TOTALLY REAL SURFACES OF THE SIX-DIMENSIONAL SPHERE

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1. Introduction. An almost Hermitian manifold $(\bar{M}, J, g)$ with Riemannian connection $\bar{\nabla}$ is called *nearly Kaehlerian* if $(\bar{\nabla}_X J)X = 0$ for any $X \in \mathscr{X}(\bar{M})$. The typical example is the sphere S^6. The nearly Kaehlerian structure J for S^6 is constructed in a natural way by making use of Cayley division algebra [**3**]. It is because of this nearly Kaehler, non-Kaehler, structure that S^6 has attracted attention. Different classes of submanifolds of S^6 have been considered by A. Gray [**4**], K. Sekigawa [**5**] and N. Ejiri [**2**]. In this paper we study 2-dimensional totally real submanifolds of S^6. These are submanifolds with the property that for every $x \in M$, $J(T_xM)$ belongs to the normal bundle ν. For this class we have obtained the following result.

THEOREM. *Let M be a complete totally real 2-dimensional submanifold of S^6. Then M is flat and minimal.*

2. Preliminaries. Let C_+ be the set of all purely imaginary Cayley numbers. Then C_+ can be viewed as a 7-dimensional linear subspace $\mathbb{R}^7$ of $\mathbb{R}^8$. Consider the unit hypersurface which is centred at the origin,

$$S^6(1) = \{x \in C_+ : \langle x, x \rangle = 1\}.$$

The tangent space T_xS^6 of $S^6(1)$ at a point x may be identified with the affine subspace of C_+ which is orthogonal to x.

On $S^6(1)$ define a $(1,1)$-tensor field J by putting

$$J_xU = x \times U,$$

where the above product is defined as in [**2**] for $x \in S^6(1)$ and $U \in T_xS^6$. This tensor field J determines an almost complex structure (i.e. $J^2 = -\mathrm{Id}$) on $S^6(1)$. The compact simple Lie group of automorphisms G_2 acts transitively on $S^6(1)$ [**3**]. Now let G be the $(2,1)$-tensor field on $S^6(1)$ defined by

$$G(X, Y) = (\bar{\nabla}_X J)Y \tag{2.1}$$

where $\bar{\nabla}$ is the Levi-Civita connection on $S^6(1)$ and $X, Y \in \mathscr{X}(S^6)$. The vector field G possesses the following properties ([**5**], [**4**]);

$$G(X, X) = 0, \tag{2.2}$$

$$G(X, Y) = -G(Y, X), \tag{2.3}$$

$$G(X, JY) = -JG(X, Y), \tag{2.4}$$

$$g(G(X, Y), Z) = -g(G(X, Z), Y), \tag{2.5}$$

$$g(G(X, Y), G(Z, W)) = g(X, Z)g(Y, W) - g(X, W)g(Z, Y) + g(JX, Z)g(Y, JW) - g(JX, W)g(Y, JZ) \tag{2.6}$$

where $X, Y, Z, W \in \mathscr{X}(S^6)$ and g is the Hermitian metric on $S^6(1)$. Note that (2.2) means that S^6 is nearly Kaehler with respect to J.

Let M be a submanifold of $S^6(1)$ and denote by ∇, $\bar{\nabla}$ and $\nabla^{\perp}$ the Riemannian connections on M, S^6 and the normal bundle respectively. These Riemannian connections are related by the Gauss formula and Weingarten formula

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.7}$$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^{\perp} N, \tag{2.8}$$

where N is a local normal vector field on M in $S^6(1)$ and $X, Y \in \mathscr{X}(M)$, and where $h(X, Y)$ and $A_N X$ are the second fundamental forms which are related by

$$g(h(X, Y), N) = g(A_N X, Y).$$

For M in $S^6(1)$ the equation of Codazzi is given by

$$(\bar{\nabla}_X h)(Y, Z) = (\bar{\nabla}_Y h)(X, Z), \tag{2.9}$$

where $(\bar{\nabla}_X h)(Y, Z) = \bar{\nabla}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z)$.

3. Totally real submanifolds of $S^6(1)$. We consider 2-dimensional totally real submanifolds of $S^6(1)$; so in the following M always denotes a 2-dimensional totally real submanifold of $S^6(1)$. For M, equations (2.7), (2.8), and (2.9) hold. Assume that X and Y are unit tangent basis vectors for the tangent space T_xM. The normal bundle ν splits as $\nu = \mu \oplus J(TM)$ where μ is an invariant subbundle of ν i.e. $J\mu = \mu$. Therefore the normal bundle ν is spanned by an orthonormal frame field of the form $\{JX, JY, N, JN\}$ for some unit vector field N in μ.

Now using (2.5) and (2.2) we get

$$g(G(X, Y), X) = 0. \tag{3.1}$$

Also, using (2.3), (2.5) and (2.2), we have

$$g(G(X, Y), Y) = 0. \tag{3.2}$$

From (2.5), (2.4) and (2.2) we get

$$g(G(X, Y), JX) = 0. \tag{3.3}$$

Switching the role of X and Y in (3.3) and using (2.3) we also get

$$g(G(X, Y), JY) = 0. \tag{3.4}$$

Equations (3.1), (3.2), (3.3) and (3.4) imply that $G(X, Y) \in \mu$.

From (2.8) with $N = JY$ we have

$$J\bar{\nabla}_X Y + (\bar{\nabla}_X J)Y = -A_{JY}X + \nabla_X^{\perp} JY. \tag{3.5}$$

Using (2.7) and (2.1) in (3.5) we get

$$Jh(X, Y) = -A_{JY}X + \nabla_X^{\perp} JY - G(X, Y) - J\nabla_X Y. \tag{3.6}$$

Assume that the orthonormal frame field $\{X, Y\}$ for TM is chosen in such a way that $\nabla_X X = 0$. Such a choice is possible since M is complete and therefore such a frame exists [**6**, p. 456]. To choose the field Y orthonormal to X one can just apply the Gram–Schmidt process to any frame field orthogonal to X. For the frame field $\{X, Y\}$ we have

$$g(\nabla^{\perp}_X JY, JY) = 0, \tag{3.7}$$

$$g(\nabla^{\perp}_X JY, JX) = 0. \tag{3.8}$$

(3.7) is trivial since the frame field is orthonormal; (3.8) follows from $g(JX, JY) = 0$, (2.8), (2.2), with the help of $\nabla_X X = 0$, and the fact that g is Hermitian.

From (3.7) and (3.8) we conclude that $\nabla^{\perp}_X JY$ belongs to μ. Since the normal bundle ν splits as $\nu = \mu \oplus J(TM)$, the vector $Jh(X, Y) \in \mu \oplus (TM)$. Hence the vector $-A_{JY}X + \nabla^{\perp}_X JY - G(X, Y) - J\nabla_X Y$ in the right hand side of (3.6) belongs to $\mu \oplus (TM)$. Since we have shown that both $G(X, Y)$ and $\nabla^{\perp}_X JY$ belong to μ, it follows that

$$\nabla_X Y = 0. \tag{3.9}$$

Switching X and Y in (3.9) we also get

$$\nabla_Y X = 0. \tag{3.10}$$

Using (3.10) and the fact that the frame is orthonormal we get

$$\langle \nabla_Y Y, Y \rangle = 0 \tag{3.11}$$

and

$$\langle \nabla_Y Y, X \rangle = 0. \tag{3.12}$$

From (3.11) and (3.12) it follows that

$$\nabla_Y Y = 0. \tag{3.13}$$

Note that the sectional curvature K of M is given by

$$K(X, Y) = R(X, Y, Y, X) = g(\nabla_X \nabla_Y Y - \nabla_Y \nabla_X Y - \nabla_{[X,Y]} Y, X).$$

Using (3.9), (3.10) and (3.13) in this equation we get $K(X, Y) = 0$ i.e. M is flat.

4. Proof of the theorem. In order to prove the theorem we need the following lemma.

LEMMA. *Let $X, Y \in \mathscr{X}(M)$. Then $h(X, Y) \in J(TM)$.*

Proof. For $Z \in \mathscr{X}(M)$ we have

$$\begin{aligned} 2g(A_{JX}Y, Z) &= g(h(Y, Z), JX) + g(h(Y, Z), JX) \\ &= g(\bar{\nabla}_Y Z, JX) + g(\bar{\nabla}_Z Y, JX) \\ &= -g(J(\bar{\nabla}_Y Z + \bar{\nabla}_Z Y), X). \end{aligned}$$

Using (2.1) and (2.3) in the equation $\bar{\nabla}_Y JZ = J\bar{\nabla}_Y Z + (\bar{\nabla}_Y J)Z$ we have

$$J(\bar{\nabla}_Y Z + \bar{\nabla}_Z Y) = \bar{\nabla}_Y JZ + \bar{\nabla}_Z JY.$$

Therefore,

$$\begin{aligned} 2g(A_{JX}Y, Z) &= -g(\bar{\nabla}_Y JZ, X) - g(\bar{\nabla}_Z JY, X) \\ &= g(JZ, \bar{\nabla}_Y X) + g(A_{JY}Z, X) \end{aligned}$$

i.e.

$$2g(A_{JX}Y, Z) = -g(J\bar{\nabla}_Y X, Z) + g(A_{JY}X, Z).$$

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Since $Z \in \mathscr{X}(M)$ is arbitrary, we have

$$2A_{JX}Y = A_{JY}X - J\bar{\nabla}_Y X = A_{JY} - Jh(X, Y) \tag{4.1}$$

where we have used $\nabla_Y X = 0$ in the last equality. Similarly we have

$$2A_{JY}X = A_{JX}Y - Jh(Y, X). \tag{4.2}$$

Subtracting (4.2) from (4.1) we get

$$3(A_{JX}Y - A_{JY}X) = 0.$$

Thus

$$A_{JX}Y = A_{JY}X. \tag{4.3}$$

Using (4.3) in (4.1) we have

$$A_{JX}Y = -Jh(X, Y). \tag{4.4}$$

It follows from (4.4) that $h(X, Y) \in J(TM)$.

We now start the proof of the theorem. In Section 3 we proved that M is flat. We know from the above lemma that $h \in J(TM)$. Considering $\{X, Y\}$ as an orthonormal frame field on M, we can write

$$h(X, X) \oplus aJX + bJY \quad \text{and} \quad h(Y, Y) = cJX + dJY \tag{4.5}$$

for some smooth functions a, b, c, d on M. Using (4.3) we have

$$g(A_{JX}Y, X) = g(A_{JY}X, X) \quad \text{and} \quad g(A_{JY}X, Y) = g(A_{JX}Y, Y)$$

which imply that

$$g(h(X, Y), JX) = g(h(X, X), JY) \quad \text{and} \quad g(h(X, Y), JY) = g(h(Y, Y), JX). \tag{4.6}$$

From equations (4.5) and (4.6) we can write

$$h(X, Y) = bJX + cJY. \tag{4.7}$$

Since M is flat and the ambient space is of constant curvature, then the Codazzi equation (2.9) becomes

$$\nabla^{\perp}_X h(Y, X) = \nabla^{\perp}_Y h(X, X) \tag{4.8}$$

and

$$\nabla^{\perp}_Y h(X, Y) = \nabla^{\perp}_X h(Y, Y). \tag{4.9}$$

Using (3.9) in (3.6) we have

$$Jh(X, Y) = -A_{JY}X + \nabla^{\perp}_X JY - G(X, Y), \tag{4.10}$$

and using (4.4) in (4.10) we get

$$\nabla^{\perp}_X JY = G(X, Y). \tag{4.11}$$

We know that $G(X, Y) \in \mu$ and, from (2.6), $\|G(X, Y)\| = 1$. Therefore $\{JX, JY, G(X, Y), JG(X, Y)\}$ is an orthonormal frame field for the normal bundle ν. Then, using (4.5), (4.7) and (4.11) in (4.8), the $G(X, Y)$-component gives $c = -a$. Also using (4.5), (4.7) and (4.11) in (4.9), the $G(X, Y)$-component gives $b = -d$. Hence $h(X, X) = -h(Y, Y)$; i.e. M is minimal.

EXAMPLE. Let $M = S\left(\frac{1}{\sqrt{2}}\right) \times S\left(\frac{1}{\sqrt{2}}\right)$ be the clifford torus. M can be imbedded in $S^3(1)$ as follows. Let (X_1, X_2) be a point of M where X_1 and X_2 are vectors in E^2 each of length $\frac{1}{\sqrt{2}}$. Then M is a flat minimal surface of $S^3(1)$. Since $S^3(1)$ is totally geodesic in $S^6(1)$, M would be flat and minimal in $S^6(1)$. M is also totally real in $S^6(1)$. To see this first note that $S^3(1)$ can be isometrically immersed in $S^6(1)$ as a totally real and totally geodesic submanifold **[1]**. Now write $TS^6(1)|_{S^3(1)} = TS^3(1) \oplus \nu_1$ and $TS^3(1)|_M = TM \oplus \nu_2$ where ν_1 is the normal bundle of $S^3(1)$ in $S^6(1)$ and ν_2 is the normal bundle of M in $S^3(1)$. For any P in M let $X \in TM$. Then $X \in TS^3(1)$. Since $S^3(1)$ is totally real in $S^6(1)$, $JX \in \nu_1$. But $TS^6(1)|_M = TM \oplus \nu_1 \oplus \nu_2$. Therefore JX belongs to the normal bundle of M in $S^6(1)$ and it follows that M is totally real in $S^6(1)$.

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