so there exists an element y ($y \neq x$) in t sets of $\mathscr{C}$, hence in t sets of $\mathscr{A}$. Let A_y be a smallest set in $\mathscr{A}$ containing y. Certainly A_x and A_y are distinct sets. Let $\mathscr{D} = \{A_1, \dots, A_n\} - \{A_x, A_y\}$. To see that $\mathscr{D}$ is union-closed, suppose to the contrary that $A_i \cup A_j$ is not in $\mathscr{D}$ for some A_i, A_j in $\mathscr{D}$. This implies that $A_i \cup A_j = A_x$ or $A_i \cup A_j = A_y$. If $A_i \cup A_j = A_x$ then either A_i or A_j contains x. But since A_i and A_j are in $\mathscr{D}$, neither A_i nor A_j can equal A_x. So by the minimal cardinality assumption about A_x, $A_i \cup A_j$ cannot equal A_x. Similarly $A_j \cup A_j \neq A_y$. So $\mathscr{D}$ is union-closed, and there exists an element z in t sets of $\mathscr{D}$. Now $z \neq y$ and $z \neq x$, because otherwise z is in $t + 1$ sets of $\mathscr{A}$. Hence there are at least three elements each of which is in exactly $(n-1)/2$ sets of $\mathscr{A}$.

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College of Charleston
Charleston, SC 29424
USA

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USA

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