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ON THE METRIC THEORY OF THE OPTIMAL CONTINUED FRACTION EXPANSION

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Suppose k_n denotes either $\phi(n)$ or $\phi(r_n)$ $(n = 1, 2, \cdots)$ where the polynomial ϕ maps the natural numbers to themselves and r_k denotes the kth rational prime. Let $(p_n/q_n)_{n=1}^{\infty}$ denote the sequence of convergents to a real numbers x for the optimal continued fraction expansion. Define the sequence of approximation constants $(\theta_n(x))_{n=1}^{\infty}$ by

$$\theta_n(x) = q_n^2 \left| x - \frac{p_n}{q_n} \right|. \qquad (n = 1, 2, \cdots).$$

In this paper we study the behaviour of the sequence $(\theta_{k_n}(x))_{n=1}^{\infty}$ for almost all x with respect to Lebesgue measure. In the special case where $k_n = n$ $(n = 1, 2, \cdots)$ these results are due to Bosma and Kraaikamp.

1. INTRODUCTION

In this paper we refine some results on the optimal continued fraction expansion of a real number proved in [**1**]. We first introduce the notion of a semi-regular continued fraction expansion, which both the regular continued fraction expansion and the optimal continued fraction expansion (our primary object of study) are examples of. For a real number x we write

$$x = c_0 + \cfrac{\varepsilon_1}{c_1 + \cfrac{\varepsilon_2}{c_2 + \cfrac{\varepsilon_3}{c_3 + \cfrac{\varepsilon_4}{c_4 \cdots}}}}$$

also sometimes written more succinctly as $[c_0; \varepsilon_1 c_1, \cdots,]$ where $(c_i)_{n=1}^{\infty}$ is a sequence of integers and $\varepsilon_i \in \{-1, 1\}$. The numbers c_i $(i = 1, 2, \cdots)$ are called the partial quotients of the expansion and for each natural number n the truncates

$$\frac{P_n}{Q_n} = [c_0; \varepsilon_1 c_1, \cdots, \varepsilon_n c_n],$$

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are called the convergents of the expansion. The expansion is called semi-regular if: (i) c_n is a natural number, for positive n; (ii) $\varepsilon_{n+1} + c_{n+1} \geqslant 1$ for all natural numbers n and (iii) $\varepsilon_{n+1} + c_{n+1} \geqslant 2$ for infinitely many n if the expansion is itself infinite. Central to the class of semi-regular continued fraction expansions is the regular continued fraction expansion which is also the most familiar and is obtained when c_n is a natural number and ε_n takes the value one for all n. Notice that for the regular continued fraction expansion $c_0 = \lfloor x \rfloor$, that is, the greatest integer not less than x. Each regular convergent is always a best approximation to x in the sense that there do not exist better approximations with smaller denominators. That is, for all integers r and s such that $0 < s \leqslant Q_n$, if for some rational r/s we have

$$\left| x - \frac{r}{s} \right| \leqslant \left| x - \frac{P_n}{Q_n} \right|$$

then $r/s = P_n/Q_n$. The converse does not hold [**13**, Section 16]. It is none the less possible to improve the approximation properties of x by convergents in other regards by looking at other continued fraction expansions in the semi-regular class. We consider two senses in which this can be done below. Firstly, as a form of Dirchlet's theorem on diophantine approximation [**6**] recall the inequality

$$\left| x - \frac{P_n}{Q_n} \right| \leqslant \frac{1}{Q_n^2},$$

satisfied by the convergents of the regular continued fraction expansion. Clearly if for each natural number n we set

$$\theta_n(x) = Q_n^2 \left| x - \frac{P_n}{Q_n} \right|, \tag{1.1}$$

then for each x the sequence $(\theta_n(x))_{n=1}^{\infty}$ lies in the interval $[0, 1]$. It turns out that because the convergents of any semi-regular continued fraction expansion are a subsequence of the sequence of convergents of the regular continued fraction expansion, the sequence $(\theta_n(x))_{n=1}^{\infty}$ may also be defined similarly for any semi-regular continued fraction expansion. In particular it was observed by Minkowski that the regular convergents for which $\theta_n(x) < 1/2$ are the convergents of a semi-regular continued fraction expansion [**13**]. In addition a theorem of Legendre tells us that if $Q\,|Qx - P| < 1/2$ then P/Q is a regular convergent [**6**]. We shall therefore confine attention henceforth to expansions for which $\theta_n(x) < 1/2$ holds for all natural numbers n. Secondly we are interested in semi-regular continued fractions with convergents, henceforth denoted $(p_k/q_k)_{n=1}^{\infty}$, which are as sparse as possible as a subsequence of the sequence of regular convergents $(P_n/Q_n)_{n=1}^{\infty}$. There is a restriction on how sparse the sequence $(p_k/q_k)_{n=1}^{\infty}$

can be in that to remain a semi-regular expansion one of any two consecutive terms of $(p_k/q_k)_{n=1}^{\infty}$ must remain in $(P_n/Q_n)_{n=1}^{\infty}$. A semi-regular continued fraction expansion is called closest if the first requirement, namely that $\theta_n(x) < 1/2$ is true for all natural numbers n and called fastest if $(p_k/q_k)_{n=1}^{\infty}$ is as sparse as a subset of $(P_n/Q_n)_{n=1}^{\infty}$. A number of semi-regular continued fraction expansions satisfy one or other of these properties. See [**7**], [**14**], [**9**] or [**10**] for details. The optimal continued fraction expansion introduced in [**3**] satisfies both. In Section 4 we shall introduce and describe in detail this expansion which is our primary object of study. In Section 2 we introduce certain general results from ergodic theory necessary for our investigation. In Section 3 we present certain information about the regular continued fraction expansion we also need for our investigation. Finally in Section 5 the results of Section 2 are applied to obtain new results on the distribution of the sequence $(\theta_n(x))_{n=1}^{\infty}$ for almost all x with respect to Lebesgue measure in the case of the optimal continued fraction expansion. These results extend earlier work contained in [**2**].

2. Basic Ergodic theory

Here and throughout the rest of the paper by a dynamical system (X, β, μ, T) we mean a set X, together with a σ-algebra β of subsets of X, a probability measure μ on the measurable space (X, β) and a measurable self map T of X that is also measure preserving. By this we mean that if given an element A of β if we set $T^{-1}A = \{x \in X : Tx \in A\}$ then $\mu(A) = \mu(T^{-1}A)$. We say a dynamical system is ergodic if $T^{-1}A = A$ for some A in β means that $\mu(A)$ is either zero or one in value. We say the dynamical system (X, β, μ, T) is weak mixing (among other equivalent formulations [**17**]) if for each pair of sets A and B in β we have

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left|\mu(T^{-n}A \cap B) - \mu(A)\mu(B)\right| = 0.$$

Weak mixing is a strictly stronger condition than ergodicity. A piece of terminology that is becoming increasingly standard is to call a sequence $\mathbf{k} = (k_n)_{n=1}^{\infty}$ of non-negative integers L^p good universal if given any dynamical system (X, β, μ, T) and any function f in $L^p(X, \beta, \mu)$ it is true that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} f(T^{k_n}x) = \ell_f(x),$$

exist almost everywhere with respect to the measure μ. Here and henceforth for each real number y, let $\langle y \rangle$ denote its fractional part, that is $y - \lfloor y \rfloor$. The following theorem is proved in [**12**].

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Theorem 2.1. *Suppose the sequence $\mathbf{k} = (k_n)_{n=1}^{\infty}$ of non-negative integers is such that for each irrational number α the sequence $(\langle k_n\alpha\rangle)_{n=1}^{\infty}$ is uniformly distributed modulo one and for a particular p greater or equal to one that $\mathbf{k} = (k_n)_{n=1}^{\infty}$ is L^p good universal. Then if the dynamical system (X, β, μ, T) is weak mixing, $\ell_f(x) = \int_X f(t)\, d\mu(t)$ almost everywhere with respect to μ.*

If k_n denotes either $\phi(n)$ or $\phi(p_n)$ where ϕ denotes any non-constant polynomial mapping the natural numbers to themselves and p_n denotes the nth rational prime then $\mathbf{k}$ is L^p good universal for any p greater than one. See [**4**] and [**11**] respectively for proofs. The fact that for each irrational number α the sequence $(\langle k_n\alpha\rangle)_{n=1}^{\infty}$ is uniformly distributed modulo one in both instances are well known classical results. See [**16**] and [**18**] respectively. Other sequences are known by the author to satisfy both hypotheses but these results have yet to appear in print. Henceforth for reasons of brevity, we shall call a sequence $\mathbf{k} = (k_n)_{n=1}^{\infty}$ p-good if it satisfies the hypothesis of Theorem 2.1 and we call it good in the special case when it is p-good for $p = \infty$.

3. Regular continued fractions

Suppose for a real number x that it has regular continued fraction expansion

$$x = c_0 + \cfrac{1}{c_1 + \cfrac{1}{c_2 + \cfrac{1}{c_3 + \cfrac{1}{c_4 \cdots}}}}.$$

Let $g\colon [0, 1] \to [0, 1]$ be the map defined by

$$gx = \left\langle \frac{1}{x} \right\rangle x \neq 0; \quad g0 = 0,$$

also known as the Gauss map. Notice that $c_n(x) = c_{n-1}(gx)$ $(n = 1, 2, \cdots)$ and recall that

$$\frac{P_n}{Q_n} = [c_0; c_1, \cdots, c_n] \qquad (n = 1, 2, \cdots).$$

We have the following classical recurrence relations [**6**]

$$P_{-1} = 1;\ P_0 = 0;\ P_n = c_nP_{n-1} + P_{n-2} \qquad (n = 1, 2, \cdots)$$

and

$$Q_{-1} = 1;\ Q_0 = 0;\ Q_n = c_nQ_{n-1} + Q_{n-2} \qquad (n = 1, 2, \cdots).$$

Set

$$T_n = g^{n-1}(x - c_0) \qquad (n = 1, 2, \cdots)$$

and

$$V_n = V_n(x) = \frac{Q_{n-1}}{Q_n}(x) \qquad (n = 1, 2, \cdots).$$

Then it is straightforward to check that

$$T_n = [0;\, c_{n+1},\, c_{n+2},\, \cdots],$$

and

$$V_n = [0;\, c_n,\, c_{n-1},\, \cdots,\, c_1].$$

From g we build a two dimensional map $\mathcal{T}$ defined on $\Omega = ([0, 1) \setminus \mathbf{Q}) \times [0, 1]$ by

$$\mathcal{T}(x, y) = \left(gx,\, \frac{1}{\lfloor 1/x \rfloor + y}\right).$$

Then for each natural number n

$$\mathcal{T}^n(x, y) = (g^n x,\, [0;\, c_n,\, c_{n-1},\, \cdots,\, c_2,\, c_1 + y])$$

and in particular for non-negative n

$$\mathcal{T}^n(x, 0) = (T_n(x),\, V_n(x)).$$

Let β denote the σ-algebra of Borel sets in Ω and η the measure on Ω defined for A in β by

$$\eta(A) = \frac{1}{(\log 2)} \int_A \frac{dxy}{(1 + xy)^2}.$$

We have the following theorem [**7**].

Theorem 3.1. *The dynamical system* $(\Omega, \beta, \eta, \mathcal{T})$ *is weak mixing.*

4. Basic theory of the optimal continued fraction expansion

Let x be an irrational real number and suppose it lies in the interval $(c_0 - 1/2,$ $c_0 - 1/2)$ for some integer c_0 and put $t_0 = x - c_0$, $\varepsilon_1(x) = \operatorname{sgn}(t_0)$ and

(4.1) $$p_1 = 1,\ p_0 = c_0,\ q_1 = 0,\ q_0 = 1,$$

and $v_0 = 0$. Suppose t_i, p_i, q_i, c_i, v_i and ε_{i+1} have been defined for $i \leqslant k$ and some positive integer k. Then define t_{k+1}, p_{k+1}, q_{k+1}, c_{k+1}, v_{k+1} and ε_{k+2} inductively as follows. Let

$$c_{k+1} = \left\lfloor |t_k|^{-1} + \frac{\lfloor |t_k|^{-1} \rfloor + \varepsilon_{k+1} v_k}{2\left(\lfloor |t_k|^{-1} \rfloor + \varepsilon_{k+1} v_{k+1}\right) + 1} \right\rfloor,$$

$$t_{k+1} = |t_k|^{-1} - c_{k+1},$$

$$\varepsilon_{k+2} = \operatorname{sgn}(t_{k+1}),$$

(4.2) $$p_{k+1} = c_{k+1}p_k + \varepsilon_{k+1}p_{k-1}\,;\; q_{k+1} = c_{k+1}q_k + \varepsilon_{k+1}q_{k-1}$$

and $v_{k+1} = q_k/q_{k+1}$. Now the optimal continued fraction expansion of x is

$$x = [c_0;\, \varepsilon_1 c_1,\, \varepsilon_2 c_2,\, \cdots].$$

One straight forwardly verifies that

$$t_k = [0;\, \varepsilon_{k+1}c_{k+1},\, \varepsilon_{k+2}c_{k+2},\, \cdots],$$

and

$$v_k = [0;\, c_k,\, \varepsilon_k c_{k-1},\, \cdots,\, \varepsilon_2 c_1].$$

The sequence $(p_k/q_k)_{k=-1}^{\infty}$ are the convergents and as we said in the introduction are a subsequence of the sequence of regular convergents $(P_n/Q_n)_{n=-1}^{\infty}$ and if we define the function $n\colon \mathbf{N} \to \mathbf{N}$ by $p_k/q_k = P_{n(k)}/Q_{n(k)}$ then $n(k+1) = n(k)+1$ if and only if $\varepsilon_{k+2} = 1$ and $n(k+1) = n(k)+2$ otherwise, once we have set $n(0) = 0$ for $x > 0$ and $n(0) = 1$ otherwise. Define $\Gamma \subset \Omega$ by

$$\Gamma = \left\{(T,\, V) \in \Omega \,:\, V < \min\left(T,\, \frac{2T-1}{1-T}\right)\right\}$$

and put $H = \Omega \setminus \Gamma$. We have the following lemma [**2**].

Lemma 4.1. *Suppose x is irrational and n a natural number. The following are equivalent:*

(i) *the regular continued fraction convergent P_n/Q_n is not an optimal continued fraction convergent;*

(ii) $c_{n+1} = 1$, $\theta_{n-1} < \theta_n$ *and* $\theta_n > \theta_{n+1}$; *and*

(iii) $(T_n,\, V_n)$ *is in* Γ.

We now define the map $U\colon H \to H$, by

$$U(T,\, V) = \begin{cases} \mathcal{T}(T,\, V) & \text{if } \mathcal{T}(T,\, V) \in H; \\ \mathcal{T}^2(T,\, V) & \text{if } \mathcal{T}(T,\, V) \notin H. \end{cases}$$

It is convenient to write $g = \left(1-\sqrt{5}\right)/2$ and $G = \left(1+\sqrt{5}\right)/2$ henceforth. Let β_H denote the σ-algebra of Borel subsets of H and μ_H the probability measure on H with density $(\log G)^{-1}(1+xy)^{-2}$. In [**8**] it is shown that the dynamical system $(H,\, \beta_H,\, \mu_H,\, U)$, which is in fact the system induced on H by $\mathcal{T}$, is exact and hence weak mixing. It is possible to describe a dynamical system explicitly which is isomorphic to $(H,\, \beta_H,\, \mu_H,\, U)$ and which is not described indirectly as an induced system. We do this as follows. Let $\Delta \subset (-1,\, 1) \times (-1,\, 1)$ be defined by

$$\Delta = \left\{(y,\, v) \in (-1,\, 1) \times (-1,\, 1)\colon v \leqslant \min\left(\frac{2t+1}{t+1},\, \frac{t+1}{t+2}\right);\, v \geqslant \max\left(0,\, \frac{2t-1}{1-t}\right)\right\}.$$

Define a map W from Δ to itself by

$$W(t, v) = \left(|t|^{-1} - \beta(t, v), \frac{1}{\beta(t, v) + \operatorname{sgn}(t)v}\right),$$

where

$$\beta(t, v) = \left\lfloor |t|^{-1} + \frac{\lfloor |t_k|^{-1} \rfloor + \operatorname{sgn}(t)v}{2\left(\lfloor |t_k|^{-1} \rfloor + \operatorname{sgn}(t)v\right) + 1} \right\rfloor.$$

Also define a measure μ_Δ on Δ by setting its Radon Nikodym derivative relative to two dimensional Lebesgue measure to be $(\log G)^{-1}(1+xy)^{-2}$. Finally note that if x is in $(-1/2, 1/2)$ then $W^k(x, 0) = (t_k, v_k)$ for all positive integers k. The dynamical system $(\Delta, \beta_\Delta, \mu_\Delta, W)$, where β_Δ is the σ-algebra of Borel sets on Δ, is Bernoulli [**8**] and hence weak mixing.

5. Statistical properties of the sequence $(\theta_n(x))_{n=1}^\infty$

We have the following theorem from which all the other results of this paper may be derived.

Theorem 5.1. *Suppose $(t_k, v_k)_{k=1}^\infty$ is as defined in Section 4. Then if $\mathbf{k} = (k_n)_{n=1}^\infty$ is good for each element A of β_H we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_A(t_{k_n}, v_{k_n}) = \frac{1}{\log G} \int_A \frac{dt dv}{(1+tv)^2},$$

almost everywhere with respect to Lebesgue measure.

Proof: Note that for all y such that (x, y) is in Δ we have

$$\lim_{n\to\infty} (W^n(x, y)) - (W^n(x, 0)) = 0,$$

and that $W^n(x, 0) = (t_n, v_n)$. Then Theorem 5.1 is an immediate consequence of Theorem 2.1. ∎

We now consider applications of this theorem. Let

$$\Pi = \{(w, z) \in \mathbf{R} \times \mathbf{R} : w > 0,\ z > 0,\ 4w^2 + z^2 < 1,\ w^2 + 4z^2 < 1\}.$$

Theorem 5.2. *Suppose A is a Borel subset of the set Π. If $\mathbf{k} = (k_n)_{n=1}^\infty$ is good we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_A(\theta_{k_n-1}(x), \theta_{k_n}(x)) = \int_{A\cap\Pi} \left(\frac{1}{\sqrt{1-4wt}} + \frac{1}{\sqrt{1+4wt}}\right) dw dz,$$

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almost everywhere with respect to Lebesgue measure.

PROOF: Let ψ denote the two to one map from Δ to Π defined by

$$\psi(t, v) = \left(\frac{v}{1+tv}, \frac{\varepsilon(t)t}{1+tv}\right),$$

where $\varepsilon(t)$ denotes the sign of t. We note that $\psi(t_k, v_k) = (\theta_{k-1}, \theta_k)$ for each natural number k. To see this note that from a standard fact from the elementary theory of continued fractions we have

$$x = \frac{p_k + t_k p_{k-1}}{q_k + t_k q_{k-1}} \tag{5.1}$$

and so

$$\theta_k = \frac{\varepsilon_{k-1} t_k}{1 + t_k v_k}. \tag{5.2}$$

Set

$$\Delta_{-1} = \{(t, v) \in \Delta \colon \varepsilon(t) = -1\}$$

and

$$\Delta_1 = \{(t, v) \in \Delta \colon \varepsilon(t) = 1\}.$$

Also let $\psi_{-1} = \psi_{|\Delta_{-1}}$ and $\psi_1 = \psi_{|\Delta_1}$. These maps are then continuously differentiable bijective maps from Δ_{-1} (respestively Δ_1) to Π. Using the coordinate change formula for measures, the image measure for

$$\mu(A) = \frac{1}{\log G} \iint_{A \cap \Pi} \frac{dtdw}{(1+tv)^2}$$

under both maps ψ_{-1} and ψ_1 is given by

$$(\psi_{-1}\mu)(B) = (\psi_1\mu)(B) = \frac{1}{\log G} \iint_{B \cap \Pi} \left(\frac{1+xy}{1-xy}\right) dxdy.$$

Since by (5.1) and (5.2) if $\varepsilon(t_k) = \varepsilon_{k+1} = 1$ then

$$\left(\frac{1 - t_k v_k}{1 + t_k v_k}\right)^2 = 1 - 4\theta_{k-1}\theta_k$$

and if $\varepsilon(t_k) = \varepsilon_{k+1} = -1$ then

$$\left(\frac{1 - t_k v_k}{1 + t_l v_k}\right)^2 = 1 + 4\theta_{k-1}\theta_k$$

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and hence the image of μ under ψ is given by

$$(\psi\mu)(A) = \int_{A\cap\Pi} \left(\frac{1}{\sqrt{1-4wt}} + \frac{1}{\sqrt{1+4wt}}\right) dwdt.$$

The result now follows from Theorem 5.1. ∎

In [**2**] it is shown that for each irrational x we have $0 < \theta_{k-1} + \theta_k < 2/\sqrt{5}$. Let

$$h(z) = \begin{cases} \left(\log\sqrt{1+z} - \log\sqrt{1-z} + \arctan z\right)/\log G & \text{if } z \in [0,\ 1/2]; \\ \left(\log\left(\dfrac{5\sqrt{5-4z^2}-5z}{\sqrt{5-4z^2}+z}\right) + 2\arctan\left(\dfrac{2\sqrt{5-4z^2}-3z}{5\sqrt{1+z^2}}\right)\right)\Big/ 2\log G & \text{if } z \in [1/2, 2/\sqrt{5}]. \end{cases}$$

THEOREM 5.3. *Let h be as just above. If $\mathbf{k} = (k_n)_{n=1}^{\infty}$ is good*

$$\lim_{N\to\infty} \frac{1}{N} \left|\{1 \leqslant n \leqslant N : \theta_{k_n-1}(x) + \theta_{k_n}(x) < a\}\right| = \int_0^a h(t)dt,$$

almost everywhere with respect to Lebesgue measure.

PROOF: The result follows immediately by applying Theorem 5.2 to the function $w + t = const.$ ∎

In [**2**] it is shown that for each irrational x we have $0 \leqslant |\theta_{n-1} - \theta_n| \leqslant 1/2$ for each natural number k. Let

$$j(z) = \frac{1}{\log G}\left(\log\left(\frac{5\sqrt{5-4z^2}-5z}{1+z}\right) - \arctan z + \arcsin\left(\frac{2\sqrt{5-4z^2}-3z}{\sqrt{1+z^2}}\right)\right).$$

We have the following theorem.

THEOREM 5.4. *Let j be as defined just above. If $\mathbf{k} = (k_n)_{n=1}^{\infty}$ is good and a is in $[0,\ 1/2)$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \left|\{1 \leqslant n \leqslant N :\ |\theta_{k_n-1}(x) - \theta_{k_n}(x)| < a\}\right| = \int_0^a j(t)dt,$$

almost everywhere with respect to Lebesgue measure.

PROOF: The proof of this result is an immediate consequence of Theorem 5.2 and the appropriate choice of A. ∎

In [**2**] it is shown that for irrational x, $\theta_k(x)$ is in $(0,\ 1/2)$. Let

$$k(z) = \begin{cases} \dfrac{1}{\log G} & \text{if } z \in \left(0,\ 1/\sqrt{5}\right); \\ \dfrac{1}{\log G}\dfrac{\sqrt{1-4z^2}}{z} & \text{if } z \in \left[1/\sqrt{5},\ 1/2\right). \end{cases}$$

We have the following result:

THEOREM 5.5. *Suppose k is defined as just above. If $\mathbf{k} = (k_n)_{n=1}^{\infty}$ is good and a is in $[0,\ 1/2)$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \chi_A(\theta_{k_n}(x)) = \int_{A\cap(0,\,1/2)} d(z)\,dz,$$

almost everywhere with respect to Lebesgue measure.

PROOF: Apply Theorem 5.2 with $w < z$. ∎

Also calculating the first moment of k we have:

THEOREM 5.6. *If $\mathbf{k} = (k_n)_{n=1}^{\infty}$ is good then*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \theta_{k_n}(x) = \frac{1}{4\log G} \arctan\frac{1}{2}$$

almost everywhere with respect to Lebesgue measure.

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