Canad. Math. Bull. Vol. **20** (1), 1976

ON PRIME GOLDIE-LIKE QUADRATIC JORDAN MATRIX ALGEBRAS[1]

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In [1] and [2], there was given a characterization for linear Jordan matrix algebras whose coordinatizing ring is *-prime Goldie or a Cayley–Dickson ring (C–D ring). If one considers the corresponding question in the more general setting of quadratic Jordan algebra as defined by McCrimmon in [11], then the result is similar. In this latter case the ample quadratic Jordan algebras, as studied by Montgomery in [12] and [13], are brought into play. Here we tie these concepts together in extending [2, Theorem 0] to its quadratic Jordan generalization. This paper also shortens some arguments of [2] and indicates some corrections necessary in [2].

The corresponding setting is that of the Coordinatization Theorem for quadratic Jordan algebras [11].

Let K be a commutative associative ring with 1 and R an alternative K-algebra with 1 and involution *. A quadratic Jordan subalgebra T of the symmetric elements of the nucleus of R which contains the norm xx^* and the trace $x+x^*$ of each element x in R is said to be ample. An ample sub-algebra T is said to be closed ample if $x^*Tx \subseteq T$ for each $x \in R$. Let R_0 be a closed ample subalgebra containing 1 and let $a = \text{diag}(a_1, \ldots, a_n) \in R_n$ with a_i invertible elements contained in R_0. Let γ_a be the diagonal involution on R_n given by $y^{\gamma_a} = a^{-1}y^{*t}a$ where y^{*t} denotes the taking of conjugate transpose. Let $J = H(R_n, R_0, \gamma_a)$ be the set of those γ_a-symmetric elements $y = y^{\gamma_a}$ whose diagonal entries y_{ii} lie in $a_i^{-1}R_0 = R_0a_i$. If R_0 is the set of all symmetric elements of R which are contained in the nucleus, then we write $H(R_n, \gamma_a)$. $J_{ii} = \{x[ii] = xe_{ii} : x \in a_i^{-1}R_0\}$ and $J_{ij} = \{x[ij] = xe_{ij} + a_j^{-1}x^*a_i : x \in R\}$, $i \neq j$, are the Pierce components of J where $\{e_{ij}\}$ is a standard set of matrix units. For a subset A of J we use A_{ij} to denote $A \cap J_{ij}$. A quadratic (inner) ideal Q is said to be ij-quadratic if $Q_{ij} \neq 0$.

We will use the definitions of: Jordan ring of quotients; common multiple property (cmp); and the Goldie-like conditions as stated in [2].

First we point out that condition (iii) of [2, Theorem 0] should have added to it the statement that $(R, *)$ is a subring of $(R', *)$. This is used, for example, in the proof of Lemma 2. It is not necessary if $n \geq 3$.

Received by the editors March 12, 1976.

(1) This paper was prepared while the author was at the 1975 Summer Research Institute at Dalhousie University and held NRC grant A8471.

https://doi.org/10.4153/CMB-1977-008-0 Published online by Cambridge University Press

Our aim here is to point out the changes necessary in [1] and [2] in order to obtain

THEOREM 0′. *Let* $J = H(R_n, R_0, \gamma_a)n \geq 2$ *be a diagonal Jordan matrix algebra as described above. Then the following are equivalent*

(i) *J is a prime Goldie-like*

(ii) *R is *-prime Goldie or* $n = 2, 3$ *and R is a* C–D *ring with its symmetric elements in its nucleus*

(iii) *J has* cmp *and a Jordan ring of quotients* $J' = H(R'_n, R'_0, \gamma_a)$ *where* $(R, *)$ *is a subring of* $(R', *)$ *and R′ is an involution simple Artinian ring on* $n = 2, 3$ *and R′ is a* C–D *algebra with standard involution.* R'_0 *is a Jordan ring of quotients for* R_0.

Implied in the use of the symbol $H(R'_n, R'_0, \gamma_a)$ in (iii) above is that R'_0 is a closed ample subalgebra of R'. If R is a C–D ring then it is easy to check that the Jordan ring of quotients R'_0, contained in the nucleus of the C–D algebra R' corresponding to R, is closed ample. By the Herstein–Kleinfeld–Osborn–McCrimmon Theorem [8, p. 3.32], $R'_0 = Z'$, the center of R'.

If R is *-prime Goldie then it follows from Montgomery's Theorem as formulated in [4, Theorem 2] that the Jordan subring of quotients R'_0, contained in the involution simple Artinian ring corresponding to R, is closed ample.

In [1] and [2], the equivalence of Theorem 0′ was shown when $\frac{1}{2} \in R$. In [1], the use of $\frac{1}{2}$ was indirect in that it was not used as such but the results of [7] were used and this assumption was made there. Thus if one uses [8] for his reference this need of $\frac{1}{2}$ is essentially eliminated. The main thrust of the arguments of [1] involved J_{ij} $i \neq j$ which are of the same form regardless of characteristic. One must change the construction of quadratic ideals given on [1, p. 89–90] so that R_0 plays its role. After making such natural changes one obtains the equivalence of (i) and (ii).

We now show that if R satisfies (ii) then J has a ring of quotients as described in (iii).

LEMMA 1. *Let R be an associative ring with involution * and* $1 \in J$ *a quadratic Jordan subalgebra of* $H(R_n, \gamma_a)$. *Then J is a closed ample subalgebra of* $H(R_n, \gamma_a)$ *if and only if J has the form* $H(R_n, R_0, \gamma_a)$.

Proof. This can be proven by straightforward calculation similar to those given in the proof of the theorem characterizing outer ideals [8, Theorem 2, p. 2.18].

LEMMA 2. *Let* $J = H(R_n, R_0, \gamma_a)n \geq 2$ *be prime Goldie-like. Then* $J' = H(R'_n, R'_0, \gamma_a)$ *is a Jordan ring of quotients for J where R′ is the ring of quotients for R and* $R'_0 \subseteq R'$ *is the Jordan ring of quotients for* R_0.

Proof. Suppose R is a C–D ring. Let Z be the center of R, Z' its field of quotients and R' the C–D algebra $Z'R$. The general form of an element of R' is given by $z^{-1}t$ where z is a norm in R and t is arbitrary in R so that all norms of R' are in its center. Thus, by [8, Theorem 8], * on R' is the standard involution. As previously pointed out, $R'_0 = Z'$ the center of R'. Hence $\{U_{ra}^{-1}(x): x \in J$ and $r = \text{diag}(r_0, \ldots, r_0)$ with $0 \neq r_0 \in R_0\} = H(R'_n, \gamma_a)$. Clearly this is a Jordan ring of quotient for J.

Now suppose that R is *-prime Goldie. By [13, Corollary 2] applied to the involution prime Goldie ring R_n, J has a Jordan ring of quotients J' which is an ample Jordan subalgebra of $H(R'_n, \gamma_a)$ where R' is the associative ring of quotients of R. It was shown [4, Theorem 2] that J' is closed ample and hence, by Lemma 1, it has the form $H(R'_n, R'_0, \gamma_a)$. Using the common multiple property of the associative ring R, it is easy to show that each element of J' has the form $\beta^{-1}x$ where $\beta = \text{diag}\,(\alpha a_1, \ldots, \alpha a_n)$ and α is a norm in R. Thus $\beta^{-1}x = \beta^{-1}(x\beta)\beta^{-1}$ and $x\beta \in J$. From this we see that the ith diagonal entry of any element in J' is in $\{a_i^{-1}\alpha^{-1}t\alpha^{-1}: \alpha, t \in R_0, \alpha \text{ regular}\}$.

By [13, Corollary 2], R_0 has a Jordan ring of quotients in R' and we have shown above that it is R'_0.

We now show that if (ii) is satisfied the J has cmp.

Let R be a C–D ring and R' its C–D algebra. Since U_x is a bijective Z'-linear transformation ($Z' =$ center R) on J', for any choice r, w, z $y \in J$, w and x regular, $r = \text{diag}(r_0, \ldots, r_0)$ with $0 \neq r_0 \in R_0$ there exist r', $v \in J$, $r' = \text{diag}(r_1, \ldots, r_1)$ with $0 \neq r_1 \in R_0$ such that $U_x(U_{r'}^{-1}(v)) = U_y(U_r^{-1}(w))$. The cmp follows if $U_y(U_r^{-1}(w)) \neq 0$ for some choice of r and w.

It is easy to check that R' has a Z' basis of regular elements and if t is invertible in R' then $t[ij](+1[k,k], i \neq k \neq j$ if $n = 3)$ is invertible in J'. Also, if $a \neq -b$ are non-zero elements of Z' then $a[ii] = ((a+b)[ii] + \sum_{j \neq i} b[jj]) - (\sum_{j=1}^{n} b[jj])$ is expressed as the difference of two invertible elements. If the center of R' has only two elements, then $u = 1[ii] + 1[ij](+1[kk], i \neq k = j$ if $n = 3)$ and $s = 1[ij](+1[kk], i \neq k \neq j$ if $n = 3)$ are invertible elements such that $1[ii] = s + u$. In any case, every element of J' may be expressed as the sum of invertible elements, and each invertible element of J' has the form $U_r^{-1}(w)$ where r and w are as described above. Since $U_y \neq 0$, and every element is the sum of invertible elements, there is some invertible element $U_r^{-1}(w)$ such that $U_y(U_r^{-1}(w)) \neq 0$ and hence we have the cmp in this case.

This argument is similar to that given in the proof of [2, Theorem 1].

When R is *-prime Goldie the proof analogous to that given in [2] would involve a "twisting" of $J' = H(R'_n, R'_0, \gamma_a)$ to obtain a capacity. Even after this is done, there seems to be other complications so that here we offer an alternate proof.

Let R be *-prime Goldie and R' its ring of quotients. Then $R' \simeq D_m$ where

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D is a division ring or $\Delta \oplus \Delta^0$ and Δ is a division ring. Let $\{f_{ij}\}$ be a standard set of matrix units for D_m. By cmp on R, $f_{ij} = d^{-1}w_{ij}d^{-1}$ for w_{ij}, $d \in R$ where d is a fixed *-norm. We may express w_{ij} as the sum of regular elements.

$$w_{ii} = \left(w_{ii} + w_{ij} + w_{ji} + \sum_{k \neq i,j} w_{kk}\right) - \left(w_{ij} + w_{ji} \times \sum_{k=i,j} w_{kk}\right)$$

$$w_{ij} = (w_{ij} + 1) - 1.$$

Apply left and right multiplication by d^{-1} to see that the elements in parenthesis are regular.

LEMMA 3. *Let R be *-prime Goldie. If $t \in R$ is such that $txt = 0$ for each regular element of R then $t = 0$.*

Proof. Let R, D, f_{ij}, w_{ij}, and d be as above. Consider $R \subseteq R' = D_m$. Now if $txt = 0$ for each regular element $x \in R$ then $tw_{ij}t = 0$ for $i, j = 1, \ldots, m$. But then $0 = dt\,dd^{-1}w_{ij}d^{-1}\,dtd = dt\,df_{ij}\,dtd$, $i, j = 1, \ldots, m$. If we express dtd as $\sum t_{ij}f_{ij}$ with $t_{ij} \in D$ then we see that $t_{ij}^2 = 0$, $i, j = 1, \ldots, m$. But in D this implies $t_{ij} = 0$, which in turn implies $dtd = 0$ and hence $t = 0$.

By using cmp on R_n, the problem of showing that $J = H(R_n, R_0, \gamma_a)$ has cmp can be reduced to showing that $0 \neq y \in J$ implies there is some regular $z \in J$ such that $U_y(z) \neq 0$ (see proof of [2, Theorem 1]).

Let $0 \neq y \in J$ be such that $U_y(z) = 0$ for all z regular in J. Set $y = \sum_1^n y_{ii}[ii] + \sum_{i \leq j < k \leq n} y_{jk}[jk]$. Let x be a regular element of R and $r_0 \in R_0 a_i$. Then $z = x[ij] + \sum_{k=i,j} 1[kk]$ and $z' = r_0[ii] + z$ are regular in J since they are invertible in R'_n. Thus since $r_0[ii]$ is the sum of regular elements $-z$ and z' we have $U_y(r_0[ii]) = 0$ and hence $y_{ii}r_0y_{ii} = 0$ for each $r_0 \in R_0 a_i$ so that, by [5, Theorem 7′], $y_{ii} = 0$ for $i = 1, \ldots, n$. Now, since $r_0[kk]$ is the sum of regular elements $1[kk]$ is and the same is true of $x[ij]$. Thus $U_y(x[ij]) = 0$ so that $y_{ij}a_j^{-1}x^*a_iy_{ij} = 0$ for all regular x in R. Thus $y_{ij} = 0$, by Lemma 3.

We have now proven

LEMMA 4. *If $J = H(R_n, R_0, \gamma_a)n \geq 2$ is prime Goldie-like then J has* cmp.

Lemmas 2 and 4 as well as the results of [1] (after the changes indicated earlier) give us the implications (i) $\Leftrightarrow$ (ii) $\Rightarrow$ (iii). To complete the proof of Theorem 0′, we now show (iii) $\Rightarrow$ (ii). [2, Lemma 2] gives us this implication in the case of R' being a C–D algebra. Therefore, we must show that if R' is an involution simple Artinian ring then R is *-prime Goldie. By the proof of [2, Theorem 3]. R satisfies ACC on left (right) annihilator ideals and by [2, Lemma 4], R is semi-prime. For the remainder of this paper we assume: R is a semi-prime associative ring with involution * and satisfies ACC on left annihilator ideals; $(R, *)$ is a subring of $(R', *)$ a *-simple Artinian ring; and $J' = H(R'_n, R'_0, \gamma_a)$ (resp. R'_0) is a Jordan ring of quotients for $J = H(R_n, R_0, \gamma_a)$ (resp. R_0).

First we show that R is *-prime. The proof we offer here is shorter than that of [2] and it does not use the Second Structure Theorem.

LEMMA 5. *R is *-prime.*

Proof. Suppose that R is not *-prime. Then R contains non-zero *-ideals A and B such that A is both the left and right annihilator for B as is B for A. By semiprimeness of R, $A \cap B = 0$. For $\alpha \in A_n \cap J$, $\beta \in B_n \cap J$, and $y \in J'$, there exist $u, w \in J$, u regular such that $u^{-1}wu^{-1} = \alpha y\beta + \beta y\alpha \in (A_nJ'B_n + B_nJ'A_n) \cap J$. Since $w \in (A_nJ'B_n + B_nJ'A_n) \cap J$, $A_nwA_n = 0 = B_nwB_n$ so that $w \in A_n \cap B_n = 0$. Thus $t = \alpha y\beta$ is γ_a-skew. Thus $ty_1t \in J'$ for $y_1 \in J'$ and hence there exist $u_1, w_1 \in J$, u_1 regular, such that $u_1^{-1}w_1u_1^{-1} = ty_1t \in A_nR'_nB_n \cap J'$. Thus $w_1 \in A_nR'_nB_n \cap J$ so that $w_1A_n = 0 = B_nw_1$ and hence $w_1 \in A_n \cap B_n = 0$. Therefore, $ty_1t = 0$. That is for any $\alpha \in A_n \cap J$, $\beta \in B_n \cap J$, $y, y_1 \in J'$ we have $\alpha y\beta y_1\alpha y\beta = 0$.

By [12, Lemma 3.1], $A \cap R_0 \neq 0 \neq B \cap R_0$. Pick $0 \neq \alpha_1 \in A \cap R_0$, $0 \neq \beta_1 \in B \cap R_0$. Then $\alpha_1[12] \in A_n \cap J$ and $\beta_1[12] \in B_n \cap J$ so that

$$\begin{aligned} 0 &= \alpha_1[12]x[21]\beta_1[12]x'[21]\alpha_1[12]x[21]\beta_1[12] \\ &= \alpha_1 x\beta_1 x'\alpha_1 x\beta_1[12] \end{aligned}$$

where x, x' are arbitrary in R'. Therefore, by the semiprimeness of R', $\alpha_1x\beta_1 = 0$. $H(R') = \{x \in R : x^* = x\}$ is a prime Jordan algebra since R' is *-prime [5, Corollary, p. 162] and by [5, Theorem 7] $\alpha_1 = 0$ or $\beta_1 = 0$, contrary to choice.

Using the lemmas of [2] we outline the proof that R is Goldie. (Note the corrected definition of τ_i.)

LEMMA 6. *R is *-prime Goldie.*

Proof. Since R is *-prime, R contains a prime ideal P such that $P \cap P^* = 0$ [10, or 3]. Suppose that R/P contains both an infinite direct sum of left ideals and an infinite direct sum of right ideals. Then R contains left ideals $\lambda_i \supset P$ such that $\sum_1^\infty (\lambda_i/P)$ is an infinite direct sum and similarly for right ideals $\rho_i \supset P$. Define $\tau_i = \lambda_i$ if $P = 0$ and if $P \neq 0$ set

$$\tau_i = \begin{cases} P^*\lambda_i & \text{if } i \text{ is even} \\ P\rho_i^* & \text{if } i \text{ is odd.} \end{cases}$$

Then $\sum \tau_i$ is an infinite direct sum of left ideals of R. Form $L_i = (\tau_i)_n \subseteq R_n$.

Since R is involution simple Artinian $R'_n \simeq D_{mn}$ where D is a division ring or $\Delta \oplus \Delta^0$ with Δ a division ring. Since the right submodule $(\sum_1^\infty L_i)D$ of the right D-module R'_n has a "finite basis" in $\sum_1^\infty L_i$, for some k there are $x_i \in L_i$, $i = 1, \ldots, k$ such that $\sum_{k+1}^\infty L_i \subseteq \sum_1^k x_iD$. Consider $x_0 = \sum_1^k x_i$. Suppose $y \in R$ is such that $yx_0 = 0$. Then $yx_i = 0$ for $i = 1, \ldots, k$ since $\sum_1^k L_i$ is a direct sum. Thus

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$y(\sum_{k+1}^{\infty} L_i)=0$ so that $yL_i=0$ for $i\geq k+1$. This implies $y\in(P\cap P^*)_n=0$ since $L_{2k}\subseteq(P^*)_n$ and $L_{2k+1}\subseteq P_n$. Therefore x_0 is not a right divisor of zero and hence by [2, Lemma 10] x_0 is regular in R_n. This contradicts [2, Lemma 11] which says that $\sum_i^{\infty} L_i$ contains no regular element.

We have now shown that R/P does not contain an infinite direct sum of left ideals and an infinite direct sum of right ideals. That is, R/P is either left or right Goldie. Therefore, we may assume that R/P is left Goldie and $P\neq 0$. If R/P is also right Goldie then R is Goldie as can be seen using the subdirect sum embedding of R in $R/P\oplus R/P^*$. Set $R_1=R/P$. If R is not Goldie, then $T=(R_1)_n$ has an infinite direct sum of right ideals, say $\sum\rho_i'$, which are arrived at from an infinite direct sum of right ideals in R_1.

Now R is a subdirect sum of $R_1\oplus R_1^0$. Letting Δ_l be the ring of left quotients for R_1, we have $R_n\subseteq(R_1)_n\oplus(R_1^0)_n\subseteq(\Delta\oplus\Delta^0)_{\text{ln}}=U$ and * transpose extends to U as γ_1, the exchange involution followed by transpose.

Let $T=(R_1)_n\oplus\{0\}\subseteq S=\Delta_{\text{ln}}\oplus 0\subseteq U$ and $\{f_{hk}\}$ be the standard set of matrix units for U so that f_{kk} is the matrix or ordered pairs with $(0,0)$ in every entry except the kkth entry and here $(1_\Delta, 1_\Delta^0)$ appears where 1_Δ (resp. 1_Δ^0) is the identity of Δ (resp. Δ^0). By [2, Lemma 15], $I=f_{kk}Sf_{kk}\cap T$ is a right order in $f_{kk}Sf_{kk}$. Since $g:\Delta\to f_{kk}Sf_{kk}$, $\delta\to\text{diag}((\delta,0),\ldots,(\delta,0))f_{kk}$ is an isomorphism, every element in Δ has the form $g^{-1}(\alpha\beta^{-1})$ for $\alpha,\beta\in I$, $\beta\neq 0$. By finite dimensionality of Δ_{ln} over Δ we may choose $x_i\in\rho_i'$, $i=1,\ldots,q$ such $\sum_{q+1}^{\infty}\rho_i'\subseteq\sum_1^q x_i\Delta$. Also, we may choose $x\in\rho_{q+1}$ and k such that $xf_{kk}\neq 0$. Using the right common multiple property of I, there are α_i, $\beta\in I$, $\beta\neq 0$ such that

$$x=\sum_1^q x_i g^{-1}(\alpha_i\beta^{-1}).$$

This implies $0\neq x\beta=\sum x_i\alpha_i\in(\rho_{q+1}'\cap\sum_1^q\rho_i')=0$. This contradiction completes the proof.

References

1. Daniel J. Britten, *Goldie-like Conditions on Jordan Matrix Rings*, Trans. A.M.S. **192** (1974), 87–98.

2. ——, *Prime Goldie-like Jordan Matrix Rings and the Common Multiple Property*, Communications in Alg. **3** (1975), 365–389.

3. ——, *On Cayley–Dickson Rings*, Canad. Math. Bull., **17** (1975), 625–627.

4. ——, *On Semiprime Ample Jordan Rings $J\subseteq H$ with Chain Condition* (to appear).

5. T. S. Erickson and S. Montgomery, *The Prime Radical in Special Jordan Rings*, Trans. A.M.S. **156** (1971) 155–164.

6. N. Jacobson, *Structure of Rings*, Amer. Math. Soc. Colloq. Publ. Vol. **37,** Amer. Math/Soc., Providence, R.I., 1964.

7. ——, *Structure and Representation of Jordan Algebras*, Amer. Math. Soc. Colloq. Publ. Vol. **39,** Amer. Math. Soc. Providence R.I., 1968.

8. ——, *Lectures on Quadratic Jordan Algebras*, Tata Institute Lecture Notes, Bombay, 1970.

9. E. Kleinfeld, *Primitive Alternative Rings and Semi-Simplicity*, Amer. J. Math. **77** (1955), 725–730.

10. W. S. Martindale, *Rings with involution and polynomial identities*, Journal of Algebra, **2** (1969), pp. 186–194.

11. K. McCrimmon, *A General Theory of Jordan Rings*, Proc. Nat. Acad. Sci. U.S.A., **56** (1966), 1072–1079.

12. Susan Montgomery, *Rings of Quotients for a Class of Special Jordan Rings*, J. Algebra **31** (1974), 325–331.

13. ——, *Chain Condition on Symmetric Elements*, Proc. A.M.S. **46** (1974), 325–331.

14. C. Tsai, *The Prime Radical in Jordan Ring*, Proc. A.M.S. **19** (1968), 1171–1175.

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