ENUMERATION OF INDICES OF GIVEN ALTITUDE AND DEGREE

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1. THIS note is a sequel to the article by Minc (**2**) on the same problem.

I described in (**1**) a notation for indices of powers in non-associative algebra, defined the degree † and altitude of a power or index, and observed that powers can be represented by bifurcating root-trees. For example, the power $xx.x$ is denoted x^{2+1}, with index $2+1$, and is represented by the tree ⋎; the degree (the number of factors, or free knots in the tree) is 3, and the altitude (the height of the tree) is 2. Multiplication being non-commutative or commutative, one maintains or ignores the distinction between left and right in the tree.

Let a_δ, p_α, $p(\alpha, \delta)$ denote the numbers of distinct indices of degree δ, of altitude α, of altitude α and degree δ, in the non-commutative case; and let b_δ, q_α, $q(\alpha, \delta)$ be the corresponding numbers in the commutative case. I discussed the enumerations a_δ, b_δ, p_α, q_α in (**1**), quoting some of the numerous writers who have considered a_δ, b_δ. In particular it is known that

$$a_\delta = (2\delta-2)!/(\delta-1)!\delta!,$$

and that for $\delta = 1, 2, 3, \ldots$

$$a_\delta = 1,\quad 1,\quad 2,\quad 5,\quad 14,\quad 42,\quad 132,\quad 429,\quad 1430,\quad 4862,\quad \ldots,$$
$$b_\delta = 1,\quad 1,\quad 1,\quad 2,\quad 3,\quad 6,\quad 11,\quad 23,\quad 46,\quad 98,\quad \ldots.$$

Minc obtains two formulæ, giving $p(\alpha, \delta)$, $q(\alpha, \delta)$ in terms of the same with smaller α and δ, and calculates these numbers as far as $\alpha = 4$. I shall show that explicit formulæ can be found for

$$p(\alpha, \alpha+k), \quad p(\alpha, 2^\alpha-k), \quad q(\alpha, \alpha+k), \quad q(\alpha, 2^\alpha-k)$$

for $k = 1, 2, 3, \ldots$ in succession. These formulæ for small values of k are given in § **6**. The initial formulæ in each group (and some others) are obvious by consideration of trees; the rest are derived via first order difference equations in α.

The following results will be used. For any index of altitude α and degree δ, $\alpha+1 \leqq \delta \leqq 2^\alpha$, so that

$$p(\alpha, \delta) = q(\alpha, \delta) = 0 \text{ if } \delta \leqq \alpha \text{ or } > 2^\alpha; \qquad \ldots\ldots(1)$$

$$a_\delta = \sum_{\alpha=[\log_2\delta]}^{\delta-1} p(\alpha, \delta) = \sum_{\alpha=0}^{\kappa} p(\alpha, \delta), \quad b_\delta = \sum_{\alpha=0}^{\kappa} q(\alpha, \delta), \quad \text{where } \kappa \geqq \delta-1; \quad \ldots(2)$$

together with Minc's two formulæ for $p(\alpha, \delta)$, $q(\alpha, \delta)$, some of his numerical results, and some of the values of a_δ, b_δ quoted above.

† Called *potency* by Minc.

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2. We seek first a connexion between $p(\alpha,\ \alpha+k)$ and $p(\alpha+1,\ \alpha+1+k)$. Minc's first formula gives

$$p(\alpha+1,\alpha+1+k)$$
$$=\sum_{d=\alpha+1}^{\alpha+k}\left[p(\alpha,\ d)\left\{2\sum_{r=0}^{\alpha-1}p(r,\ \alpha+1+k-d)+p(\alpha,\ \alpha+1+k-d)\right\}\right].$$

Here, by (2), $\sum_{r=0}^{\alpha-1} p(r,\ \alpha+1+k-d)=a_{\alpha+1+k-d}$ provided that

$$\alpha-1\geqq\alpha+k-d,\quad \text{i.e. } d-1\geqq k,\quad \text{i.e. } \alpha\geqq k.$$

Under the same condition $\alpha\geqq k$, we have $\alpha+1+k-d\leqq\alpha$ and therefore, by (1), $p(\alpha,\ \alpha+1+k-d)=0$. Thus if $\alpha\geqq k$

$$p(\alpha+1,\ \alpha+1+k)=2\sum_{d=\alpha+1}^{\alpha+k}a_{\alpha+1+k-d}\,p(\alpha,\ d).$$

Since $a_1=1$, the last term in this sum is $2p(\alpha,\ \alpha+k)$. Hence

$$p(\alpha+1,\ \alpha+1+k)-2p(\alpha,\ \alpha+k)$$
$$=2\{a_k p(\alpha,\ \alpha+1)+a_{k-1}p(\alpha,\ \alpha+2)+\ldots+a_2 p(\alpha,\ \alpha+k-1)\}\quad(\alpha\geqq k).$$

If we assume that explicit formulæ are already known for $p(\alpha,\ \alpha+1)$, $p(\alpha,\ \alpha+2)$, ..., $p(\alpha,\ \alpha+k-1)$, then we have a linear difference equation with constant coefficients from which to determine $p(\alpha,\ \alpha+k)$. The arbitrary constant in the solution is to be adjusted to give the right result when $\alpha=k$.

It is easily seen by consideration of trees that

$$p(\alpha,\ \alpha+1)=2^{\alpha-1}\quad(\alpha\geqq1).$$

Hence formulæ for $p(\alpha,\alpha+2), p(\alpha,\alpha+3), p(\alpha,\alpha+4)$, ... can be found in succession by solving difference equations, provided that the values of $p(2,\ 4)$, $p(3,\ 6)$, $p(4,\ 8)$, ... are known. (These values are 1, 6, 68, ...).

Calculation is facilitated by putting

$$p(\alpha,\ \alpha+k)=f_k(\alpha)2^\alpha\qquad\ldots\ldots(3)$$

The difference equation becomes

$$\Delta f_k(\alpha)\equiv f_k(\alpha+1)-f_k(\alpha)=a_k f_1(\alpha)+a_{k-1}f_2(\alpha)+\ldots+a_2 f_{k-1}(\alpha),$$

so that

$$f_k(\alpha)=\Delta^{-1}\{a_k f_1(\alpha)+a_{k-1}f_2(\alpha)+\ldots+a_2 f_{k-1}(\alpha)\}\quad(\alpha\geqq k).$$

For illustration, let us suppose that the formulæ for $p(\alpha,\ \alpha+2)$ and $p(\alpha,\ \alpha+3)$ have been obtained (see § **6**), so that we have

$$f_1(\alpha)=\tfrac12,\quad f_2(\alpha)=\tfrac12\alpha-\tfrac34,\quad f_3(\alpha)=\tfrac14\alpha^2-\tfrac32.$$

Then

$$f_4(\alpha)=\Delta^{-1}\{5\,.\,\tfrac12+2(\tfrac12\alpha-\tfrac34)+1(\tfrac14\alpha^2-\tfrac32)\}$$
$$=\Delta^{-1}\{\tfrac14\alpha(\alpha-1)+\tfrac54\alpha-\tfrac12\}$$
$$=\tfrac1{12}\alpha(\alpha-1)(\alpha-2)+\tfrac58\alpha(\alpha-1)-\tfrac12\alpha+C,$$

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where C is such that $f_4(4)2^4 = p(4, 8) = 68$. This gives $C = -\frac{13}{4}$, and finally

$$f_4(\alpha) = \tfrac{1}{24}(2\alpha^3+9\alpha^2-23\alpha-78),$$
$$p(\alpha, \alpha+4) = \tfrac{1}{3}(2\alpha^3+9\alpha^2-23\alpha-78)2^{\alpha-3} \quad (\alpha \geqq 4).$$

Since Δ^{-1} raises the degree of a polynomial by 1, the general result is of the form (3) $(\alpha \geqq k)$, in which $f_k(\alpha)$ is a polynomial in α of degree $k-1$.

3. We seek next a difference equation for $p(\alpha, 2^\alpha - k)$. Minc's first formula gives

$$p(\alpha+1, 2^{\alpha+1}-k)$$
$$= \sum_{d=\alpha+1}^{2^{\alpha+1}-k-1} \left[p(\alpha, d) \left\{ 2 \sum_{r=0}^{\alpha-1} p(r, 2^{\alpha+1}-k-d) + p(\alpha, 2^{\alpha+1}-k-d) \right\} \right]. \quad \ldots\ldots(4)$$

Now, by (1), $p(\alpha, d) = 0$ if $d > 2^\alpha$. Hence in the inner summation we may assume $d \leqq 2^\alpha$, so that

$$2^{\alpha+1}-k-d \geqq 2^{\alpha+1}-k-2^\alpha = 2^\alpha - k.$$

Using this, and again using (1), we see that every term in the inner summation vanishes if

$$2^\alpha - k > 2^{\alpha-1}, \quad \text{i.e. } 2^{\alpha-1} > k, \quad \text{i.e. } \alpha \geqq 2 + [\log_2 k]. \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

We assume this condition satisfied, and note that it implies that $\alpha \geqq 2$, hence $2^{\alpha-1} \geqq \alpha$; and since by (5) $2^\alpha - k \geqq 2^{\alpha-1}+1$, this implies that

$$2^\alpha - k \geqq \alpha+1; \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

it can also be deduced that

$$2^\alpha < 2^{\alpha+1}-k-1. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

(4) now reduces to

$$p(\alpha+1, 2^{\alpha+1}-k) = \sum_{d=\alpha+1}^{2^{\alpha+1}-k-1} p(\alpha, d) p(\alpha, 2^{\alpha+1}-k-d).$$

As already observed, the first factor in the summation is zero if $d > 2^\alpha$; the second factor is zero if $2^{\alpha+1}-k-d > 2^\alpha$, i.e. $d < 2^\alpha - k$. Hence, in view of (6) and (7),

$$p(\alpha+1, 2^{\alpha+1}-k) = \sum_{d=2^\alpha-k}^{2^\alpha} p(\alpha, d) p(\alpha, 2^{\alpha+1}-k-d). \quad \ldots\ldots\ldots\ldots(8)$$

Now $p(\alpha, 2^\alpha) = 1$, the only index of altitude α and degree 2^α being 2^α. So the first and last terms in the summation (8) are both $p(\alpha, 2^\alpha - k)$. In the case $k = 1$, this exhausts the summation, and we have the difference equation

$$p(\alpha+1, 2^{\alpha+1}-1) = 2p(\alpha, 2^\alpha - 1) \quad (\alpha \geqq 2),$$

whose solution $p(\alpha, 2^\alpha - 1) = C \,.\, 2^\alpha$ with initial condition $p(2, 3) = 2$ yields the formula

$$p(\alpha, 2^\alpha - 1) = 2^{\alpha-1} \quad (\alpha \geqq 2).$$

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If $k>1$, (8) can be written

$$p(\alpha+1,\ 2^{\alpha+1}-k)-2p(\alpha,\ 2^{\alpha}-k) = \sum_{d=2^{\alpha}-k+1}^{2^{\alpha}-1} p(\alpha,\ d)p(\alpha,\ 2^{\alpha+1}-k-d);$$

or finally, putting $d = 2^{\alpha}-k+r$,

$$p(\alpha+1,\ 2^{\alpha+1}-k)-2p(\alpha,\ 2^{\alpha}-k) = \sum_{r=1}^{k-1} p(\alpha,\ 2^{\alpha}-k+r)p(\alpha,\ 2^{\alpha}-r)$$
$$(k>1,\ \alpha \geqq 2+[\log_2 k]).$$

If we write $p(\alpha,\ 2^{\alpha}-k) = u_k(\alpha)$, the equation is

$$u_k(\alpha+1)-2u_k(\alpha) = u_{k-1}(\alpha)u_1(\alpha)+u_{k-2}(\alpha)u_2(\alpha)+\ldots+u_1(\alpha)u_{k-1}(\alpha).$$

We know $u_0 = 1$, $u_1 = 2^{\alpha-1}$, and so by solving difference equations we can calculate u_2, u_3, ... in succession. At each stage the arbitrary constant is to be adjusted to give the right result when $\alpha = 2+[\log_2 k]$. It can be shown inductively that the general form of the result is

$$p(\alpha,\ 2^{\alpha}-k) = \text{a polynomial of degree } k \text{ in } 2^{\alpha} \quad (\alpha \geqq 2+[\log_2 k]).$$

4. By similar methods difference equations can be found for $q(\alpha,\ \alpha+k)$ and $q(\alpha,\ 2^{\alpha}-k)$. The details are rather tedious and I will merely quote the results which I have obtained.

If $\alpha \geqq k$,
$$\Delta q(\alpha,\ \alpha+k) \equiv q(\alpha+1,\ \alpha+1+k)-q(\alpha,\ \alpha+k)$$
$$= b_k q(\alpha,\ \alpha+1)+b_{k-1}q(\alpha,\ \alpha+2)+\ldots+b_2 q(\alpha,\ \alpha+k-1).$$

Hence, for $\alpha \geqq k$,

$$q(\alpha,\ \alpha+k) = \Delta^{-1}\{b_k q(\alpha,\ \alpha+1)+b_{k-1}q(\alpha,\ \alpha+2)+\ldots+b_2 q(\alpha,\ \alpha+k-1)\}$$

where the arbitrary constant is to be chosen to fit $q(k,\ 2k)$; and $q(\alpha,\ \alpha+k)$ is a polynomial in α of degree $k-1$. The formulæ given in § **6** (iii), going as far as $k = 4$, proved thus for $\alpha \geqq k$, are found to be in fact true for $\alpha \geqq k-1$.

5. If $\alpha \geqq 2+[\log_2 k]$, and if $q(\alpha,\ 2^{\alpha}-k)$ is denoted v_k, then

(i) if k is odd,

$$v_k = \Delta^{-1}\{v_{k-1}v_1+v_{k-2}v_2+\ldots+v_{(k+1)/2}v_{(k-1)/2}\};$$

(ii) if k is even,

$$v_k = \Delta^{-1}\{v_{k-1}v_1+v_{k-2}v_2+\ldots+v_{k/2+1}v_{k/2-1}+\tfrac{1}{2}v_{k/2}(v_{k/2}+1)\}.$$

The arbitrary constant is to be chosen to give the right result when $\alpha = 2+[\log_2 k]$. For $\alpha \geqq 2+[\log_2 k]$, $k \geqq 1$, $q(\alpha,\ 2^{\alpha}-k)$ is a polynomial in α of degree $k-1$.

6. Conclusions

(i)
$$p(\alpha,\ \alpha+1) = 2^{\alpha-1} \quad (\alpha \geqq 1)$$
$$p(\alpha,\ \alpha+2) = (2\alpha-3)2^{\alpha-2} \quad (\alpha \geqq 2)$$
$$p(\alpha,\ \alpha+3) = (\alpha^2-6)2^{\alpha-2} \quad (\alpha \geqq 3)$$
$$p(\alpha,\ \alpha+4) = \tfrac{1}{3}(2\alpha^3+9\alpha^2-23\alpha-78)2^{\alpha-3} \quad (\alpha \geqq 4)$$

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(ii) $p(\alpha, 2^\alpha) = 1$

$p(\alpha, 2^\alpha-1) = 2^{\alpha-1} \quad (\alpha \geqq 2)$

$p(\alpha, 2^\alpha-2) = 2^{\alpha-2}(2^{\alpha-1}-1) \quad (\alpha \geqq 3)$

$p(\alpha, 2^\alpha-3) = \frac{1}{3}2^{\alpha-2}(2^{2\alpha-2}-3.2^{\alpha-1}+5) \quad (\alpha \geqq 3)$

(iii) $q(\alpha, \alpha+1) = 1$

$q(\alpha, \alpha+2) = \alpha-1 \quad (\alpha \geqq 1)$

$q(\alpha, \alpha+3) = \frac{1}{2}(\alpha^2-\alpha-2) \quad (\alpha \geqq 2)$

$q(\alpha, \alpha+4) = \frac{1}{6}(\alpha^3-\alpha-18) \quad (\alpha \geqq 3)$

(iv) $q(\alpha, 2^\alpha) = 1$

$q(\alpha, 2^\alpha-1) = 1 \quad (\alpha \geqq 2)$

$q(\alpha, 2^\alpha-2) = \alpha-1 \quad (\alpha \geqq 3)$

$q(\alpha, 2^\alpha-3) = \frac{1}{2}(\alpha^2-3\alpha+4) \quad (\alpha \geqq 3).$

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