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ON ARC-REGULAR FROBENIUS METACIRCULANTS

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Abstract

A graph is called *arc-regular* if its full automorphism group acts regularly on its arc set. In this paper, we completely determine all the arc-regular Frobenius metacirculants of prime valency.

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1. Introduction

In this paper, graphs are assumed to be finite, simple and undirected.

A graph Γ is called *arc-regular* if its full automorphism group $\mathsf{Aut}\,\Gamma$ acts regularly on its arc set. Since determining the full automorphism groups of graphs is one of the fundamental topics in the field of algebraic graph theory, characterising arc-regular graphs has received much attention (see, for example, [6, 19, 20, 22]).

For a graph Γ, if $\mathsf{Aut}\,\Gamma$ contains a metacyclic subgroup R which is transitive on the vertex set of Γ, then Γ is called a *metacirculant* (this definition is slightly more general than the original definition of Alspach and Parsons in [1]); if R is also a Frobenius group, then Γ is called a *Frobenius metacirculant* (recall that a group R is *metacyclic* if R has a normal cyclic subgroup N such that the quotient group R/N is also cyclic). In some cases, to emphasise the metacyclic group, we say that Γ is a metacirculant of R. As usual, a graph Γ is called a *circulant* or a *dihedrant* if $\mathsf{Aut}\,\Gamma$ has a regular cyclic subgroup or a dihedral subgroup, respectively.

The class of metacirculants provides a rich source of many interesting families of graphs and has been extensively studied. For example, some special edge-transitive metacirculants have been characterised (see [12, 14] for circulants, [5, 21] for the case of order a product of two primes, [18] for the case of prime-power order, [15] for the vertex-primitive case and [17, 24] for the case of Frobenius metacirculants with small valency). Moreover, an infinite family of arc-regular dihedrants of any prescribed valency is constructed in [11] and arc-regular dihedrants of prime valency are classified in [7].

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The main purpose of this paper is to determine arc-regular Frobenius metacirculants of any prime valency.

A graph Γ is called a *Cayley graph* if there exist a group G and a subset $S \subseteq G\backslash\{1\}$ with $S = S^{-1} := \{s^{-1} \mid s \in S\}$ such that the vertex set $V\Gamma = G$ and a vertex x is adjacent to a vertex y if and only if $yx^{-1} \in S$. This Cayley graph is denoted by $\mathsf{Cay}(G, S)$. It is well known that a graph Γ is isomorphic to a Cayley graph if and only if $\mathsf{Aut}\,\Gamma$ contains a subgroup R which acts regularly on $V\Gamma$ (see [2, Proposition 16.3]). In particular, if R is normal in $\mathsf{Aut}\,\Gamma$, then Γ is called a *normal Cayley graph* of R. Since a metacirculant is not necessarily a Cayley graph (for example, the Petersen graph), Cayley graphs of metacyclic groups form a proper subfamily of metacirculants.

For a positive integer m, we denote by D_{2m} the dihedral group of order $2m$ and by $\mathbb{Z}_m$ the cyclic group of order m. Given two groups N and H, denote by $N \times H$ the direct product of N and H and by $N \cdot H$ an extension of N by H. If such an extension is split, we write $N : H$ instead of $N \cdot H$.

Example 1.1. Let $G = \langle a, b \mid a^m = b^2 = 1, a^b = a^{-1}\rangle \cong \mathrm{D}_{2m}$ be a dihedral group and let p be an odd prime. Suppose that k is a solution of the equation

$$x^{p-1} + x^{p-2} + \cdots + x + 1 \equiv 0 \pmod{m}$$

and set

$$S = \{b, ab, a^{k+1}b, \ldots, a^{k^{p-2}+k^{p-3}+\cdots+1}b\}, \quad \mathsf{CD}_{2m,p,k} = \mathsf{Cay}(G, S).$$

The letters 'CD' stand for 'Cayley graphs of dihedral groups'.

The main result of this paper is the following assertion, which completely determines arc-regular Frobenius metacirculants of odd prime valency.

Theorem 1.2. *Let R be a Frobenius metacyclic group and Γ be a connected arc-regular metacirculant of R of odd prime valency p. Then Γ is a dihedrant and one of the following statements holds:*

(1) $\Gamma \cong \mathsf{CD}_{2n,p,k}$ *is not a Cayley graph of R, and $R \cong \mathbb{Z}_n : \mathbb{Z}_{2p}$ with $n \geq 13$;*
(2) $\Gamma \cong \mathsf{CD}_{2p^t n,p,k}$ *is a normal Cayley graph of R, and $R \cong \mathrm{D}_{2p^t n}$ with $p^t n \geq 13$ and $t \leq 1$;*
(3) $\Gamma \cong \mathsf{CD}_{2pn,p,k}$ *is a Cayley graph of R but not normal, and $R \cong \mathbb{Z}_n : \mathbb{Z}_{2p}$,*

where, in parts (1)–(3), $n = p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s}$ with $s \geq 1$ and $p_1, p_2, \ldots, p_s$ are distinct primes such that $p \mid p_i - 1$ for $i = 1, 2, \ldots, s$.

Up to isomorphism, there are exactly $(p-1)^{s-1}$ graphs in each part of Theorem 1.2.

This paper is organised as follows. After this introductory section, we give some preliminary results in Section 2. We then prove some technical lemmas in Section 3 and complete the proof of Theorem 1.2 in Section 4.

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2. Preliminaries

In this section, we quote certain preliminary results that will be used later. The first one is a classification of transitive permutation groups of prime degree.

Lemma 2.1 [4, page 99]. *Let $G \le \mathrm{Sym}(\Omega)$ be a transitive permutation group of prime degree p. Then G is either affine or almost simple, and one of the following statements is true, where $\alpha \in \Omega$:*

(1) $\mathbb{Z}_p \le G \le \mathbb{Z}_p : \mathbb{Z}_{p-1}$;
(2) $(G, G_\alpha) = (\mathrm{A}_p, \mathrm{A}_{p-1})$ *or* $(\mathrm{S}_p, \mathrm{S}_{p-1})$ *with* $p \ge 5$;
(3) $(G, G_\alpha) = (\mathrm{PSL}(2, 11), \mathrm{A}_5)$, $(\mathsf{M}_{11}, \mathsf{M}_{10})$ *or* $(\mathsf{M}_{23}, \mathsf{M}_{22})$;
(4) *G is a projective group with* $\mathrm{PSL}(d, q) \lhd G \le \mathrm{P\Gamma L}(d, q)$*, acting naturally on the set of projective points of degree* $p = (q^d - 1)/(q - 1)$.

For a group G and its subgroup H, let $C_G(H)$ and $N_G(H)$ denote the centraliser and normaliser of H in G, respectively. The following result is the well-known 'N/C' theorem.

Lemma 2.2 [10, Ch. I, Lemma 4.5]. *Let G be a group and H a subgroup of G. Then* $N_G(H)/C_G(H) \le \mathsf{Aut}(H)$.

Let $\Gamma = \mathsf{Cay}(G, S)$ be a Cayley graph of a group G. Let

$$\hat{G} = \{\hat{g} \mid \hat{g} : \ x \mapsto xg, \text{for all } g, x \in G\},$$
$$\mathsf{Aut}(G, S) = \{\sigma \in \mathsf{Aut}(G) \mid S^\sigma = S\}.$$

Then both $\hat{G}$ and $\mathsf{Aut}(G, S)$ are subgroups of $\mathsf{Aut}\,\Gamma$. Further, the following lemma holds.

Lemma 2.3 [9, Lemma 2.1]. *Let $\Gamma = \mathsf{Cay}(G, S)$ be a Cayley graph. Then the normaliser* $N_{\mathsf{Aut}\,\Gamma}(\hat{G}) = \hat{G} : \mathsf{Aut}(G, S)$.

We remark that the regular subgroup $\hat{G}$ is isomorphic (but not equal) to the defining group G. However, for convenience, we will often write $\hat{G}$ as G.

For a group G and a subgroup H, the *core* of H in G, denoted by $\mathsf{core}_G(H)$, is the largest normal subgroup of G contained in H. In particular, if $\mathsf{core}_G(H) = 1$, then H is called *core-free* in G.

Given a group X and two subgroups L and R such that $L \cap R$ is core-free in X, define a bipartite graph, $\mathsf{Cos}(X, L, R)$, with vertex set $[X : L] \cup [X : R]$, and Lx, Ry with $x, y \in X$ adjacent if and only if $yx^{-1} \in RL$. This graph is called a *bi-coset graph*.

Lemma 2.4 [8, Lemma 3.7]. *Let $\Gamma = \mathsf{Cos}(X, L, R)$ be as above. Then:*

(1) $X \le \mathsf{Aut}\,\Gamma$*, and Γ is X-vertex-intransitive and X-edge-transitive;*
(2) *Γ is connected if and only if* $\langle L, R\rangle = X$.

Conversely, each X-vertex-intransitive and X-edge-transitive graph is isomorphic to $\mathsf{Cos}(X, X_\alpha, X_\beta)$*, where α and β are adjacent vertices.*

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A typical method for studying graphs is taking normal quotient graphs. Suppose that Γ is a graph, and $X \leq \mathsf{Aut}\,\Gamma$ has an intransitive normal subgroup N. Denote by $V\Gamma_N$ the set of N-orbits in $V\Gamma$. The *normal quotient graph* Γ_N of Γ induced by N has vertex set $V\Gamma_N$, and two vertices $B, C \in V\Gamma_N$ are adjacent if and only if some vertex in B is adjacent in Γ to some vertex in C. If further Γ and Γ_N have the same valency, then Γ is called a *normal N-cover* (or *regular N-cover*) of Γ_N.

Theorem 2.5 [16, Lemma 2.5]. *Let Γ be an X-arc-transitive graph of odd prime valency and let $N \lhd X$ have at least three orbits on $V\Gamma$. Then N is semiregular on $V\Gamma$, $X/N \leq \mathsf{Aut}\,\Gamma_N$, Γ_N is X/N-arc-transitive and Γ is a normal N-cover of Γ_N.*

Lemma 2.6 [7, Theorem 3.1]. *Let Γ be a connected arc-regular Cayley graph of a dihedral group $R \cong \mathrm{D}_{2m}$ of prime valency p. Then $\Gamma \cong \mathsf{CD}_{2m,p,k}$ is a normal Cayley graph of R, and $m = p^t p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} \geq 13$, with $p_1, p_2, \ldots, p_s$ distinct primes, $t \leq 1$, $s \geq 1$ and $p \mid p_i - 1$ for $i = 1, 2, \ldots, s$.*

3. Technical lemmas

For a graph Γ, if there is an automorphism group $X \leq \mathsf{Aut}\,\Gamma$ which acts regularly on the arc set of Γ, then Γ is called X-arc-regular.

Lemma 3.1. *Let Γ be an X-arc-regular Cayley graph of odd prime valency of a metacyclic group R, where $R \leq X \leq \mathsf{Aut}\,\Gamma$. Then R is not core-free in X.*

Proof. Suppose that $\mathsf{val}(\Gamma) = p$. Then $|X : R| = p$, $X_\alpha \cong \mathbb{Z}_p$ and $X = RX_\alpha$, where $\alpha \in V\Gamma$.

Assume that, on the contrary, R is core-free in X. Then X acts faithfully on $[X : R]$ (the set of right cosets of R in X) by the coset action:

$$(Rx)^y = Rxy \quad \text{where } Rx \in [X : R],\ y \in X.$$

(This action was introduced by Li [13] and is often called a *'dual-action'* of the action of X on $V\Gamma$.) Thus, X can be viewed as a transitive permutation group of prime degree p on the set $[X : R]$ and hence X satisfies Lemma 2.1.

If $X \leq \mathbb{Z}_p : \mathbb{Z}_{p-1}$ is an affine group, then $\mathbb{Z}_p \cong X_\alpha \lhd X$, which is a contradiction.

Suppose that X is an almost simple group. View R as a point of $[X : R]$. Then the point stabiliser $X_R = R$ and so the tuple (X, R) (as (G, G_α) there) satisfies parts (2)–(4) of Lemma 2.1. For parts (2) and (3), R is not a metacyclic group, which is a contradiction.

Consider the case that (X, R) satisfies part (4) of Lemma 2.1. Then

$$R \geq (\mathrm{PSL}(d, q))_R \cong [q^{d-1}] \cdot \mathbb{Z}_{(q-1)/(d,q-1)} \cdot \mathrm{PSL}(d-1, q) \cdot \mathbb{Z}_{(d-1,q-1)},$$

where $[q^{d-1}]$ denotes an elementary abelian group of order q^{d-1}. Assume first that $d \geq 3$. If $(d, q) = (3, 2)$ or $(3, 3)$, by the Atlas [3], $R \geq \mathrm{S}_4$ or $\mathbb{Z}_3^2 : 2\mathrm{S}_4$, respectively, and, for the other cases, R has a nonabelian simple section $\mathrm{PSL}(d-1, q)$. So R is never metacyclic in the case $d \geq 3$, yielding a contradiction. Assume that $d = 2$. Then $q = p - 1$ is even, so $q = 2^e$ with $e \geq 2$. It follows that $R \cong \mathbb{Z}_2^e : \mathbb{Z}_{2^e-1}$ is not metacyclic, also yielding a contradiction. □

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For a Cayley graph $\Gamma = \mathsf{Cay}(G, S)$, where G is a cyclic group and $|S| \geq 3$ is odd, since G has an automorphism $\sigma : x \to x^{-1}$ which preserves S (noting that Γ is assumed to be undirected, so $S = S^{-1}$), we see that $\sigma \in \mathsf{Aut}(G, S) \leq (\mathsf{Aut}\,\Gamma)_{\mathbf{1}}$, where $\mathbf{1}$ denotes the vertex of Γ corresponding to the identity element of G. So $(\mathsf{Aut}\,\Gamma)_{\mathbf{1}} \geq \langle\sigma\rangle \cong \mathbb{Z}_2$ is not regular on $\Gamma(\mathbf{1}) = S$. Thus, a circulant of odd valency is never an arc-regular graph. However, Γ may be X-arc-regular for some proper subgroup X of $\mathsf{Aut}\,\Gamma$.

The next lemma characterises a class of X-arc-regular circulants.

Lemma 3.2. *Let Γ be a connected X-arc-regular Cayley graph of odd prime valency p of a cyclic group $\langle a\rangle$, where $\langle a\rangle \leq X \leq \mathsf{Aut}\,\Gamma$. Suppose further that $\mathsf{core}_X(\langle a\rangle) = \langle a^2\rangle$. Then $\Gamma \cong \mathsf{K}_{p,p}$ is a complete bipartite graph and $\langle a\rangle \cong \mathbb{Z}_{2p}$.*

Proof. Suppose that $\langle a\rangle \cong \mathbb{Z}_m$. Since Γ is of odd valency, $m = |V\Gamma|$ is even. Set $m = 2n$ and $K = \mathsf{core}_X(\langle a\rangle)$.

Assume that n is even. Because $\langle a^4\rangle$ is a characteristic subgroup of K, it is normal in X. Since $\langle a\rangle$ is regular on $V\Gamma$, $\langle a^4\rangle$ has four orbits on $V\Gamma$, by Theorem 2.5, and the quotient graph $\Gamma_{\langle a^4\rangle}$ is an $X/\langle a^4\rangle$-arc-regular Cayley graph of $\langle a\rangle/\langle a^4\rangle \cong \mathbb{Z}_4$, so $\mathsf{val}(\Gamma_{\langle a^4\rangle}) = 3$ and $\Gamma_{\langle a^4\rangle} \cong \mathsf{K}_4$. Since $|X/\langle a^4\rangle| = 12$ and $\mathsf{Aut}(\mathsf{K}_4) \cong \mathrm{S}_4$, we conclude that $\mathbb{Z}_4 \cong R/\langle a^4\rangle \leq X/\langle a^4\rangle \cong \mathrm{A}_4$, which is a contradiction.

Thus, n is odd. Since $|X| = 2np$, $|X/\langle a^2\rangle| = 2p$, and $X/\langle a^2\rangle \cong \mathbb{Z}_{2p}$ or D_{2p}. If $X/\langle a^2\rangle \cong \mathbb{Z}_{2p}$, then $\langle a\rangle \lhd X$, which contradicts the assumption $\mathsf{core}_X(\langle a\rangle) = \langle a^2\rangle$.

Consider the case $X/\langle a^2\rangle \cong \mathrm{D}_{2p}$. Let $C = C_X(\langle a^2\rangle)$. If $C = \langle a^2\rangle$, Lemma 2.2 implies that $\mathrm{D}_{2p} \cong X/\langle a^2\rangle = X/C \leq \mathsf{Aut}(\langle a^2\rangle)$ is abelian, which is a contradiction. Hence, $C \supset \langle a^2\rangle$. Since $1 \neq C/\langle a^2\rangle \lhd X/\langle a^2\rangle \cong \mathrm{D}_{2p}$, C has an abelian subgroup isomorphic to $\langle a^2\rangle \cdot \mathbb{Z}_p$, so $X \cong \mathbb{Z}_{pn} \cdot \mathbb{Z}_2$, or $(\mathbb{Z}_n \times \mathbb{Z}_p) \cdot \mathbb{Z}_2$ with $p \mid n$. For the former case, since $X_\alpha \cong \mathbb{Z}_p$ for $\alpha \in V\Gamma$, we have $X_\alpha \lhd X$, which is not possible. For the latter case, we may write $n = p^e q$, where $e \geq 1$ and $(p, q) = 1$. If $q \neq 1$, then X has a normal Sylow p-subgroup X_p, by [25, Theorem 3.4], and X_p has $2q$ orbits on $V\Gamma$. It then follows from Theorem 2.5 that X_p is semiregular on $V\Gamma$, so $|X_p| = p^{e+1}$ divides $|V\Gamma| = 2p^e q$, yielding a contradiction. Hence, $q = 1$, $n = p^e$ and, by [25, Theorem 3.4], Γ is X_p-edge-transitive and X_p-vertex-intransitive. If $e \geq 2$, by Lemma 2.4, $\Gamma \cong \mathsf{Cos}(X_p, (X_p)_\alpha, (X_p)_\beta)$, where α and β are adjacent vertices of Γ; however, as X_p is abelian, and $(X_p)_\alpha \cong (X_p)_\beta \cong \mathbb{Z}_p$, we have $\langle (X_p)_\alpha, (X_p)_\beta\rangle < G_p$, so Γ is disconnected, which is a contradiction. Therefore, $e = 1$ and $|V\Gamma| = 2p$. Now, since $\langle a^2\rangle \lhd X$ has exactly two orbits on $V\Gamma$, Γ is a bipartite graph of order $2p$ and valency p; hence, $\Gamma \cong \mathsf{K}_{p,p}$. □

We remark that the complete bipartite graph $\Gamma = \mathsf{K}_{p,p}$ is really an example satisfying the assumptions of Lemma 3.2. Let $\{1, 2, \ldots, p\}$ and $\{1', 2', \ldots, p'\}$ be the two parts of Γ. Then

$$X := \langle (12 \cdots p), (1'2' \cdots p'), (11')(22') \cdots (pp')\rangle \cong \mathbb{Z}_p \wr \mathbb{Z}_2$$

is an arc-regular automorphism group of Γ and X has a cyclic subgroup

$$R := \langle (12 \cdots p)(1'2' \cdots p'), (11')(22') \cdots (pp')\rangle = \langle (12'3 \ldots p1'2 \ldots p')\rangle \cong \mathbb{Z}_{2p}$$

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which is regular on $V\Gamma$ (so Γ is a Cayley graph of R). Set $a = (12'3 \dots p1'2 \dots p')$. Since $R = \langle a\rangle$ is not normal in X and $\langle a^2\rangle = \langle (12 \cdots p)(1'2' \cdots p')\rangle \lhd X$, we see that $\mathsf{core}_X(R) = \langle a^2\rangle$.

Generalised dihedral groups are natural generalisations of dihedral groups. A group G is called a *generalised dihedral group* if $G = H : \langle g\rangle$ for some abelian subgroup H and an involution g such that $h^g = h^{-1}$ for each $h \in H$. This generalised dihedral group is denoted by $\mathsf{Dih}(H)$. Clearly, $\mathsf{Dih}(\mathbb{Z}_m) \cong \mathrm{D}_{2m}$.

Lemma 3.3. *Let Γ be an X-edge-transitive graph with $X \le \mathsf{Aut}\,\Gamma$, and suppose that X has an abelian normal subgroup H which acts semiregularly and has exactly two orbits on $V\Gamma$. Then Γ is a Cayley graph of the generalised dihedral group $\mathsf{Dih}(H)$.*

Proof. Let $H = \{h_1, h_2, \dots, h_n\}$, and let Δ_1 and Δ_2 be the two orbits of H on $V\Gamma$. Then $\Delta_1 = \{u^{h_i} \mid 1 \le i \le n\}$ and $\Delta_2 = \{v^{h_i} \mid 1 \le i \le n\}$, where $u \in \Delta_1$ and $v \in \Delta_2$. Since Γ is X-edge-transitive and $H \lhd X$, it is easy to show that there is no edge in both Δ_1 and Δ_2. Suppose that $\mathsf{val}(\Gamma) = k$. Then $k \le n$ and, without loss of generality, we may assume that $\Gamma(u) = \{v^{h_i} \mid 1 \le i \le k\}$. Then $\Gamma(v) = \{u^{h_i^{-1}} \mid 1 \le i \le k\}$. Define

$$\sigma : u^{h_i} \to v^{h_i^{-1}},\ v^{h_i} \to u^{h_i^{-1}} \quad \text{for } 1 \le i \le n.$$

Clearly, σ is a permutation on $V\Gamma$ with order two. Since H is abelian,

$$\begin{aligned}
u^{h_i} \sim v^{h_j} &\Longleftrightarrow u \sim v^{h_j h_i^{-1}} \\
&\Longleftrightarrow h_i^{-1}h_j = h_j h_i^{-1} \in \{h_1, h_2, \dots, h_k\} \\
&\Longleftrightarrow h_j^{-1}h_i \in \{h_1^{-1}, h_2^{-1}, \dots, h_k^{-1}\} \\
&\Longleftrightarrow v \sim u^{h_j^{-1}h_i} \\
&\Longleftrightarrow v^{h_i^{-1}} \sim u^{h_j^{-1}} \\
&\Longleftrightarrow (u^{h_i})^{\sigma} \sim (v^{h_j})^{\sigma}.
\end{aligned}$$

Thus, $\sigma \in \mathsf{Aut}\,\Gamma$. Further, for any $h, h_i \in H$, we have $(u^{h_i})^{\sigma h \sigma} = (v^{h_i^{-1}})^{h\sigma} = (v^{h_i^{-1}h})^{\sigma} = u^{h^{-1}h_i} = (u^{h_i})^{h^{-1}}$ and, similarly, $(v^{h_i})^{\sigma h \sigma} = (v^{h_i})^{h^{-1}}$. Consequently, $h^{\sigma} = \sigma h \sigma = h^{-1}$, and $\langle H, \sigma\rangle \cong \mathsf{Dih}(H)$ is a generalised dihedral group. Because $\langle H, \sigma\rangle$ is regular on $V\Gamma$, Γ is a Cayley graph of $\langle H, \sigma\rangle \cong \mathsf{Dih}(H)$. □

We end this section with an observation on the automorphism groups of Frobenius metacyclic groups.

Lemma 3.4. *Let $R = \langle a\rangle : \langle b\rangle \cong \mathbb{Z}_m : \mathbb{Z}_n$ be a Frobenius group. Then $\mathsf{Aut}(R) \cong \mathbb{Z}_m : \mathbb{Z}_m^*$, where $\mathbb{Z}_m^*$ denotes the multiplicative group of the residue class ring modulo m, and each automorphism σ of R has the form*

$$\sigma : a \to a^i, b \to a^j b \quad \text{where } (i, m) = 1,\ 0 \le j \le m - 1.$$

Proof. Since R is a Frobenius group, $n \mid m - 1$, so $\langle a\rangle$ is a characteristic subgroup of R, and $\sigma(a) = a^i$ with $(i, m) = 1$.

Suppose that $a^b = a^r$ and $\sigma(b) = a^j b^k$. Then $\sigma(ab) = a^{i+j} b^k$; also, as $ab = ba^r$, we have $\sigma(ab) = \sigma(ba^r) = a^j b^k a^{ir}$, so $a^{i+j} b^k = a^j b^k a^{ir}$ and $a^i b^k = b^k a^{ir}$. It follows that $a^{ir^k} = b^{-k} a^i b^k = a^{ir}$. Hence, $ir^k \equiv ir \pmod{m}$ and, in turn, $r^{k-1} \equiv 1 \pmod{m}$ as $(ir, m) = 1$. Because R is a Frobenius group, n is the smallest positive integer solution of the equation $r^x \equiv 1 \pmod{m}$ (for, otherwise, R has a nontrivial centre). We conclude that $n \mid k - 1$ and $\sigma(b) = a^j b$, that is, σ has the form stated in Lemma 3.4. Conversely, it is straightforward to verify that the map σ given in the lemma really determines an automorphism of R.

Now, let

$$\tau : a \to a,\ b \to ab, \quad M = \{\phi \in \mathsf{Aut}(R) \mid \phi : a \to a^i, b \to b,\ (i, m) = 1\}.$$

Then it is easy to show that $\mathsf{Aut}(R) = \langle\tau\rangle M$, $\mathbb{Z}_m \cong \langle\tau\rangle \lhd \mathsf{Aut}(R)$, $M \cong \mathbb{Z}_m^*$ and $\langle\tau\rangle \cap M = 1$. Hence, $\mathsf{Aut}(R) \cong \mathbb{Z}_m : \mathbb{Z}_m^*$. □

4. Proof of Theorem 1.2

Let Γ be a connected arc-regular metacirculant of R, with odd prime valency p, and suppose that $R = \langle a\rangle : \langle b\rangle \cong \mathbb{Z}_m : \mathbb{Z}_l$ is a Frobenius metacyclic group and transitive on $V\Gamma$. Then $l \mid m - 1$ and, as Γ is of odd valency and ml is even, it follows that m is odd and l is even.

Let $\mathrm{A} = \mathsf{Aut}\,\Gamma$. Then $|\mathrm{A}| = p|V\Gamma|$, $\mathrm{A} = R\mathrm{A}_\alpha$ and $\mathrm{A}_\alpha \cong \mathbb{Z}_p$, where $\alpha \in V\Gamma$.

Lemma 4.1. *Suppose that R is not regular on $V\Gamma$. Then Γ and R satisfy part (1) of Theorem 1.2.*

Proof. Since R is not regular on $V\Gamma$ and $R_\alpha \le \mathrm{A}_\alpha \cong \mathbb{Z}_p$, we have $R_\alpha = \mathrm{A}_\alpha$ and $R = \mathrm{A}$. Because $\langle a\rangle$ is a characteristic subgroup of R, $\langle a\rangle_\alpha \le R_\alpha$ is normal in R and, as R_α is core-free in R, we conclude that $\langle a\rangle_\alpha = 1$, that is, $\langle a\rangle$ is semiregular on $V\Gamma$.

Suppose that $\langle a\rangle$ has at least three orbits on $V\Gamma$. By Theorem 2.5, the normal quotient graph $\Gamma_{\langle a\rangle}$ is an $R/\langle a\rangle$-arc-regular graph of valency p, which is impossible as $R/\langle a\rangle \cong \langle b\rangle$ is a cyclic group, so $\langle a\rangle$ has at most two orbits on $V\Gamma$. If $\langle a\rangle$ is transitive, then $\langle a\rangle$ is regular on $V\Gamma$, that is, Γ is an arc-regular circulant, which is not possible by the remarks before Lemma 3.2.

Thus, $\langle a\rangle$ has exactly two orbits on $V\Gamma$, $m = \frac{1}{2}|V\Gamma|$ and $l = 2p$. By Lemma 3.3, Γ is a Cayley graph of a dihedral group isomorphic to D_{2m}. It then follows from Lemma 2.6 that $\Gamma \cong \mathsf{CD}_{2m,p,k}$, where $m = p^t p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} \ge 13$ with $p_1, p_2, \ldots, p_s$ distinct primes, $t \le 1$, $s \ge 1$ and $p \mid p_i - 1$ for each i; further, if $t = 1$, then $(m, l) \ne 1$, which is impossible as $l \mid m - 1$. Hence, $t = 0$ and Γ and R are as in part (1) of Theorem 1.2. □

The next lemma treats the case where R is regular on $V\Gamma$ and normal in A.

Lemma 4.2. *Suppose that R is normal in A and regular on $V\Gamma$. Then Γ and R satisfy part (2) of Theorem 1.2.*

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Proof. Since R is regular on $V\Gamma$, Γ is a Cayley graph of R, so $\Gamma = \mathsf{Cay}(R, S)$ for some $S = S^{-1} \subseteq R\backslash\{1\}$. Since R is normal in A, by Lemma 2.3, $\mathrm{A} = R : \mathsf{Aut}(R, S)$, and $\mathrm{A}_{\mathbf{1}} = \mathsf{Aut}(R, S) \cong \mathbb{Z}_p$ is regular on $\Gamma(\mathbf{1}) = S$, where $\mathbf{1}$ denotes the vertex corresponding to the identity element of R. So $S = s^{\langle\sigma\rangle}$, where $\sigma \in \mathsf{Aut}(R, S)$ is of order p and $s \in S$. In particular, all elements in S have the same order. Further, if $o(s) > 2$, then, for each $s' \in S$, as $S = S^{-1}$, we have that $(s')^{-1} \neq s'$ is also in S. It follows that $|S| = \mathsf{val}(\Gamma)$ is even, which is a contradiction. Hence, s is an involution.

Recall that l is even and set $l = 2l'$. Since m is odd, R has a cyclic Sylow 2-subgroup and s is conjugate to $b^{l'}$ in R. Since $\mathsf{Cay}(R, S) \cong \mathsf{Cay}(R, S^{\phi})$ for each $\phi \in \mathsf{Aut}(R)$, up to isomorphism, we may assume that $s = b^{l'}$. By Lemma 3.4, σ has the form

$$\sigma : a \to a^i,\ b \to a^j b \quad \text{where } (i, m) = 1 \text{ and } 0 \leq j \leq m - 1.$$

Noting that $\sigma(s) = (\sigma(b))^{l'} = (a^j b)^{l'} = a^{j'} b^{l'}$ for some integer j', we conclude that $\sigma^k(s) \in \langle a, b^{l'}\rangle$ for each integer k. Therefore, if $l' > 1$, we have that $\langle S\rangle \subsetneq \langle a, b^{l'}\rangle \cong \mathbb{Z}_m : \mathbb{Z}_2$ is a proper subgroup of R, which contradicts the connectivity of Γ. Thus, $l' = 1$ and $l = 2$. Now, $R \cong \mathbb{Z}_m : \mathbb{Z}_2$ is a Frobenius group, so $R \cong \mathrm{D}_{2m}$ and, by Lemma 2.6, Γ satisfies part (2) of Theorem 1.2. □

We finally consider the case where R is regular on $V\Gamma$ but not normal in A. Let $K = \mathsf{core}_{\mathrm{A}}(R)$, so that $K < R$.

Note that a Cayley graph may be expressed as a Cayley graph of different groups. It can be a normal Cayley graph for one of them, but not for another. A simple example is the complete graph K_4, which is a normal Cayley graph of $\mathbb{Z}_2^2$ and a Cayley graph (but not normal) of $\mathbb{Z}_4$.

Lemma 4.3. *Suppose that K has exactly two orbits on $V\Gamma$. Then Γ and R satisfy part (3) of Theorem 1.2.*

Proof. By assumption, K has index two in R. Since $\langle a\rangle/(K \cap \langle a\rangle) \cong K\langle a\rangle/K \leq R/K \cong \mathbb{Z}_2$ and $|\langle a\rangle| = m$ is odd, we conclude that $\langle a\rangle \lhd K \cong \mathbb{Z}_m : \mathbb{Z}_{l/2}$. Since $(m, l/2) = 1$, $\langle a\rangle$ is a characteristic subgroup of K and hence a normal subgroup of A.

Assume that $K = \langle a\rangle$. By Lemma 3.3, Γ is a Cayley graph of a dihedral group $H := \langle a\rangle : \langle c\rangle \cong \mathrm{D}_{2m}$, where $c \in \mathrm{A}$ is an involution such that $a^c = a^{-1}$. Then Lemma 2.6 implies that $H \lhd \mathrm{A}$ and $m = p^t p_1^{e_1} p_2^{e_2} \cdots p_s^{e_s} \geq 13$, where $t \leq 1$, $s \geq 1$ and $p_1, p_2, \ldots, p_s$ are distinct primes such that $p \mid p_i - 1$ for each i. Let x be an involution of R. Then $\langle x\rangle \cong \langle c\rangle \cong \mathbb{Z}_2$ are Sylow 2-subgroups of A, so x is conjugate to c in A, say $x = c^y$ for some $y \in \mathrm{A}$. It follows that $R = \langle a\rangle : \langle x\rangle = \langle a\rangle^y : \langle c^y\rangle = H^y = H \lhd \mathrm{A}$, which is a contradiction as $K = \mathsf{core}_{\mathrm{A}}(R) < R$.

Therefore, $\langle a\rangle < K$ has at least four orbits on $V\Gamma$. As $\langle a\rangle \lhd \mathrm{A}$, by Theorem 2.5, $\Gamma_{\langle a\rangle}$ is an $\mathrm{A}/\langle a\rangle$-arc-regular Cayley graph of $R/\langle a\rangle \cong \mathbb{Z}_l$ with $l \geq 4$. Since $K/\langle a\rangle \cong \mathbb{Z}_{l/2}$ is normal in $\mathrm{A}/\langle a\rangle$ and $R/\langle a\rangle \cong \mathbb{Z}_l$ is not normal in $\mathrm{A}/\langle a\rangle$, $\mathsf{core}_{\mathrm{A}/\langle a\rangle}(R/\langle a\rangle) \cong \mathbb{Z}_{l/2}$. Then, by Lemma 3.2, we have $l = 2p$, $R \cong \mathbb{Z}_m : \mathbb{Z}_{2p}$ and $\Gamma_{\langle a\rangle} \cong \mathsf{K}_{p,p}$. Further, by Theorem 2.5, Γ is a normal $\mathbb{Z}_m$-cover of $\mathsf{K}_{p,p}$. Note that p does not divide m. By [23, Theorem 1.1], Γ is a normal Cayley graph of a dihedral group D_{2pm} and hence satisfies part (3) of Theorem 1.2, by Lemma 2.6. □

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Proof of Theorem 1.2. If R is not regular on $V\Gamma$, by Lemma 4.1, part (1) of Theorem 1.2 holds.

Suppose that R is regular on $V\Gamma$. Let $\mathrm{A} = \mathsf{Aut}\,\Gamma$. By Lemma 3.1, R is not core-free in A, that is, $K := \mathsf{core}_{\mathrm{A}}(R) \neq 1$.

Assume that K has at least three orbits on $V\Gamma$. By Theorem 2.5, Γ_K is an A/K-arc-regular Cayley graph of R/K with valency p. But R/K is core-free in A/K, and R/K is metacyclic by Lemma 3.1, which is a contradiction. Hence, K has at most two orbits on $V\Gamma$.

If K is transitive on $V\Gamma$, then $R = K \lhd \mathrm{A}$, by Lemma 4.2, and part (2) of Theorem 1.2 holds. If K has two orbits on $V\Gamma$, by Lemma 4.3, part (3) of Theorem 1.2 holds. Finally, the last statement in Theorem 1.2 holds by [7, Theorem 3.1]. □

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