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Sliding methods for tempered fractional parabolic problem

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Abstract. In this article, we are concerned with the tempered fractional parabolic problem

$$\frac{\partial u}{\partial t}(x,t) - (\Delta + \lambda)^{\frac{\alpha}{2}} u(x,t) = f(u(x,t)),$$

where $-(\Delta+\lambda)^{\frac{\alpha}{2}}$ is a tempered fractional operator with $\alpha \in (0,2)$ and λ is a sufficiently small positive constant. We first establish maximum principle principles for problems involving tempered fractional parabolic operators. And then, we develop the direct sliding methods for the tempered fractional parabolic problem, and discuss how they can be used to establish monotonicity results of solutions to the tempered fractional parabolic problem in various domains. We believe that our theory and methods can be conveniently applied to study parabolic problems involving other nonlocal operators.

1 Introduction

In this article, we are concerned with the following parabolic equation involving tempered fractional operator:

$$\frac{\partial u}{\partial t}(x,t) - (\Delta + \lambda)^{\frac{\alpha}{2}} u(x,t) = f(u(x,t)), \tag{1.1}$$

where the tempered fractional Laplacian operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$ is defined as

$$(\Delta + \lambda)^{\frac{\alpha}{2}} u(x,t) := -c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{u(x,t) - u(y,t)}{e^{\lambda|x-y|}|x-y|^{n+\alpha}} dy, \tag{1.2}$$

where $P.V.$ stands for Cauchy principal value, and

$$c_{n,\alpha} = \begin{cases} \frac{\alpha \Gamma(\frac{n+\alpha}{2})}{2^{1-\alpha}\pi^{\frac{n}{2}}|\Gamma(1-\frac{\alpha}{2})|}, & \text{for } \lambda = 0 \text{ or } \alpha = 1, \\ \frac{\Gamma(\frac{n}{2})}{2\pi^{\frac{n}{2}}|\Gamma(-\alpha)|}, & \text{for } \lambda > 0 \text{ and } \alpha \neq 1, \end{cases} \tag{1.3}$$

and Γ denotes the Gamma function.

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To ensure that the right-hand side of the definition (1.3) is well-defined, we require that $u(x,t) \in \{\mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})\} \times C^1(\mathbb{R})$ with

$$\mathscr{L}_\alpha(\mathbb{R}^n) := \left\{ u(\cdot,t) \in L^1_{loc}(\mathbb{R}^n) \,\middle|\, \int_{\mathbb{R}^n} \frac{e^{-\lambda|x|}|u(x,t)|}{1+|x|^{n+\alpha}} dx < +\infty \right\}. \tag{1.4}$$

Throughout this article, we say that u is an classical entire solution of problem (1.1) if

$$u(x,t) \in \{\mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n \backslash \{0\})\} \times C^1(\mathbb{R}).$$

When $\lambda \to 0+$, the tempered fractional operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$ degenerate into the familiar fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$, which is also a nonlocal integro-differential operator given by

$$(-\Delta)^{\frac{\alpha}{2}} u(x,t) = C_{n,\alpha}\, P.V. \int_{\mathbb{R}^N} \frac{u(x,t)-u(y,t)}{|x-y|^{N+\alpha}} dy, \tag{1.5}$$

where $0<\alpha<2$, $C_{n,\alpha} = (\int_{\mathbb{R}^n} \frac{1-\cos(2\pi y_1)}{|y|^{n+\alpha}} dy)^{-1}$. Fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ is well-defined for any $u(x,t) \in \{C^{1,1}_{loc}(\mathbb{R}^n) \cap \dot{L}_\alpha(\mathbb{R}^n)\} \times C^1(\mathbb{R})$ with the function spaces

$$\dot{L}_\alpha(\mathbb{R}^n) := \left\{ u(\cdot,t) \in L^1_{\mathrm{loc}}(\mathbb{R}^n) \,\middle|\, \int_{\mathbb{R}^n} \frac{|u(x,t)|}{1+|x|^{n+\alpha}} dx < +\infty \right\}.$$

It can also be defined equivalently through Caffarelli and Silvestre's extension method (see [5]).

In recent years, problems involving fractional operators have attracted more attention due to their various applications in mathematical modeling, such as fluid mechanics, molecular dynamics, relativistic quantum mechanics of stars (see, e.g., [6, 20]), conformal geometry (see, e.g., [11]) and probability, and finance (see [3, 4]).

It should be pointed out that the nonlocal operator $(-\Delta)^{\frac{\alpha}{2}}$ and $-(\Delta+\lambda)^{\frac{\alpha}{2}}$ can also be defined equivalently through Caffarelli and Silvestre's extension method, we refer to [5] and the references therein for more details.

From the viewpoint of mathematics, the elliptic problem involving fractional operator has been considered by many people. Caffarelli and Silvestre [5] define the nonlocal operators via the extension method and reduce the nonlocal problem into a local problem in higher dimensions. Later on, Chen and Li [10, 11] give another approach, which considers the equivalent IEs instead of PDEs by deriving the integral representation formulae of solutions to the equations involving fractional operators (see, e.g., [21, 22, 28, 30, 33, 37, 38, 43, 44]). However, this method applies only to the operator $(-\Delta)^{\frac{\alpha}{2}}$. In order to study other non-local operators, Chen, Li, and Li [9] recently presented a direct method of moving planes. After that, Chen and Wu [13] developed the direct sliding methods. The key to these two direct methods is to establish various maximum principles, especially maximum principles in unbounded regions. Although the results on maximum principles and the consequential qualitative properties of solutions (such as symmetry and monotonicity) are currently extensive, different techniques are developed in the present article to overcome technical difficulties arising from the particular nature of the operators under study.

The method of moving planes can be traced back to the early 1950s. It was invented by Alexandroff to study surfaces with constant average curvature. An in-depth understanding of this method has become a potent tool for studying other fields, such as geometrical analysis, geometrical inequalities, conformal geometry, and PDEs. For more literature on moving plane (spheres) methods, please refer to [7, 9–11, 19, 22, 29, 32, 37, 39, 44] and the references therein.

The sliding method developed by Berestycki and Nirenberg [1, 2] provides a flexible alternative to approach symmetry and related issues. The main idea of sliding lies in comparing values of the solution for the equation at two different points, between which one point is obtained from the other by sliding the domain in a given direction, and then the domain is slid back to the limiting position. It has been adapted to the nonlocal setting in the papers cited above. For more kinds of literature on the sliding methods for nonlocal operators, such as for $-\Delta$, $(-\Delta)^s$ with $s \in (0,1)$, $(-\Delta)_p^s$ with $s \in (0,1)$ and $p \geq 2$, $(-\Delta + m^2)^s$ with $s \in (0,1)$ and the mass $m > 0$, or the nonlocal Bellman operator F_s, please refer to [8, 13, 19, 23, 24].

Recently, a series of results have been achieved with regard to the tempered fractional Laplacian $(\Delta + \lambda)^{\frac{\alpha}{2}}$. Now, let us recall the work achieved on tempered fractional Laplacian. For instance, Zhang, Deng, and Karniadakis [47] developed numerical methods for the tempered fractional Laplacian in the Riesz basis Galerkin framework. Zhang, Deng, and Fan [46] designed the finite difference schemes for the tempered fractional Laplacian equation with the generalized Dirichlet-type boundary condition. Duo and Zhang [27] proposed a finite difference method to discretize the n-dimensional (for $n \geq 1$) tempered integral fractional Laplacian and applied it to study the tempered effects on the solution of problems arising in various applications. Shiri, Wu, and Baleanu [42] proposed an collocation methods for terminal value problems of tempered fractional differential equations. Not long ago, Guo and Peng [31] developed the method of moving planes and direct sliding methods for the operator $-(\Delta + \lambda)^{\frac{\alpha}{2}}$, and established symmetry, monotonicity, Liouville-type results and uniqueness results for solutions of different tempered fractional problems (including static nonlinear Schrödinger equations and tempered fractional Choquard equations). For more works on tempered fractional Laplacian, please refer to [25, 36, 45] and the references therein.

For parabolic equations, there have been some results for local operators. For example, Li [35] obtained symmetry for positive solutions in situations where the initial data are symmetric; Hess and Poláčik [34] established asymptotic symmetry of positive solutions to parabolic equation in bounded domains. Subsequently, Poláčik made some progress in such directions for parabolic equations involving local operators in both bounded and unbounded domains, please refer to [18, 40, 41]. There also have been some results for nonlocal parabolic equations. For instance, Chen et al. [12] study the following nonlinear fractional parabolic equation on the whole space:

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,u), \quad (x,t) \in \mathbb{R}^N \times (0,+\infty). \tag{1.6}$$

They developed a systematical approach in applying an asymptotic method of moving planes to investigate qualitative properties of positive solutions for problems (1.6). Then, Chen et al. [17] consider fractional parabolic equations with indefinite nonlinearities:

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = x_1 u^p(x,t), \quad (x,t) \in \mathbb{R}^N \times \mathbb{R}, \tag{1.7}$$

they derived nonexistence of solutions to the above problem with $1 < p < +\infty$. Next, Chen and Wu [14] derived Liouville-type theorems for fractional parabolic problems in $\mathbb{R}^n_+ \times \mathbb{R}$ under some assumptions. Afterward, Chen and Wu [15] considered the ancient solutions to

$$\frac{\partial u}{\partial t}(x,t) + (-\Delta)^s u(x,t) = f(t,u), \quad (x,t) \in \mathbb{R}^N \times (-\infty, T], \tag{1.8}$$

they developed a systematical approach in applying the method of moving planes to study qualitative properties of solutions for problem (1.8). In the four references mentioned above, they used the method of moving planes. Recently, Chen and Wu [16] developed sliding methods and obtained the one-dimensional symmetry and monotonicity of entire positive solutions to fractional reaction–diffusion equations.

Inspired by work [14–16], in this article, we develop the sliding methods for the tempered fractional parabolic problem (1.1). Before we start, we give the following maximum principle in unbounded domains, which plays an important role in applying sliding methods.

Theorem 1.1 (Maximum principles in unbounded open sets) *Assume that Ω is an open set in $\mathbb{R}^n$, possibly unbounded and disconnected and satisfying*

$$\liminf_{R\to\infty} \frac{|\Omega^c \cap B_R(x)|}{|B_R(x)|} \geq c_0 > 0, \qquad \forall\ x \in \Omega, \tag{1.9}$$

for some positive constant c_0. Suppose that $u \in \{\mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{\mathrm{loc}}(\Omega)\} \times C^1(\mathbb{R})$ is bounded from above, and solves

$$\begin{cases} \dfrac{\partial u}{\partial t} - (\Delta + \lambda)^{\frac{\alpha}{2}} u(x,t) + c(x,t)u(x,t) \leq 0, \textit{at points } x \in \Omega \textit{ where } u(x,t) > 0, \\ u(x,t) \leq 0, \quad \textit{in } \Omega^c \times \mathbb{R}, \end{cases} \tag{1.10}$$

where $c(x,t)$ is nonnegative in the set $\{x \in \Omega | u(x) > 0\}$.

Then, we must have

$$u \leq 0, \quad \textit{in } \Omega \times \mathbb{R}. \tag{1.11}$$

Next, we will illustrate how these key ingredients in the above can be used in the sliding methods to establish monotonicity of solutions of tempered fractional parabolic problem (1.1).

Here is the precise description of our main result.

Theorem 1.2 *Suppose that* $u(x,t) \in \left(\mathscr{L}_\alpha \cap C^{1,1}_{\mathrm{loc}}(\mathbb{R}^n)\right) \times C^1(\mathbb{R})$ *is a bounded solution of*

$$\frac{\partial u}{\partial t}(x,t) - (\Delta + \lambda)^{\frac{\alpha}{2}} u(x,t) = f(t, u(x,t)), \quad (x,t) \in \mathbb{R}^n \times \mathbb{R} \tag{1.12}$$

with

$$|u(x,t)| \le 1,$$

and

$$u(x', x_n, t) \underset{x_n \to \pm\infty}{\longrightarrow} \pm 1 \text{ uniformly in } x' = (x_1, \dots, x_{n-1}) \text{ and in } t. \tag{1.13}$$

Assume that $f(t,u)$ *is continuous in* $\mathbb{R} \times ([-1,1])$ *and, for any fixed* $t \in \mathbb{R}$,

$$f(t,u) \text{ is non-increasing for } |u| \ge 1 - \delta \text{ with some } \delta > 0. \tag{1.14}$$

Then, $u(x,t)$ *is strictly increasing with respect to* x_n*, and furthermore, it depends on* x_n *only:*

$$u(x,t) = u(x_n, t).$$

Remark 1.3 The admissible choices of the nonlinearity $f(t,u)$ include: real fractional Ginzburg–Landau nonlinearity $f(t,u) = u - u^3$ and the Zeldovich nonlinearity $f(t,u) = u^2 - u^3$.

Lastly, we will consider the following monotonicity result on solutions to the problem (1.1) on the epigraph E, where the epigraph (refer to [26])

$$E := \{x = (x', x_n) \in \mathbb{R}^n \mid x_n > \varphi(x')\},$$

where $\varphi : \mathbb{R}^{n-1} \to \mathbb{R}$ is a continuous function. A typical example of epigraph E is the upper half-space $\mathbb{R}^n_+$ ($\varphi \equiv 0$).

Theorem 1.4 *Let* $u \in \left(\mathscr{L}_\alpha \cap C^{1,1}_{\mathrm{loc}}(E)\right) \times C^1(\mathbb{R})$ *be a bounded solution of*

$$\begin{cases} \frac{\partial u}{\partial t}(x,t) - (\Delta + \lambda)^{\frac{\alpha}{2}} u(x,t) = f(u(x,t)), & (x,t) \in E \times \mathbb{R}, \\ u(x,t) = 0, & (x,t) \notin E \times \mathbb{R}, \end{cases} \tag{1.15}$$

where $f(\cdot)$ *is non-increasing in the range of* u*. Assume that there exists* $l > 0$ *such that*

$$u \ge 0 \ \text{ in } \{(x,t) = (x', x_n, t) \in E \times \mathbb{R} \,|\, \varphi(x') < x_n < \varphi(x') + l\}. \tag{1.16}$$

Then, either $u \equiv 0$ *in* $\mathbb{R}^n \times \mathbb{R}$ *and* $f(0) = 0$*, or* u *is strictly monotone increasing in the* x_n *direction; and hence* $u > 0$ *in* $E \times \mathbb{R}$.

Furthermore, suppose E is contained in a half-space, the same conclusion can be reached without the assumption (1.16). Furthermore, if E itself is exactly a half-space, then

$$u(x,t) = u(\langle (x', x_n - \varphi(0')), \nu \rangle, t),$$

where ν *is the unit inner normal vector to the hyper-plane* ∂E *and* $\langle \cdot, \cdot \rangle$ *denotes the inner product in Euclidean space. In particular, if* $E = \mathbb{R}^n_+$*, then* $u(x,t) = u(x_n, t)$.

The article is organized as follows: In Section 2, we establish maximum principles for tempered fractional operators unbounded domains, which is Theorem 1.1. As applications, in Section 3, after extending the sliding methods, we show monotonicity of solutions to the tempered fractional parabolic problem, which are Theorem 1.2. In Section 4, we will prove Theorem 1.4.

From now on and in the following of the article, we always use the same C to denote a constant whose value may be different from line-to-line, and only the relevant dependence is specified.

2 Proof of Theorem 1.1

In this section, we shall establish maximum principles for the parabolic problem involving tempered fractional Laplacian operator $-(\Delta+\lambda)^{\frac{\alpha}{2}}$ in unbounded domains. These maximum principles are key ingredients in applying the sliding method. We begin with the following a generalized average inequality, which becomes an effective tool in establishing maximum principles in unbounded domains.

Lemma 2.1 (A generalized average inequality) *Assume $u \in \mathscr{L}_\alpha(\mathbb{R}^n) \cap C^{1,1}_{loc}(\mathbb{R}^n)$. For any $r>0$, if $\bar{x}$ is a maximum point of u in $B_r(\bar{x})$. Then, we have*

$$\frac{\left[-(\Delta+\lambda)^{\frac{\alpha}{2}}\right]u(\bar{x},t)}{c_{n,\alpha}I(r)}+\frac{1}{I(r)}\int_{B_r^c(\bar{x})}\frac{u(y,t)}{e^{\lambda|\bar{x}-y|}|\bar{x}-y|^{n+\alpha}}dy \geq u(\bar{x}), \tag{2.1}$$

where the function $I(r)$ is defined by

$$I(r):=\frac{1}{r^\alpha}\int_{B_1^c(0)}\frac{1}{|y|^{n+\alpha}e^{\lambda r|y|}}dy. \tag{2.2}$$

Proof By definition of $-(\Delta+\lambda)^{\frac{\alpha}{2}}$, we have, for any $r>0$,

$$\begin{aligned}
&-(\Delta+\lambda)^{\frac{\alpha}{2}}u(\bar{x},t)\\
&=c_{n,\alpha}P.V.\int_{\mathbb{R}^n}\frac{u(\bar{x},t)-u(y,t)}{e^{\lambda|\bar{x}-y|}|\bar{x}-y|^{n+\alpha}}dy\\
&\geq c_{n,\alpha}\int_{B_r^c(\bar{x})}\frac{u(\bar{x},t)-u(y,t)}{e^{\lambda|\bar{x}-y|}|\bar{x}-y|^{n+\alpha}}dy\\
&=c_{n,\alpha}u(\bar{x},t)\int_{B_r^c(\bar{x})}\frac{1}{e^{\lambda|\bar{x}-y|}|\bar{x}-y|^{n+\alpha}}dy-c_{n,\alpha}u(y,t)\int_{B_r^c(\bar{x})}\frac{1}{e^{\lambda|\bar{x}-y|}|\bar{x}-y|^{n+\alpha}}dy\\
&=-c_{n,\alpha}u(y,t)\int_{B_r^c(\bar{x})}\frac{1}{e^{\lambda|\bar{x}-y|}|\bar{x}-y|^{n+\alpha}}dy+c_{n,\alpha}u(\bar{x},t)I(r),
\end{aligned}$$

provided that

$$I(r):=\frac{1}{r^\alpha}\int_{B_1^c(0)}\frac{1}{|y|^{n+\alpha}e^{\lambda r|y|}}dy. \qquad \blacksquare$$

Remark 2.2 In the special case, when u satisfies the following s-subharmonic property at the point $(\bar{x}, t)$:

$$\left[-(\Delta+\lambda)^{\frac{\alpha}{2}}\right] u(\bar{x}, t) \leq 0, \tag{2.3}$$

the average inequality (2.1) becomes

$$u(\bar{x}) \leq \int_{B_r^c(\bar{x})} u(y)\, d\mu(y) \tag{2.4}$$

with

$$\int_{B_r^c(\bar{x})} d\mu(y) = 1.$$

Next, we will prove the *maximum principles in unbounded open sets* Theorem 1.1.

Proof Suppose that (1.11) is not true, again with $u(x,t)$ is bounded from above in $\Omega \times \mathbb{R}$, then there exists a positive constant A such that

$$\sup_{(x,t)\in\Omega\times\mathbb{R}} u(x,t) := A > 0. \tag{2.5}$$

On the other hand, due to the domain $\Omega \times \mathbb{R}$ is unbounded, the supremum of $u(x,t)$ may not be attained, then there exists a sequence $\left\{\left(x^k, t_k\right)\right\} \subset \Omega \times \mathbb{R}$ such that

$$u\left(x^k, t_k\right) \to A, \text{ as } k \to \infty.$$

More accurately, there exits a nonnegative sequence $\{\varepsilon_k\} \searrow 0$ such that

$$u\left(x^k, t_k\right) = A - \varepsilon_k > 0. \tag{2.6}$$

From the assumption that $u \leq 0$ in $\Omega^c \times \mathbb{R}$ and the continuity of u, without loss of generality, we may assume that dist $\left\{x^k, \Omega^c\right\} \geq 1$. Now, we define the following auxiliary function:

$$v_k(x,t) = u(x,t) + \varepsilon_k \psi_k(x,t),$$

where $\psi_k(x,t) = \psi\left(\frac{x-x^k}{r\cdot e^{\lambda r}}, \frac{t-t_k}{r^\alpha \cdot e^{3\lambda r}}\right)$ with any fixed $r > 0$ and $\psi(x,t)$ is given by

$$\psi(x,t) = \begin{cases} 1, & \text{if } |(x,t)| \leq \frac{1}{2}, \\ 0, & \text{if } |(x,t)| \geq 1. \end{cases}$$

It is well known that $\psi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$, therefore $|(\Delta+\lambda)^{\frac{\alpha}{2}}\psi(x,t)| \leq C_0$ for any $x \in \mathbb{R}^n \times \mathbb{R}$.

Next, for convenience, we denote

$$Q_{re^{\lambda r}}\left(x^k, t_k\right) := \left\{(x,t) \middle| \left(\frac{x-x^k}{r\cdot e^{\lambda r}}, \frac{t-t_k}{r^\alpha e^{3\lambda r}}\right) \middle| < 1\right\}.$$

It is easy to see that

$$\max_{x\in Q_{re^{\lambda r}}(x^k,t_k)} v_k(x,t) \geq \max_{x\in Q^c_{re^{\lambda r}}(x^k,t_k)} v_k(x,t),$$

which implies the maximum value of $v_k(x,t)$ in $\mathbb{R}^n \times \mathbb{R}$ is attained in $Q_{re^{\lambda r}}\left(x^k, t_k\right)$ along which we will be able to derive a contradiction. More precisely, one can infer from the definition of $v_k(x,t)$ and (2.6), we know that

$$v_k\left(x^k, t_k\right) = A > 0.$$

On the other hand, for any $(x,t) \in (\mathbb{R}^n \times \mathbb{R}) \setminus Q_{re^{\lambda r}}\left(x^k, t_k\right)$, we have

$$v_k(x,t) \leq A.$$

Consequently, there exists $\left(\bar{x}^k, \bar{t}_k\right) \in Q_{re^{\lambda r}}\left(x^k, t_k\right)$ such that

$$A + \varepsilon_k \geq v_k\left(\bar{x}^k, \bar{t}_k\right) = \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} v_k(x,t) \geq A > 0. \tag{2.7}$$

Through direct computation, it follows that

$$-(\Delta+\lambda)^{\frac{\alpha}{2}} v_k\left(\bar{x}^k, \bar{t}_k\right) = c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{v_k\left(\bar{x}^k, \bar{t}_k\right) - v_k\left(y, \bar{t}_k\right)}{e^{\lambda|\bar{x}^k - y|}\left|\bar{x}^k - y\right|^{n+\alpha}} dy \geq 0,$$

and

$$\frac{\partial v_k}{\partial t}\left(\bar{x}^k, \bar{t}_k\right) = 0.$$

And hence,

$$\left|\frac{\partial u}{\partial t}\left(\bar{x}^k, \bar{t}_k\right)\right| = \left|\frac{\partial v_k}{\partial t}\left(\bar{x}^k, \bar{t}_k\right) - \varepsilon_k \frac{\partial \psi_k}{\partial t}\left(\bar{x}^k, \bar{t}_k\right)\right| \leq \frac{C\varepsilon_k}{r^\alpha e^{3\lambda r}}.$$

Collecting the above estimates, we obtain

$$\begin{aligned} 0 &\leq -(\Delta+\lambda)^{\frac{\alpha}{2}} v_k\left(\bar{x}^k, \bar{t}_k\right) \\ &= -(\Delta+\lambda)^{\frac{\alpha}{2}} u\left(\bar{x}^k, \bar{t}_k\right) + \varepsilon_k \cdot [-(\Delta+\lambda)^{\frac{\alpha}{2}}]\psi_k\left(\bar{x}^k, \bar{t}_k\right) \\ &\leq -\frac{\partial u}{\partial t}\left(\bar{x}^k, \bar{t}_k\right) + \frac{C_1\varepsilon_k}{e^{3\lambda r} r^\alpha} \\ &\leq \frac{C}{e^{3\lambda r} r^\alpha}, \end{aligned} \tag{2.8}$$

where we have use the truth of

$$\begin{aligned} &\left|(\Delta+\lambda)^{\frac{\alpha}{2}} \psi_k(x)\right| \\ &= \left|-c_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{\psi_k(x) - \psi_k(y)}{e^{\lambda|x-y|}|x-y|^{n+\alpha}} dy\right| \\ &= \left|-c_{n,\alpha} P.V. \int_{B_r(x)} \frac{\psi_k(x) - \psi_k(y)}{e^{3\lambda|x-y|}|x-y|^{n+\alpha}} dy\right| + \left|-c_{n,\alpha} P.V. \int_{B_r^c(x)} \frac{\psi_k(x) - \psi_k(y)}{e^{3\lambda|x-y|}|x-y|^{n+\alpha}} dy\right| \\ &\leq \left|\int_{B_r(x)} \frac{2c_{n,\alpha}||\psi||_{C^{1,1}(\mathbb{R}^n)}\left|\frac{x}{e^{3\lambda r} r} - \frac{y}{e^{3\lambda r} r}\right|^2}{|x-y|^{n+\alpha}} dy\right| + \left|\int_{B_r^c(x)} \frac{2c_{n,\alpha}}{e^{3\lambda r}|x-y|^{n+\alpha}} dy\right| \\ &\leq \frac{C}{e^{6\lambda r} r^\alpha} + \frac{C}{e^{3\lambda r} r^\alpha} \leq \frac{C_1}{e^{3\lambda r} r^\alpha}. \end{aligned} \tag{2.9}$$

Recalling Theorem 2.1, we know that, for any $r > 0$,

$$\frac{\left[-(\Delta+\lambda)^{\frac{\alpha}{2}}\right] v_k\left(\bar{x}^k, \bar{t}_k\right)}{c_{n,\alpha} I(r)} + \frac{1}{I(r)} \int_{B_r^c(\bar{x}^k)} \frac{u(y, \bar{t}_k)}{e^{\lambda|\bar{x}^k - y|}|\bar{x}^k - y|^{n+\alpha}} dy \geq u(\bar{x}^k), \tag{2.10}$$

where

$$I(r) := \frac{1}{r^\alpha} \int_{B_1^c(0)} \frac{1}{|y|^{n+\alpha} e^{\lambda r|y|}} dy.$$

Next, we shall prove that the left-hand side of the inequality (2.10) is strictly less than A. Indeed, after a simple calculation, we get

$$\begin{aligned} I(r) &:= \frac{1}{r^\alpha} \int_{B_1^c(0)} \frac{1}{|y|^{n+\alpha} e^{\lambda r|y|}} dy = \frac{1}{r^\alpha} \int_{B_1^c(0)} \frac{1}{|y|^{n-1}|y|^{1+\alpha} e^{\lambda r|y|}} dy \\ &\geq \frac{1}{r^\alpha} \int_{B_1^c(0)} \frac{1}{|y|^{n-1} e^{2\lambda r|y|}} dy \\ &=: \frac{C_\lambda}{r^{\alpha+1} e^{2\lambda r}} \geq \frac{C_\lambda}{r^\alpha e^{3\lambda r}}. \end{aligned} \tag{2.11}$$

Again with (2.8), we derive that

$$\frac{\left[-(\Delta+\lambda)^{\frac{\alpha}{2}}\right] v_k\left(\bar{x}^k, \bar{t}_k\right)}{c_{n,\alpha} I(r)} \leq \frac{c_\lambda}{c_{n,\alpha}} r^\alpha e^{3\lambda r} \frac{C_\lambda \varepsilon_k}{r^\alpha e^{3\lambda r}} \leq C\varepsilon_k.$$

Therefore, our goals is to estimate the upper bound of the second term in (2.10). First, since $R > R/\sqrt[n]{2}$, combine this with assumption (1.9), we know that

$$\varliminf_{R\to\infty} \frac{\left|\left(B_R\left(\bar{x}^k\right) \backslash B_{R/\sqrt[n]{2}}\left(\bar{x}^k\right)\right) \cap \Omega^c\right|}{\left|B_R\left(\bar{x}^k\right)\right|} \geq c_0 > 0, \tag{2.12}$$

which indicates that there exist two positive constants $\bar{C}$ and sufficiently large R_k such that

$$\frac{\left|\left(B_R\left(\bar{x}^k\right) \backslash B_{R/\sqrt[n]{2}}\left(\bar{x}^k\right)\right) \cap \Omega^c\right|}{\left|B_R\left(\bar{x}^k\right)\right|} \geq \bar{C} > 0, \quad R \geq R_k. \tag{2.13}$$

Combining (2.5) and (2.13), taking $r = R_k/\sqrt[n]{2}$ and noting the fact

$$v_k\left(y, \bar{t}_k\right) = u\left(y, \bar{t}_k\right) \leq 0, y \in B_r^c\left(\bar{x}^k\right) \cap \Omega^c,$$

we obtain

(2.14)

$$
\begin{aligned}
&\frac{1}{I(r)}\int_{B_r^c(\bar{x}^k)}\frac{u(y,\bar{t}_k)}{e^{\lambda|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}dy\\
&=\frac{1}{I\left((\frac{R_k}{\sqrt[n]{2}})^\alpha\right)}\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)}\frac{v_k\left(y,\bar{t}_k\right)}{e^{\lambda|\bar{x}^k-y|}\left|\bar{x}^k-y\right|^{n+\alpha}}dy\\
&=\frac{1}{I\left((R_k/\sqrt[n]{2})^\alpha\right)}\left(\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)\cap\Omega}\frac{v_k\left(y,\bar{t}_k\right)}{e^{\lambda|\bar{x}^k-y|}\left|\bar{x}^k-y\right|^{n+\alpha}}dy+\int_{B_{R_k/k_2}(\bar{x}^k)\cap\Omega^c}\frac{A+\varepsilon_k}{e^{\lambda|\bar{x}^k-y|}\left|\bar{x}^k-y\right|^{n+\alpha}}dy\right.\\
&\quad\left.-\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)\cap\Omega^c}\frac{A+\varepsilon_k}{e^{\lambda|\bar{x}^k-y|}\left|\bar{x}^k-y\right|^{n+\alpha}}dy+\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)\cap\Omega^c}\frac{v_k\left(y,\bar{t}_k\right)}{e^{\lambda|\bar{x}^k-y|}\left|\bar{x}^k-y\right|^{n+\alpha}}dy\right)\\
&\leq\frac{1}{I\left((R_k/\sqrt[n]{2})^\alpha\right)}\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)}\frac{A+\varepsilon_k}{e^{\lambda|\bar{x}^k-y|}\left|\bar{x}^k-y\right|^{n+\alpha}}dy\\
&\quad-\frac{1}{I\left((R_k/\sqrt[n]{2})^\alpha\right)}\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)\cap\Omega^c}\frac{A+\varepsilon_k}{e^{\lambda|\bar{x}^k-y|}\left|\bar{x}^k-y\right|^{n+\alpha}}dy\\
&=A+\varepsilon_k-\frac{1}{I\left((R_k/\sqrt[n]{2})^\alpha\right)}\int_{B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)\cap\Omega^c}\frac{A+\varepsilon_k}{e^{\lambda|\bar{x}^k-y|}\left|\bar{x}^k-y\right|^{n+\alpha}}dy\\
&\leq A+\varepsilon_k-\frac{1}{I\left((R_k/\sqrt[n]{2})^\alpha\right)}\int_{[B_{R_k}(\bar{x}^k)\backslash B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)]\cap\Omega^c}\frac{A+\varepsilon_k}{e^{\lambda|\bar{x}^k-y|}\left|\bar{x}^k-y\right|^{n+\alpha}}dy\\
&\leq A+\varepsilon_k-\frac{A+\varepsilon_k}{I\left((R_k/\sqrt[n]{2})^\alpha\right)}\left|[B_{R_k}(\bar{x}^k)\backslash B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)]\cap\Omega^c\right|\frac{1}{e^{\lambda\frac{R_k}{\sqrt[n]{2}}}\left|\frac{R_k}{\sqrt[n]{2}}\right|^{n+\alpha}}\\
&\leq A+\varepsilon_k-\frac{A+\varepsilon_k}{C_\lambda}\left(\frac{R_k}{\sqrt[n]{2}}\right)^{\alpha+1}e^{2\lambda\cdot\frac{R_k}{\sqrt[n]{2}}}\left|[B_{R_k}(\bar{x}^k)\backslash B^c_{R_k/\sqrt[n]{2}}(\bar{x}^k)]\cap\Omega^c\right|\frac{1}{e^{\lambda\frac{R_k}{\sqrt[n]{2}}}\left|\frac{R_k}{\sqrt[n]{2}}\right|^{n+\alpha}}\\
&\leq(A+\varepsilon_k)\left(1-C_\lambda\right).
\end{aligned}
$$

Combining (2.8)–(2.14), we know that

(2.15)
$$
\begin{aligned}
A&\leq\frac{\left[-(\Delta+\lambda)^{\frac{\alpha}{2}}\right]v_k(\bar{x}^k,\bar{t}_k)}{c_{n,\alpha}I(r)}+\frac{1}{I(r)}\int_{B_r^c(\bar{x}^k)}\frac{v_k(y,\bar{t}_k)}{e^{\lambda|\bar{x}^k-y|}|\bar{x}^k-y|^{n+\alpha}}dy\\
&\leq C\varepsilon_k+\left(A+\varepsilon_k\right)\left(1-C_\lambda\right).
\end{aligned}
$$

We can immediately get the contradiction as $k\to\infty$. Therefore, conclusion (1.11) must holds. ∎

3 Proof of Theorem 1.4

In this section, based on Theorem 1.1, combining with direct sliding method, we shall prove Theorem 1.4.

First of all, we give some useful notations, for any $x=(x',x_n)$ with $x':=(x_1,...,x_{n-1})\in\mathbb{R}^{n-1}$ and $\tau\in\mathbb{R}$, we denote:

- $x^\tau=x+\tau e_n,\quad$ with $\quad e_n=(0',1)$,
- $u_\tau(x,t):=u(x',x_n+\tau,t)$,
- $w_\tau(x,t):=u(x,t)-u_\tau(x,t)$.

Proof *Step 1.* We prove that for τ sufficiently large, we have

$$w_\tau(x,t) \le 0, \qquad (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{3.1}$$

Recalling assumption (1.13), then there exists a sufficiently large $a > 0$ such that

$$|u(x,t)| \ge 1-\delta \text{ for } |x_n| \ge a, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{3.2}$$

Suppose (3.1) is violated, then there exists a constant $A > 0$ such that

$$\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} w_\tau(x,t) = A > 0.$$

In order to derive a contradiction with (3.5), we consider the function

$$\bar{w}_\tau(x,t) = w_\tau(x,t) - \frac{A}{2}.$$

Our aim is to show that

$$\bar{w}_\tau(x,t) \le 0, (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

One can infer from assumption (1.13) that, for all $t \in \mathbb{R}$, we can choose a sufficiently large constant $M > a$ such that

$$\bar{w}_\tau(x,t) \le 0, \quad \text{for } x_n \ge M, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{3.3}$$

Denote

$$D = \mathbb{R}^{n-1} \times (-\infty, M).$$

Then equation (3.3) implies

$$\bar{w}_\tau(x,t) \le 0, (x,t) \in D^c \times \mathbb{R}. \tag{3.4}$$

That implies $\bar{w}_\tau(x,t)$ satisfies the exterior condition in Theorem 1.1.

Consequently, equation (2.13) implies that $\bar{w}_\tau(x,t)$ satisfies

$$\begin{aligned} &\frac{\partial \bar{w}_\tau}{\partial t}(x,t) - (\Delta+\lambda)^{\frac{\alpha}{2}} \bar{w}_\tau(x,t) \\ &= \frac{\partial w_\tau}{\partial t}(x,t) - (\Delta+\lambda)^{\frac{\alpha}{2}} w_\tau(x,t) \\ &= f(t,u(x,t)) - f(t,u_\tau(x,t)). \end{aligned} \tag{3.5}$$

Next, we claim that, for any $(x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, we have

$$f(t,u(x,t)) \le f(t,u_\tau(x,t)), \text{ at the points where } w_\tau(x,t) > 0. \tag{3.6}$$

We will prove our claim (3.6) by discussing three different cases.

Case (i): $|x_n| \le a$. For any $\tau \ge 2a$, then $x_n + \tau \ge a$, again with (3.2), at the points where $w_\tau(x,t) > 0$, we have

$$u(x,t) > u_\tau(x,t) \ge 1-\delta.$$

Combine this truth with the monotonicity assumption (1.14) on the function f, one can immediately get claim (3.6).

Case (ii): $x_n < -a$. For any $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, by assumption (1.13) and (3.2), we get

$$u(x,t) \le -1 + \delta,$$

and therefore at the points where $w_\tau(x,t) > 0$,

$$u_\tau(x,t) < u(x,t) \le -1 + \delta,$$

which implies that we can use the monotonicity assumption (1.14) on the function f to derive claim (3.6).

Case (iii): $x_n > a$. For any $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, by (3.2), we have, at the points where $w_\tau(x,t) > 0$,

$$u(x,t) > u_\tau(x,t) \ge 1 - \delta,$$

and therefore, we use the monotonicity assumption (1.14) on the function f again to derive claim (3.6). Thus, our claim (3.6) must hold.

One can infer from (3.6) that

$$\frac{\partial \bar{w}_\tau}{\partial t}(x,t) - (\Delta + \lambda)^{\frac{\alpha}{2}}\, \bar{w}_\tau(x,t) \le 0, \text{ at the points in } D \times \mathbb{R}, \text{ where } w_\tau(x,t) > 0.$$

This is also valid at the points in $D \times \mathbb{R}$, where $\bar{w}_\tau(x,t) > 0$, more precisely,

$$\frac{\partial \bar{w}_\tau}{\partial t}(x,t) - (\Delta + \lambda)^{\frac{\alpha}{2}}\, \bar{w}_\tau(x,t) \le 0, \text{ at the points in } D \times \mathbb{R}, \text{ where } \bar{w}_\tau(x,t) > 0,$$

which together with the exterior condition on $\bar{w}_\tau(x,t)$ (3.4), and the maximum principle in unbounded domains (Theorem 1.1) implies

$$\bar{w}_\tau(x,t) \le 0, (x,t) \in \mathbb{R}^n \times \mathbb{R}.$$

It follows that

$$w_\tau(x,t) \le \frac{A}{2}, (x,t) \in \mathbb{R}^n \times \mathbb{R},$$

which contradicts the truth of $\sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} w_\tau(x,t) = A > 0$. Therefore, $w_\tau(x,t) \le 0$, for any $\tau \ge 2a$ and $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. This completes the proof in Step 1.

Step 2. Inequality (3.1) provides a starting point for us to carry out the sliding procedure. In this step, we decrease τ from close to $\tau = 2a$ to 0, and prove that for any $0 < \tau < 2a$, we still have

$$w_\tau(x,t) \le 0, (x,t) \in \mathbb{R}^n \times \mathbb{R}. \tag{3.7}$$

To this end, we define

$$\tau_0 = \inf\left\{\tau \mid w_\tau(x,t) \le 0, (x,t) \in \mathbb{R}^n \times \mathbb{R}\right\},$$

and prove that $\tau_0 = 0$. Otherwise, we show that τ_0 can be decreased a little bit while inequality

$$w_\tau(x,t) \le 0, (x,t) \in \mathbb{R}^n \times \mathbb{R}, \tau \in (\tau_0 - \varepsilon, \tau_0]$$

is still valid, which contradicts the definition of τ_0.

(I) We first show that

$$\sup_{|x_n|\le a,(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}} w_{\tau_0}(x,t) < 0. \tag{3.8}$$

Suppose (3.14) is false, then

$$\sup_{|x_n|\le a_1(x',t)\in\mathbb{R}^{n-1}\times\mathbb{R}} w_{\tau_0}(x,t) = 0,$$

and there exists a sequence

$$\left\{\left(x^k,t_k\right)\right\} \subset \left(\mathbb{R}^{n-1}\times[-a,a]\right)\times\mathbb{R},\ k=1,2,\dots,$$

such that

$$w_{\tau_0}\left(x^k,t_k\right) \to 0, \quad \text{as } k\to\infty.$$

More precisely, there exist a nonnegative sequence $\{\varepsilon_k\} \searrow 0$ such that

$$w_{\tau_0}\left(x^k,t_k\right) = -\varepsilon_k. \tag{3.9}$$

In order to obtain more information from the supremum of $w_{\tau_0}(x,t)$, we introduce the following auxiliary function:

$$w_k(x,t) = w_{\tau_0}(x,t) + \varepsilon_k\eta_k(x,t),$$

where

$$\eta_k(x,t) = \eta\left(x-x^k,t-t_k\right)$$

with $\eta(x,t)\in C_0^\infty(\mathbb{R}^n\times\mathbb{R})$ and

$$\eta(x,t) = \begin{cases} 1, & \text{if } |(x,t)| \le \frac{1}{2}, \\ 0, & \text{if } |(x,t)| \ge 1. \end{cases}$$

Denote

$$Q_1\left(x^k,t_k\right) := \left\{(x,t) \| (x,t)-\left(x^k,t_k\right)| < 1\right\}.$$

We can observe that

$$\max_{x\in Q_1(x^k,t_k)} v_k(x,t) \ge \max_{x\in Q_1^c(x^k,t_k)} v_k(x,t).$$

Therefore, the maximum value of $w_k(x,t)$ in $\mathbb{R}^n\times\mathbb{R}$ is attained in $Q_1\left(x^k,t_k\right)$, along which we will be able to derive a contradiction. More precisely, from the definition of $w_k(x,t)$ and (3.9), one has

$$w_k\left(x^k,t_k\right) = 0.$$

On the other hand, for $(x,t)\in(\mathbb{R}^n\times\mathbb{R})\setminus Q_1\left(x^k,t_k\right)$, since $\eta_k(x,t)=0$, we get

$$w_k(x,t) \le 0.$$

Therefore, $w_k(x,t)$ attains its maximum value in $Q_1\left(x^k, t_k\right)$, say at $\left(\bar{x}^k, \bar{t}_k\right)$, i.e.,

$$\varepsilon_k \geq w_k\left(\bar{x}^k, \bar{t}_k\right) = \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} w_k(x,t) \geq 0.$$

To simplify the notion, we introduce the following auxiliary function:

$$\bar{w}_k(x,t) = w_k\left(x+\bar{x}^k, t+\bar{t}_k\right), \tag{3.10}$$

which implies

$$\varepsilon_k \geq \bar{w}_k(0,0) = \sup_{(x,t)\in\mathbb{R}^n\times\mathbb{R}} \bar{w}_k(x,t) \geq 0, \tag{3.11}$$

it follows that

$$-(\Delta+\lambda)^{\frac{\alpha}{2}}\bar{w}_k(0,0) = C_{n,s}P.V.\int_{\mathbb{R}^n}\frac{\bar{w}_k(0,0)-\bar{w}_k(y,0)}{e^{\lambda|y|}|y|^{n+\alpha}}dy \geq 0$$

and

$$\frac{\partial \bar{w}_k}{\partial t}(0,0) = 0.$$

Through direct calculations, combine with the truth of $w_{\tau_0}\left(\bar{x}^k, \bar{t}_k\right) \to 0$, as $k\to\infty$, we derive that

$$\begin{aligned}
0 &\leq -(\Delta+\lambda)^{\frac{\alpha}{2}}\bar{w}_k(0,0)\\
&= -(\Delta+\lambda)^{\frac{\alpha}{2}}w_{\tau_0}\left(\bar{x}^k,\bar{t}_k\right) + \varepsilon_k[-(\Delta+\lambda)^{\frac{\alpha}{2}}]\eta_k\left(\bar{x}^k,\bar{t}_k\right)\\
&= -\frac{\partial w_{\tau_0}}{\partial t}\left(\bar{x}^k,\bar{t}_k\right) + f\left(\bar{t}_k, u\left(\bar{x}^k,\bar{t}_k\right)\right) - f\left(\bar{t}_k, u_{\tau_0}\left(\bar{x}^k,\bar{t}_k\right)\right) + \varepsilon_k[-(\Delta+\lambda)^{\frac{\alpha}{2}}]\eta_k\left(\bar{x}^k,\bar{t}_k\right)\\
&= -\frac{\partial \bar{w}_k}{\partial t}(0,0) - \varepsilon_k\frac{\partial \eta_k}{\partial t}\left(\bar{x}^k,\bar{t}_k\right) + f\left(\bar{t}_k, u\left(\bar{x}^k,\bar{t}_k\right)\right)\\
&\quad - f\left(\bar{t}_k, u_{\tau_0}\left(\bar{x}^k,\bar{t}_k\right)\right) + \varepsilon_k[-(\Delta+\lambda)^{\frac{\alpha}{2}}]\eta_k\left(\bar{x}^k,\bar{t}_k\right)\\
&\leq C\varepsilon_k + f\left(\bar{t}_k, u\left(\bar{x}^k,\bar{t}_k\right)\right) - f\left(\bar{t}_k, u_{\tau_0}\left(\bar{x}^k,\bar{t}_k\right)\right) \to 0, \text{ as } k\to\infty.
\end{aligned} \tag{3.12}$$

Putting (3.10) and Theorem 1.1 together, we deduce

$$\frac{\left[-(\Delta+\lambda)^{\frac{\alpha}{2}}\right]\bar{w}_k(0,0)}{c_{n,\alpha}I(r)} + \frac{1}{I(r)}\int_{B_r^c(\bar{x})}\frac{\bar{w}_k(y,0)}{e^{\lambda|y|}|y|^{n+\alpha}}dy \geq \bar{w}_k(0,0), \text{ for any } r>0. \tag{3.13}$$

Therefore, we can deduce by using (3.12) and the above average inequality (3.13) that for any finite $r>0$,

$$\frac{1}{I(r)}\int_{B_r^c(0)}\frac{\bar{w}_k(y,0)}{e^{\lambda|y|}|y|^{n+\alpha}}dy \to 0, \text{ as } k\to\infty,$$

which implies that for any fixed $r>0$,

$$\bar{w}_k(y,0)\to 0,\ y\in B_r^c(0), \text{ as } k\to\infty. \tag{3.14}$$

Due to $u(x,t)$ is uniformly continuous, by Arzelà–Ascoli theorem, up to extraction of a subsequence of $u_k(x,t) := u\left(x+\bar{x}^k, t+\bar{t}_k\right)$ (still denoted by itself), we obtain

$$u_k(x,t) \to u_\infty(x,t), (x,t) \in \mathbb{R}^n \times \mathbb{R}, \text{ as } k \to \infty,$$

which together with (3.14), yields

$$u_\infty(x,0) - (u_\infty)_{\tau_0}(x,0) \equiv 0, x \in B_r^c(0).$$

Therefore, for any $j \in \mathbb{N}$ and any fixed $r > 0$, we obtain

$$\begin{aligned} u_\infty\left(x', x_n, 0\right) &= u_\infty\left(x', x_n+\tau_0, 0\right) = u_\infty\left(x', x_n+2\tau_0, 0\right) \\ &= \cdots = u_\infty\left(x', x_n+j\tau_0, 0\right), x \in B_r^c(0). \end{aligned} \tag{3.15}$$

Since the nth variable $\bar{x}_n^k$ of $\bar{x}^k$ is bounded, we deduce from the asymptotic condition of (1.13) that

$$u_\infty\left(x', x_n, 0\right) \underset{x_n \to \pm\infty}{\longrightarrow} \pm 1 \text{ uniformly in } x' = (x_1, \ldots, x_{n-1}),$$

as a consequence, one can take x_n sufficiently negative to let $u_\infty\left(x', x_n, 0\right)$ close to -1, and then take j sufficiently large to let $u_\infty\left(x', x_n+j\tau_0, 0\right)$ close to 1, this is a contradiction with (3.15). Therefore, conclusion (3.8) must hold.

(II) Suppose $\tau_0 > 0$, we are to show that there exists an $\varepsilon > 0$ such that

$$w_\tau(x,t) \le 0, \quad \forall (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}, \quad \forall \tau \in (\tau_0 - \varepsilon, \tau_0], \tag{3.16}$$

which would contradict the definition of τ_0.

First, equation (3.8) implies that there exists a small constant $\varepsilon > 0$ such that

$$\sup_{|x_n| \le a, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}} w_\tau(x,t) \le 0, \forall \tau \in (\tau_0 - \varepsilon, \tau_0]. \tag{3.17}$$

Consequently, we only need to show that

$$\sup_{|x_n| > a, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}} w_\tau(x,t) \le 0, \forall \tau \in (\tau_0 - \varepsilon, \tau_0]. \tag{3.18}$$

In fact, if (3.18) is not valid, then there exists some $\tau \in (\tau_0 - \varepsilon, \tau_0]$ and a constant $A > 0$ such that

$$\sup_{|x_n| > a, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}} w_\tau(x,t) := A > 0. \tag{3.19}$$

Applying the asymptotic condition on u in (1.13), there exists a constant $M > a$ such that

$$w_\tau(x,t) \le \frac{A}{2}, |x_n| \ge M, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{3.20}$$

To this end, we define

$$E = \left\{ (x', x_n, t) \in \mathbb{R}^{n-1} \times \mathbb{R} \times \mathbb{R} \,\middle|\, a < |x_n| < M, (x',t) \in \mathbb{R}^{n-1} \times \mathbb{R} \right\}$$

and consider the differential inequality in E satisfied by the function

$$v_\tau(x,t) := w_\tau(x,t) - \frac{A}{2}.$$

For any $\tau \in (\tau_0 - \varepsilon, \tau_0]$, $(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}$, if $a < x_n < M$ at the points in $E \times \mathbb{R}$, where $v_\tau(x,t) > 0$, we have

$$u(x,t) > u_\tau(x,t) + \frac{A}{2} \geq u_\tau(x,t) \geq 1 - \delta,$$

which implies

$$f(t, u(x,t)) - f(t, u_\tau(x,t)) \leq 0, a < x_n < M, (x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{3.21}$$

Another, using the monotonicity of *f*. If $-M < x_n < -a$, at the points in $E \times \mathbb{R}$, where $v_\tau(x,t) > 0$, we have

$$u_\tau(x,t) < u(x,t) \leq -1 + \delta.$$

Again with the assumption (1.14), one has

$$f(u(x,t)) - f(u_\tau(x,t)) \leq 0, -M < x_n < -a, (x', t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{3.22}$$

Therefore, in view of (3.21) and (3.22), at the points in $E \times \mathbb{R}$, where $v_\tau(x,t) > 0$, we have

$$\begin{aligned} \frac{\partial v_\tau}{\partial t}(x,t) - (\Delta + \lambda)^{\frac{\alpha}{2}} v_\tau(x,t) &= \frac{\partial u_\tau}{\partial t}(x,t) - (\Delta + \lambda)^{\frac{\alpha}{2}} u_\tau(x,t) \\ &= f(t, u(x,t)) - f(t, u_\tau(x,t)) \leq 0. \end{aligned}$$

Noting by (3.17) and (3.20), we have the following exterior condition:

$$v_\tau(x,t) \leq 0, (x,t) \in E^c \times \mathbb{R},$$

which together with (3.18) and the maximum principle in unbounded domains Theorem 1.1, we obtain

$$v_\tau(x,t) \leq 0, (x,t) \in \mathbb{R}^n \times \mathbb{R}, \forall \tau \in (\tau_0 - \varepsilon, \tau_0].$$

This contradicts the assumption (3.19), and it follows that (3.18) is valid, which also yield (3.7). Therefore, we complete the proof in Step 2.

Step 3. In this step, we will show that $u(x,t)$ is strictly increasing with respect to x_n, and

$$u(x,t) = u(x_n, t). \tag{3.23}$$

From Steps 1 and 2, we have derived that

$$w_\tau(x,t) \leq 0, (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}, \ \forall \tau > 0.$$

In order to show that $u(x,t)$ is strictly increasing with respect to x_n, we only need to show

$$w_\tau(x,t) < 0, (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}, \ \forall \tau > 0. \tag{3.24}$$

In fact, if equation (3.24) is not true, then there exists a point $(x^0, t_0) \in \mathbb{R}^{n-1} \times \mathbb{R}$ and $\tau_0 > 0$ such that

$$w_{\tau_0}(x^0, t_0) = 0,$$

which means $\left(x^0, t_0\right)$ is a maximum point of $w_{\tau_0}(x,t)$ in $\mathbb{R}^n \times \mathbb{R}$, and

$$\frac{\partial w_{\tau_0}}{\partial t}\left(x^0, t_0\right) = 0.$$

Through a direct calculation, we have

$$\begin{aligned}
0 &= f\left(t_0, u\left(x^0, t_0\right)\right) - f\left(t_0, u_{\tau_0}\left(x^0, t_0\right)\right) \\
&= \frac{\partial w_{\tau_0}}{\partial t}\left(x^0, t_0\right) + \left[-(\Delta + \lambda)\right]^{\frac{\alpha}{2}} w_{\tau_0}\left(x^0, t_0\right) \\
&= C_{n,\alpha} PV \int_{\mathbb{R}^n} \frac{-w_{\tau_0}\left(y, t_0\right)}{e^{\lambda |x_0 - y|} \left|x^0 - y\right|^{n+\alpha}} dy,
\end{aligned}$$

which implies immediately that

$$w_{\tau_0}\left(y, t_0\right) \equiv 0, \quad y \in \mathbb{R}^n.$$

Therefore, for any $j \in \mathbb{N}$, we have

$$\begin{aligned}
u\left(y', y_n, t_0\right) &= u\left(y', y_n + \tau_0, t_0\right) = u\left(y', y_n + 2\tau_0, t_0\right) \\
&= \cdots = u\left(y', y_n + j\tau_0, t_0\right), y \in \mathbb{R}^n.
\end{aligned}$$

Recalling asymptotic condition, we can let y_n sufficiently negative such that $u\left(y', y_n, t_0\right)$ is close to -1, and then take j sufficiently large such that $u\left(x', x_n + j\tau_0, 0\right)$ is close to 1; thus, we derive a contradiction and obtain (3.24), which yields that $u(x,t)$ is strictly increasing with respect to x_n.

Next, our goal is to prove that $u(x)$ depends on x_n only. Indeed, it can be seen from the above sliding procedure that the methods should still be valid if we replace $u_\tau(x,t)$ by $u(x + \tau\nu, t)$ with $\nu = (\nu_1, \ldots, \nu_n)$ and $\nu_n > 0$. More accurately, applying similar sliding methods as in Steps 1 and 2, for each ν with $\nu_n > 0$, yields

$$u(x + \tau\nu, t) > u(x,t), \forall \tau > 0, (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}.$$

Let $\nu_n \to 0$. We deduce from the continuity of $u(x,t)$ that

$$u(x + \tau\nu, t) \geq u(x,t)$$

for arbitrary ν with $\nu_n = 0$. Replacing ν by $-\nu$, for arbitrary ν with $\nu_n = 0$, we find that

$$u(x + \tau\nu, t) \leq u(x,t).$$

Therefore,

$$u(x + \tau\nu, t) = u(x,t),$$

which indicates that $u\left(x', x_n, t\right)$ is independent of x', hence (3.23) is true. Therefore, we have proved Step 3. This completes the proof of Theorem 1.2. ■

4 Monotonicity in epigraph *E*

In this section, combined the maximum principles Theorem 1.1 and sliding methods, we shall prove the monotonicity result on solutions to tempered fractional parabolic problem on epigraph *E*.

Proof For any $0 < \tau < l$, let

$$u^\tau(x,t) := u(x', x_n + \tau, t)$$

and

$$w^\tau(x,t) := u(x,t) - u^\tau(x,t).$$

Since $f(\cdot)$ is non-increasing, we have

$$\frac{\partial w^\tau}{\partial t}(x,t) - (\Delta + \lambda)^{\frac{\alpha}{2}} w^\tau(x,t) = f(u(x,t)) - f(u^\tau(x,t)) \le 0$$

at points $x \in E$, where $w^\tau(x,t) < 0$. In addition, for any $0 < \tau < l$, we have

$$w^\tau(x,t) \le 0, \quad \forall \ (x,t) \in \{\mathbb{R}^n \backslash E\} \times \mathbb{R}.$$

Thus, it follows immediately from Theorem 1.1 that, for any $0 < \tau < l$,

$$w^\tau(x,t) \le 0, \quad \forall \ (x,t) \in E \times \mathbb{R}.$$

Now, suppose that $u \not\equiv 0$ in $E \times \mathbb{R}$, then there exists a $(\hat{x}, \hat{t}) \in E \times \mathbb{R}$ such that $u(\hat{x}, \hat{t}) > 0$. We are to show that, for any $0 < \tau < l$,

$$w^\tau(x,t) < 0, \quad \forall \ x \in E \times \mathbb{R}. \tag{4.1}$$

If not, there exists a point $(x^\tau, t_0) \in E \times \mathbb{R}$ such that

$$w^\tau(x^\tau, t_0) = 0 = \max_{\mathbb{R}^n \times \mathbb{R}} w^\tau(x,t),$$

and hence

$$\frac{\partial w^\tau}{\partial t}(x^\tau, t_0) = 0.$$

One one hand, recalling the definition of $-(\Delta + \lambda)^{\frac{\alpha}{2}}$, we know that

$$\frac{\partial w^\tau}{\partial t}(x^\tau, t_0) - (\Delta + \lambda)^{\frac{\alpha}{2}} w^\tau(x^\tau, t_0) = C_{n,\alpha} P.V. \int_{\mathbb{R}^n} \frac{-w^\tau(y, t_0)}{e^{\lambda|x^\tau - y|} |x^\tau - y|^{n+\alpha}} dy > 0.$$

On the other hand, one has

$$\frac{\partial w^\tau}{\partial t}(x^\tau, t_0) - (\Delta + \lambda)^{\frac{\alpha}{2}} w^\tau(x^\tau, t_0) = f(u(x^\tau, t_0)) - f(u^\tau(x^\tau, t_0)) = 0.$$

That is impossible! Therefore, (4.1) holds; and hence, u is strictly monotone increasing in the x_n direction. In particular, $u > 0$ in E.

If, in addition, E is contained in a half-space, we will prove that

$$u(x,t) \ge 0, \qquad \text{in } E \times \mathbb{R},$$

and hence, the assumption (1.16) is redundant.

Without loss of generalities, we may assume that $E \subseteq \mathbb{R}^n_+$, let

$$T_0 := \{x \in \mathbb{R}^n | x_n = 0\}, \tag{4.2}$$

$$\Sigma_0 := \{x \in \mathbb{R}^n | x_n > 0\} \tag{4.3}$$

be the region above the plane T_0, and

$$x^0 := (x_1, x_2, \ldots, -x_n)$$

be the reflection of x about the plane T_0. We denote $u_0(x,t) := u\left(x^0, t\right)$ and $w_0(x,t) = u_0(x,t) - u(x,t)$. For $(x,t) \in \Sigma_0 \times \mathbb{R}$, where $w_0(x,t) > 0$, we derive from (1.15) that, for any $(x,t) \in E \times \mathbb{R}$, where $w_0(x,t) > 0$, one has

$$\frac{\partial w_0}{\partial t}(x,t) - (\Delta + \lambda)^{\frac{\alpha}{2}} w_0(x,t) = f(u_0(x,t)) - f(u(x,t)) \leq 0.$$

Hence, we obtain from theorem that $w_0 \leq 0$ in $\Sigma_0 \times \mathbb{R}$, which implies immediately $u \geq 0$ in $E \times \mathbb{R}$.

Furthermore, suppose E itself is exactly a half-space. Without loss of generalities, we may assume that $E = \mathbb{R}^n_+$. We will show that $u(x,t)$ depends on x_n only.

In fact, when $E = \mathbb{R}^n_+$, it can be seen from the above sliding procedure that the methods should still be valid if we replace $u^\tau(x,t) := u(x + \tau e_n, t)$ by $u(x + \tau \nu, t)$, where $\nu = (\nu_1, \ldots, \nu_n)$ is an arbitrary vector such that $\langle \nu, e_n \rangle = \nu_n > 0$. Applying similar sliding methods as above, we can derive that, for arbitrary such vector ν,

$$u(x + \tau\nu, t) > u(x,t) \quad \text{in } \mathbb{R}^n_+ \times \mathbb{R}, \quad \forall\ \tau > 0.$$

Let $\nu_n \to 0+$, from the continuity of u, we deduce that

$$u(x + \tau\nu, t) \geq u(x,t)$$

for arbitrary vector ν with $\nu_n = 0$. By replacing ν by $-\nu$, we arrive at

$$u(x + \tau\nu, t) = u(x,t)$$

for arbitrary vector ν with $\nu_n = 0$, this means that $u(x,t)$ is independent of x', hence $u(x,t) = u(x_n, t)$. This finishes the proof of Theorem 1.4. ■

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