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A LOCALLY SMOOTHING METHOD FOR MATHEMATICAL PROGRAMS WITH COMPLEMENTARITY CONSTRAINTS

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Abstract

We propose a locally smoothing method for some mathematical programs with complementarity constraints, which only incurs a local perturbation on these constraints. For the approximate problem obtained from the smoothing method, we show that the Mangasarian–Fromovitz constraints qualification holds under certain conditions. We also analyse the convergence behaviour of the smoothing method, and present some sufficient conditions such that an accumulation point of a sequence of stationary points for the approximate problems is a C-stationary point, an M-stationary point or a strongly stationary point. Numerical experiments are employed to test the performance of the algorithm developed. The results obtained demonstrate that our algorithm is much more promising than the similar ones in the literature.

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1. Introduction

In the field of economics and engineering sciences, it is well known that the Cournot equilibrium problem and the generalized Nash equilibrium problem are typical two-level mathematical programming problems [11]. With some mild assumptions such as convexity or concavity in the lower-level problem, all two-level programming problems can be formulated as a mathematical program with complementarity constraints (MPCC) (see, for example, [11]). In the past two decades, efficient numerical algorithms for the solution of MPCCs have been attracting the attention of applied mathematicians and experts in engineering, because the existing theory

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and algorithms for the standard optimization problems cannot be directly applied to MPCCs (see [6, 19, 20] and the references therein).

The MPCC model is written as follows:

$$\begin{array}{ll} \text{minimize} & f(z) \\ \text{subject to} & g(z) \leq 0, \quad h(z) = 0, \\ & G(z) \geq 0, \quad H(z) \geq 0, \\ & G(z)^T H(z) = 0, \end{array} \tag{1.1}$$

where $f : R^n \to R$, $g : R^n \to R^m$, $h : R^n \to R^p$ and $G, H : R^n \to R^l$ are continuously differentiable functions.

One of the major challenges in solving problem (1.1) is that some of the popular constraint qualification conditions for a standard optimization problem are not satisfied for the MPCC (1.1) in general. Thus, new optimality conditions as well as new efficient numerical algorithms on the basis of these conditions should be investigated to identify and find the optimal solution of problem (1.1). The results available in the literature include the sequential quadratic programming methods [4], the interior methods [12], the penalty function methods [14], the lifting method [17], the relaxation methods [9, 18] and the smoothing methods [3, 20].

As one of the most fundamental methods, the smoothing methods in [3, 5, 13] use different smooth constraints to approximate the nonsmooth complementarity constraints in problem (1.1). Then, by analysing the behaviour of the solutions to the approximate optimization problems, the corresponding algorithms for solving problem (1.1) are developed. Note that in the existing smoothing results, the complementarity constraints $G(z) \geq 0$, $H(z) \geq 0$, $G(z)^T H(z) = 0$ in problem (1.1) are overall replaced by a system of approximately smooth equations.

However, the complementarity constraints $G(z) \geq 0$, $H(z) \geq 0$, $G(z)^T H(z) = 0$ in problem (1.1) are not smooth at some feasible points only. Thus, it seems unnecessary to make an overall substitution for these constraints. In this paper, we intend to present a locally smoothing method for the complementarity constraints, which only incurs a local perturbation on the original model (1.1). We prove that the convergence of the solutions of the perturbed problems improves by finding a solution to the original problem (1.1). Actually, for the new smoothing method, we present some conditions such that the Mangasarian–Fromovitz constraints qualification (MFCQ) holds for the perturbed problem. Under certain conditions, we prove that any accumulation point of a sequence of stationary points for the perturbed problems is a C-stationary point, an M-stationary point or a strongly stationary point. Numerical experiments are used to show the efficiency of the algorithm developed.

The rest of the paper is organized as follows. In the next section we introduce some concepts in nonlinear programs and MPCCs. A new locally smoothing method is proposed, then the corresponding algorithm is developed in Section 3. In Section 4 we carry out an analysis of convergence with the perturbation parameter tending to zero. Section 5 is devoted to testing the efficiency of the algorithm developed. Some final remarks draw the paper to a close.

For convenience, we use the following notation. The ith component of a vector G is denoted by G_i, and F denotes the feasible set of problem (1.1). For a constraint function $g : R^n \to R^m$ and a given point $z \in F \subseteq R^n$, we denote by $I_g(z) = \{i : g_i(z) = 0\}$ the active index set of g at z. For a vector $\alpha \in R^n$, $\mathrm{supp}(\alpha) = \{i : \alpha_i \neq 0\}$ stands for the support of α.

2. Preliminaries

In this section, some of the basic concepts which are relevant to the convergence results are stated.

For the ordinary nonlinear programming problem (NLP) given by

$$\begin{array}{ll} \text{minimize} & f(z) \\ \text{subject to} & g(z) \leq 0, \\ & h(z) = 0, \end{array} \tag{2.1}$$

stationary points play a fundamental role in finding a local minimizer. Denote by F the feasible set of problem (2.1).

Definition 2.1. A point $\bar{z} \in F$ is called a *stationary point* of the NLP, if there are multiplier vectors $\lambda \in R^m_+$ and $\mu \in R^p$ such that $(\bar{z}, \lambda, \mu)$ is a Karush–Kuhn–Tucker (KKT) point of (2.1), that is, λ and μ satisfy $\lambda_i g_i(\bar{z}) = 0$, and

$$\nabla f(\bar{z}) + \sum_{i=1}^{m} \lambda_i \nabla g_i(\bar{z}) + \sum_{i=1}^{p} \mu_i \nabla h_i(\bar{z}) = 0$$

for all $i = 1, 2, \ldots, m$.

For MPCC (1.1), some different concepts of stationary points have also been introduced in the literature. In this paper, the following types of stationary points are relevant to the description of convergence results.

Definition 2.2. Let $\bar{z}$ be a feasible point of problem (1.1). Then $\bar{z}$ is said to be:

(a) *weakly stationary*, if there exist multiplier vectors $\bar{\lambda} \in R^m$, $\bar{\mu} \in R^p$ and $\bar{u}, \bar{v} \in R^l$ such that

$$\begin{gathered} \nabla f(\bar{z}) + \nabla g(\bar{z})\bar{\lambda} + \nabla h(\bar{z})\bar{\mu} - \nabla G(\bar{z})\bar{u} - \nabla H(\bar{z})\bar{v} = 0, \\ \bar{\lambda} \geq 0, \quad \bar{z} \in F, \quad \bar{\lambda}^T g(\bar{z}) = 0, \\ \bar{u}_i = 0, \quad i \in I_{+0}(\bar{z}), \\ \bar{v}_i = 0, \quad i \in I_{0+}(\bar{z}); \end{gathered}$$

(b) *C-stationary*, if it is weakly stationary and

$$\bar{u}_i \bar{v}_i \geq 0, \quad i \in I_{00}(\bar{z});$$

(c) *M-stationary*, if it is weakly stationary and

$$\bar{u}_i > 0, \quad \bar{v}_i > 0, \quad \text{or} \quad \bar{u}_i \bar{v}_i = 0, \quad i \in I_{00}(\bar{z});$$

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(d) *strongly stationary*, if it is weakly stationary and

$$\bar{u}_i \geq 0, \quad \bar{v}_i \geq 0, \quad i \in I_{00}(\bar{z}),$$

where

$$I_{+0}(\bar{z}) = \{i : G_i(\bar{z}) > 0, H_i(\bar{z}) = 0\},$$
$$I_{0+}(\bar{z}) = \{i : G_i(\bar{z}) = 0, H_i(\bar{z}) > 0\},$$
$$I_{00}(\bar{z}) = \{i : G_i(\bar{z}) = 0, H_i(\bar{z}) = 0\}.$$

Similar to that in the NLP, a suitable constraint qualification (CQ) is necessary for a local minimizer $\bar{z}$ of MPCC (1.1) to be a stationary point. However, as pointed out by Luo et al. [11], the CQs in the standard NLP are often violated for MPCC (1.1). Thus, one of the key theoretical issues in MPCCs is to study new CQs such that an MPCC local minimizer satisfies conditions on the stationary points. In this paper, we will mainly use the concept of mathematical programs with equilibrium constraints – Mangasarian–Fromovitz constraints qualification (MPEC–MFCQs).

Definition 2.3. Let $\bar{z} \in F$. $\bar{z}$ is said to satisfy an MPEC–MFCQ, if the gradients

$$\{\nabla g_i(\bar{z}) \mid i \in I_g(\bar{z})\} \cup \{\{\nabla h_i(\bar{z}) \mid i = 1, 2, \ldots, p\} \cup \{\nabla G_i(\bar{z}) \mid i \in I_{00}(\bar{z}) \cup I_{0+}(\bar{z})\}$$
$$\cup \{\nabla H_i(\bar{z}) \mid i \in I_{00}(\bar{z}) \cup I_{+0}(\bar{z})\}\}$$

are positive linearly independent. In other words, $\bar{z}$ is said to satisfy an MPEC–MFCQ, if and only if there does not exist a vector $(\lambda_{I_g(\bar{z})}, \mu, \alpha_{I_{00}(\bar{z})\cup I_{0+}(\bar{z})}, \beta_{I_{00}(\bar{z})\cup I_{+0}(\bar{z})}) \neq 0$ with $\lambda_i \geq 0$ for all $i \in I_g(\bar{z})$ such that

$$\sum_{i\in I_g(\bar{z})} \lambda_i \nabla g_i(\bar{z}) + \sum_{i=1}^{p} \mu_i \nabla h_i(\bar{z}) - \sum_{i\in I_{00}(\bar{z})\cup I_{0+}(\bar{z})} \alpha_i \nabla G_i(\bar{z}) - \sum_{i\in I_{00}(\bar{z})\cup I_{+0}(\bar{z})} \beta_i \nabla H_i(\bar{z}) = 0.$$

3. New locally smoothing method and algorithm

In this section, we propose a new locally smoothing method for MPCC (1.1) and develop an algorithm. The complementarity conditions

$$a \geq 0, \quad b \geq 0, \quad ab = 0$$

can be written as

$$a \geq 0, \quad b \geq 0, \quad a + b \leq |a - b|.$$

Thus, with any kind of smooth approximation to the absolute value function $|\cdot|$, we can construct a smooth approximation to the complementarity conditions.

Steffensen and Ulbrich [10] approximated $\varphi(t) = |t|$ by

$$\psi_\epsilon(t) = \frac{2t}{\pi} \arctan\Big(\frac{t}{\varepsilon}\Big)$$

for solving the complementarity problems. We attempt to construct a new locally smoothing method for the complementarity constraints in MPCC (1.1) to obtain a

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perturbed problem for (1.1). Unlike the existing approaches in the literature, we replace the complementarity constraints

$$G(z) \geq 0, \quad H(z) \geq 0, \quad G(z)^T H(z) = 0 \tag{3.1}$$

with

$$G(z) \geq 0, \quad H(z) \geq 0, \quad \Phi_\varepsilon(z) \leq 0, \tag{3.2}$$

where

$$\Phi_\varepsilon(z) = \begin{pmatrix} \phi_{\varepsilon,1}(z) \\ \vdots \\ \phi_{\varepsilon,l}(z) \end{pmatrix}, \quad \phi_{\varepsilon,i}(z) = \tfrac{1}{2}\{G_i(z) + H_i(z) - \psi_\epsilon(G_i(z) - H_i(z))\}. \tag{3.3}$$

Thus, MPCC (1.1) is reformulated as a standard optimization problem as follows:

$$\begin{array}{ll} \text{minimize} & f(z) \\ \text{subject to} & g(z) \leq 0, \quad h(z) = 0, \\ & G(z) \geq 0, \quad H(z) \geq 0, \\ & \Phi_\varepsilon(z) \leq 0. \end{array} \tag{3.4}$$

Let F_ε denote the feasible region of problem (3.4).

REMARK 3.1. In the existing smoothing methods in articles such as [3, 5, 13], the equilibrium constraints (3.1) are overall substituted by a system of smooth equations. In our smoothing method (3.2)–(3.3), the inequality constraints $G(z) \geq 0$ and $H(z) \geq 0$ do not change, except for the equality $G(z)^T H(z) = 0$ being approximated by the inequality $\Phi_\varepsilon(z) \leq 0$.

REMARK 3.2. For the standard smooth optimization problem (3.4), we prove in Section 4 that it satisfies the standard MFCQ at any feasible point (see Theorem 4.1). Thus, there are many efficient algorithms that can be used to find its stationary point for each smoothing parameter ε (see [2, 7, 8]). One of our main interests is in proving that the sequence of stationary points generated by solving NLP(ε_k) with $\varepsilon_k \downarrow 0$, converges to the stationary point of problem (1.1).

We now present some results that are used in developing our smoothing method.

LEMMA 3.1. *The function $\psi_\epsilon : R \to R$ has the following properties.*

(1) *For arbitrary given positive constants a and b ($b \geq a$), there exists a constant scalar $T_{ab} > 0$ such that*

$$0 \leq |t| - \psi_\epsilon(t) \leq T_{ab}\varepsilon \quad \textit{for all } t \in [a, b] \cup [-b, -a].$$

(2) *Let $\partial\varphi(t)$ be the generalized Clarke gradient [1] of $\varphi(t)$. Then, for an arbitrary t,*

$$\lim_{\varepsilon \downarrow 0} \operatorname{dist}(\psi'_\epsilon(t), \partial\varphi(t)) = 0,$$

where $\operatorname{dist}(v, S)$ *is the distance of the point v from the set S.*

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The proof of Lemma 3.1 follows directly from a result of Li and Li [10].

LEMMA 3.2. *Let $m : R \to R$ be defined by*

$$m(t) = \frac{1}{\pi}\arctan t + \frac{1}{\pi}\frac{t}{1+t^2}$$

for all $t \in R$. Then, (1) *$m(t)$ is increasing;* (2) $-1/2 < m(t) < 1/2$.

PROOF. (1) Since $m'(t) = 2/\pi(1+t^2)^2 > 0$, $m(t)$ is increasing on R.

(2) Since $\arctan t \to \pi/2$ as $t \to +\infty$, and $\arctan t \to -\pi/2$ as $t \to -\infty$, $t/(1+t^2) \to 0$ as $t \to \pm\infty$. Thus, $m(t) \to 1/2$ as $t \to +\infty$, and $m(t) \to -1/2$ as $t \to -\infty$. From the first result, we conclude that $-1/2 < m(t) < 1/2$ for all $t \in R$. □

LEMMA 3.3. *Let $\phi_{\varepsilon,i}$ be defined as in* (3.3). *Then the following statements are true.*

(1) *For all $i = 1, 2, \ldots, l$, $\phi_{\varepsilon,i}$ is continuously differentiable.*
(2) $\nabla\phi_{\varepsilon,i}(z) = \eta_i^{\Phi_\varepsilon}\nabla G_i(z) + \zeta_i^{\Phi_\varepsilon}\nabla H_i(z)$, *where*

$$\eta_i^{\Phi_\varepsilon} = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{G_i(z)-H_i(z)}{\varepsilon} - \frac{1}{\pi}\frac{\{G_i(z)-H_i(z)\}/\varepsilon}{1+[\{G_i(z)-H_i(z)\}/\varepsilon]^2}$$

and

$$\zeta_i^{\Phi_\varepsilon} = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{G_i(z)-H_i(z)}{\varepsilon} + \frac{1}{\pi}\frac{\{G_i(z)-H_i(z)\}/\varepsilon}{1+[\{G_i(z)-H_i(z)\}/\varepsilon]^2}.$$

(3) $\eta_i^{\Phi_\varepsilon} + \zeta_i^{\Phi_\varepsilon} = 1$. *Furthermore, $\eta_i^{\Phi_\varepsilon} \in (0,1)$, $\zeta_i^{\Phi_\varepsilon} \in (0,1)$.*
(4) *Let $\bar{z}$ be a feasible point of problem* (1.1). *If $i \in I_{+0}(\bar{z})$, then $\eta_i^{\Phi_\varepsilon} \to 0$, $\zeta_i^{\Phi_\varepsilon} \to 1$ as $z \to \bar{z}$ and $\varepsilon \downarrow 0$. If $i \in I_{0+}(\bar{z})$, then $\eta_i^{\Phi_\varepsilon} \to 1$, $\zeta_i^{\Phi_\varepsilon} \to 0$ as $z \to \bar{z}$ and $\varepsilon \downarrow 0$.*

PROOF. The first part follows from the definition of $\phi_{\varepsilon,i}$. By direct calculation, we can obtain the second part. The third part directly follows from the second one and Lemma 3.2, so it remains to prove only the last part.

Since $i \in I_{+0}(\bar{z})$,

$$\eta_i^{\Phi_\varepsilon} = \frac{1}{2} - \frac{1}{\pi}\arctan\frac{G_i(z)-H_i(z)}{\varepsilon} - \frac{1}{\pi}\frac{\{G_i(z)-H_i(z)\}/\varepsilon}{1+[\{G_i(z)-H_i(z)\}/\varepsilon]^2},$$
$$\zeta_i^{\Phi_\varepsilon} = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{G_i(z)-H_i(z)}{\varepsilon} + \frac{1}{\pi}\frac{\{G_i(z)-H_i(z)\}/\varepsilon}{1+[\{G_i(z)-H_i(z)\}/\varepsilon]^2}.$$

We know that $\{G_i(z)-H_i(z)\}/\varepsilon \to +\infty$ as $z \to \bar{z}$ and $\varepsilon \downarrow 0$. Thus, it is clear that $\eta_i^{\Phi_\varepsilon} \to 0$, $\zeta_i^{\Phi_\varepsilon} \to 1$ as $z \to \bar{z}$ and $\varepsilon \downarrow 0$. Similarly, when $i \in I_{0+}(\bar{z})$, we get $\{G_i(z)-H_i(z)\}/\varepsilon \to -\infty$ as $z \to \bar{z}$ and $\varepsilon \downarrow 0$. Therefore,

$$\eta_i^{\Phi_\varepsilon} \to 1, \quad \zeta_i^{\Phi_\varepsilon} \to 0\ (z \to \bar{z},\ \varepsilon \downarrow 0).$$

This completes the proof of the lemma. □

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In virtue of the perturbed smoothing problem (3.4), we now develop an efficient algorithm to solve MPCC (1.1).

Algorithm 1: Algorithm to solve MPCC (1.1)

1. Given an initial point z_1, $\varepsilon_1 > 0$, $\epsilon_{\text{stop}}, \beta \in (0, 1)$, $k = 1$.
2. Let ε_k be the current parameter. Solve subproblem (3.4) with $\varepsilon = \varepsilon_k$ by a smooth NLP solver. The optimal solution is referred to as $\bar{z}_k$.
3. If $\text{maxvio}(\bar{z}_k) < \epsilon_{\text{stop}}$, then $z_k = \bar{z}_k$ is the approximate solution of MPCC (1.1). The algorithm stops. Otherwise, set $\varepsilon_{k+1} = \beta\varepsilon_k$, $z_{k+1} = \bar{z}_k$, $k = k + 1$. Return to step 2.

Remark 3.3. In step 3 of Algorithm 1, to measure the violation degree at the final iterate $\bar{z}_k$, we denote the maximal violation of all constraints by

$$\text{maxvio}(\bar{z}_k) = \max\{\|\max\{g(\bar{z}_k), 0\}\|, \|h(\bar{z}_k)\|, \|\min\{G(\bar{z}_k), H(\bar{z}_k)\}\|\}.$$

If $\text{maxvio}(\bar{z}_k)$ is small enough, then the solution of the original problem (1.1) is obtained.

4. Convergence analysis

In this section, we will consider the limiting behaviour of a sequence of stationary points of the subproblems. Write

$$I_{\Phi_\varepsilon}(z) = \{i \mid \phi_{\varepsilon,i}(z) = 0\}.$$

We first study the constraint qualification of problem (3.4). The following lemma presents the conditions to guarantee MFCQ.

Lemma 4.1. *Let $\bar{z}$ be a feasible point of problem* (1.1)*. Suppose that the MPEC–MFCQ is satisfied at $\bar{z}$. Then there is a neighbourhood $U(\bar{z})$ and an $\bar{\varepsilon} > 0$ such that the vectors*

$$\begin{cases} \nabla g_i(z), & i \in I_g(\bar{z}), \\ \nabla h_i(z), & i = 1, 2, \ldots, p, \\ \nabla G_i(z), & i \in I_G(z), \\ \nabla H_i(z), & i \in I_H(z), \\ \eta_i^{\Phi_\varepsilon}\nabla G_i(z) + \zeta_i^{\Phi_\varepsilon}\nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{0+}(\bar{z}), \\ \eta_i^{\Phi_\varepsilon}\nabla G_i(z) + \zeta_i^{\Phi_\varepsilon}\nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{+0}(\bar{z}), \\ \nabla G_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar{z}), \\ \nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar{z}), \end{cases}$$

are positive linearly independent for all $z \in U(\bar{z}) \cap F_\varepsilon$ and $\varepsilon \in (0, \bar{\varepsilon})$.

Proof. Since g, h, G, H are all continuous, there exist a neighbourhood $U_1(\bar{z})$ and a positive constant $\bar{\varepsilon}_1$ such that for any $\varepsilon \in (0, \bar{\varepsilon}_1)$ and any point $z \in U_1(\bar{z}) \cap F_\varepsilon$

$$\begin{cases} I_g(z) \subseteq I_g(\bar{z}), & I_G(z) \subseteq I_{00}(\bar{z}) \cup I_{0+}(\bar{z}), \\ I_h(z) \subseteq I_h(\bar{z}), & I_H(z) \subseteq I_{00}(\bar{z}) \cup I_{+0}(\bar{z}), \end{cases}$$

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which yields

$$I_{\Phi_\varepsilon}(z) \cap I_G(z) = \emptyset, \quad I_{\Phi_\varepsilon}(z) \cap I_H(z) = \emptyset. \tag{4.1}$$

In fact, if $i \in I_G(z)$, then $G_i(z) = 0$ and

$$\begin{aligned} 2\phi_{\varepsilon,i}(z) &= G_i(z) + H_i(z) - \frac{2\{G_i(z) - H_i(z)\}}{\pi} \arctan\Big(\frac{G_i(z) - H_i(z)}{\varepsilon}\Big) \\ &= H_i(z) + \frac{2H_i(z)}{\pi} \arctan\Big(\frac{-H_i(z)}{\varepsilon}\Big) \\ &> H_i(z) - \frac{2H_i(z)}{\pi}\frac{\pi}{2} = 0. \end{aligned}$$

Therefore, $i \notin I_{\Phi_\varepsilon}(z)$, that is, $I_{\Phi_\varepsilon}(z) \cap I_G(z) = \emptyset$. Similarly, we obtain $I_{\Phi_\varepsilon}(z) \cap I_H(z) = \emptyset$. Note that the MPEC–MFCQ holds, so the gradients

$$\begin{gathered} \{\nabla g_i(\bar z) \mid i \in I_g(\bar z)\} \cup \{\{\nabla h_i(\bar z) \mid i = 1, 2, \ldots, p\} \cup \{\nabla G_i(\bar z) \mid i \in I_{00}(\bar z) \cup I_{0+}(\bar z)\} \\ \cup \{\nabla H_i(\bar z) \mid i \in I_{00}(\bar z) \cup I_{+0}(\bar z)\}\} \end{gathered}$$

are positive linearly independent by Definition 2.3.

We have

$$\begin{cases} I_G(z) \cup (I_{\Phi_\varepsilon}(z) \cap I_{0+}(\bar z)) \cup (I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar z)) \subseteq I_{00}(\bar z) \cup I_{0+}(\bar z) \\ I_H(z) \cup (I_{\Phi_\varepsilon}(z) \cap I_{+0}(\bar z)) \cup (I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar z)) \subseteq I_{00}(\bar z) \cup I_{+0}(\bar z). \end{cases}$$

On the other hand, $G_i(z) > 0$, $H_i(z)$ is sufficiently close to zero for all $i \in I_{+0}(\bar z)$, and $G_i(z)$ is sufficiently close to zero, $H_i(z) > 0$ for all $i \in I_{0+}(\bar z)$ with z being close to $\bar z$. Thus, it is clear from Lemma 3.3(4) that

$$\begin{cases} \eta_i^{\Phi_\varepsilon} \to 0, & \zeta_i^{\Phi_\varepsilon} \to 1 \quad \text{for all } i \in I_{+0}(\bar z), \\ \eta_i^{\Phi_\varepsilon} \to 1, & \zeta_i^{\Phi_\varepsilon} \to 0 \quad \text{for all } i \in I_{0+}(\bar z). \end{cases}$$

Similar to the proof of Proposition 2.2 given by Qi and Wei [16], we know that there exist a neighbourhood $U_2(\bar z)$ and a sufficiently small $\bar\varepsilon_2 > 0$ such that the set of vectors

$$\begin{cases} \nabla g_i(z), & i \in I_g(\bar z), \\ \nabla h_i(z), & i = 1, 2, \ldots, p, \\ \nabla G_i(z), & i \in I_G(z), \\ \nabla H_i(z), & i \in I_H(z), \\ \eta_i^{\Phi_\varepsilon} \nabla G_i(z) + \zeta_i^{\Phi_\varepsilon} \nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{0+}(\bar z), \\ \eta_i^{\Phi_\varepsilon} \nabla G_i(z) + \zeta_i^{\Phi_\varepsilon} \nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{+0}(\bar z), \\ \nabla G_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar z), \\ \nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar z), \end{cases}$$

is positive linearly independent for the given $z \in U_2(\bar z) \cap F_\varepsilon$ and $\varepsilon \in (0, \bar\varepsilon_2)$.

Denote $U(\bar{z}) = U_1(\bar{z}) \cap U_2(\bar{z})$ and $\bar{\varepsilon} = \min\{\bar{\varepsilon}_1, \ \bar{\varepsilon}_2\}$. Then, for all $z \in U(\bar{z}) \cap F_\varepsilon$ and $\varepsilon \in (0, \bar{\varepsilon})$, the vectors

$$\begin{cases} \nabla g_i(z), & i \in I_g(\bar{z}), \\ \nabla h_i(z), & i = 1, 2, \ldots, p, \\ \nabla G_i(z), & i \in I_G(z), \\ \nabla H_i(z), & i \in I_H(z), \\ \eta_i^{\Phi_\varepsilon} \nabla G_i(z) + \zeta_i^{\Phi_\varepsilon} \nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{0+}(\bar{z}), \\ \eta_i^{\Phi_\varepsilon} \nabla G_i(z) + \zeta_i^{\Phi_\varepsilon} \nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{+0}(\bar{z}), \\ \nabla G_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar{z}), \\ \nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar{z}), \end{cases}$$

are positive linearly independent. This completes the proof of the lemma. □

THEOREM 4.1. *Let $\bar{z}$ be a feasible point of problem (1.1) and suppose that the MPEC–MFCQ is satisfied at this point. Then there exist a neighbourhood $U(\bar{z})$ of $\bar{z}$ and an $\bar{\varepsilon} > 0$ small enough such that problem (3.4) satisfies the standard MFCQ at any point $z \in U(\bar{z}) \cap F_\varepsilon$, where $\varepsilon \in (0, \bar{\varepsilon})$.*

PROOF. From Lemma 4.1, it follows that there exist a neighbourhood $U(\bar{z})$ and an $\bar{\varepsilon} > 0$ such that the vectors

$$\begin{cases} \nabla g_i(z), & i \in I_g(\bar{z}), \\ \nabla h_i(z), & i = 1, 2, \ldots, p, \\ \nabla G_i(z), & i \in I_G(z), \\ \nabla H_i(z), & i \in I_H(z), \\ \eta_i^{\Phi_\varepsilon} \nabla G_i(z) + \zeta_i^{\Phi_\varepsilon} \nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{0+}(\bar{z}), \\ \eta_i^{\Phi_\varepsilon} \nabla G_i(z) + \zeta_i^{\Phi_\varepsilon} \nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{+0}(\bar{z}), \\ \nabla G_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar{z}), \\ \nabla H_i(z), & i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar{z}), \end{cases} \tag{4.2}$$

are positive linearly independent, if $z \in U(\bar{z}) \cap F_\varepsilon$ and $\varepsilon \in (0, \bar{\varepsilon})$.

We now claim that the standard MFCQ holds for problem (3.4), if $z \in U(\bar{z}) \cap F_\varepsilon$ and $\varepsilon \in (0, \bar{\varepsilon})$. In view of Definition 2.3, we show that for any given $z \in U(\bar{z}) \cap F_\varepsilon$,

$$\begin{aligned} \sum_{i \in I_g(z)} \lambda_i \nabla g_i(z) + \sum_{i=1}^{p} \mu_i \nabla h_i(z) - \sum_{i \in I_G(z)} \alpha_i \nabla G_i(z) - \sum_{i \in I_H(z)} \beta_i \nabla H_i(z) \\ + \sum_{i=1}^{l} \gamma_i (\eta_i^{\Phi_\varepsilon} \nabla G_i(z) + \zeta_i^{\Phi_\varepsilon} \nabla H_i(z)) = 0, \end{aligned} \tag{4.3}$$

if and only if all the multiplier vectors $\mu \in R^p$, λ, α, β and γ are zero.

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We rewrite equation (4.3) as

$$\begin{aligned}0 = &\sum_{i\in I_g(z)} \lambda_i \nabla g_i(z) + \sum_{i=1}^{p} \mu_i \nabla h_i(z) - \sum_{i\in I_G(z)} \alpha_i \nabla G_i(z) - \sum_{i\in I_H(z)} \beta_i \nabla H_i(z)\\ &+ \sum_{i\in I_{\Phi_\varepsilon}(z)\cap I_{+0}(\bar z)} \gamma_i [\eta_i^{\Phi_\varepsilon} \nabla G_i(z) + \zeta_i^{\Phi_\varepsilon} \nabla H_i(z)]\\ &+ \sum_{i\in I_{\Phi_\varepsilon}(z)\cap I_{0+}(\bar z)} \gamma_i [\eta_i^{\Phi_\varepsilon} \nabla G_i(z) + \zeta_i^{\Phi_\varepsilon} \nabla H_i(z)]\\ &+ \sum_{i\in I_{\Phi_\varepsilon}(z)\cap I_{00}(\bar z)} \gamma_i \eta_i^{\Phi_\varepsilon} \nabla G_i(z) + \sum_{i\in I_{\Phi_\varepsilon}(z)\cap I_{00}(\bar z)} \gamma_i \zeta_i^{\Phi_\varepsilon} \nabla H_i(z).\end{aligned} \tag{4.4}$$

By positive linear independence, it follows from (4.2) and (4.4) that

$$\begin{cases}\lambda_i = 0\ (i \in I_g(z)), \quad \mu_i = 0\ (i = 1, 2, \ldots, p), \quad \alpha_i = 0\ (i \in I_G(z)), \quad \beta_i = 0\ (i \in I_H(z)),\\ \gamma_i = 0\ (i \in I_{\Phi_\varepsilon}(z) \cap (I_{+0}(\bar z) \cup I_{0+}(\bar z))), \quad \gamma_i \eta_i^{\Phi_\varepsilon} = \gamma_i \zeta_i^{\Phi_\varepsilon} = 0, \quad i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar z).\end{cases}$$

From Lemma 3.3(3), it follows that for all $i \in I_{00}(\bar z)$, $\eta_i^{\Phi_\varepsilon} + \zeta_i^{\Phi_\varepsilon} = 1$. Thus, $\gamma_i = 0$ for all $i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar z)$. Since $\gamma_i = 0$ $(i \in I_{\Phi_\varepsilon}(z) \cap (I_{+0}(\bar z) \cup I_{0+}(\bar z)))$ and $\gamma_i = 0$ $(i \in I_{\Phi_\varepsilon}(z) \cap I_{00}(\bar z))$, we get $\gamma_i = 0$, $i \in I_{\Phi_\varepsilon}(z)$. Therefore, the desired result follows directly from Definition 2.3. This completes the proof. □

The following theorems establish the relations between the solutions of the original problem and the smoothing subproblem under the MPEC–MFCQ.

Theorem 4.2. *Let $\{\varepsilon_k\}$ be a positive sequence which converges to zero. Suppose that $\{z_k\}$ is a sequence of stationary points of the subproblems (3.4) with $\varepsilon = \varepsilon_k$. If $\bar z$ is an accumulation point of the sequence $\{z_k\}$ such that the MPEC–MFCQ holds at $\bar z$, then $\bar z$ is a C-stationary point of problem (1.1).*

Proof. It follows from Theorem 4.1 that there exist Lagrangian multiplier vectors λ^k, μ^k, α^k, β^k and γ^k such that

$$\begin{aligned}\nabla f(z_k) + \sum_{i=1}^{m} \lambda_i^k \nabla g_i(z_k) + \sum_{i=1}^{p} \mu_i^k \nabla h_i(z_k) - \sum_{i\in I_G(z_k)} \alpha_i^k \nabla G_i(z_k)&\\ - \sum_{i\in I_H(z_k)} \beta_i^k \nabla H_i(z_k) + \sum_{i=1}^{l} \gamma_i^k \nabla \phi_{\varepsilon_k,i}(z_k) = 0,&\end{aligned} \tag{4.5}$$

$$\begin{aligned}\lambda^k \geq 0, &\quad \mathrm{supp}(\lambda^k) \subseteq I_g(z_k),\\ \alpha^k \geq 0, &\quad \mathrm{supp}(\alpha^k) \subseteq I_G(z_k),\\ \beta^k \geq 0, &\quad \mathrm{supp}(\beta^k) \subseteq I_H(z_k),\\ \gamma^k \geq 0, &\quad \mathrm{supp}(\gamma^k) \subseteq I_{\Phi_{\varepsilon_k}}(z_k).\end{aligned} \tag{4.6}$$

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From equation (4.5), we have

$$\begin{aligned}\nabla f(z_k) &+ \sum_{i\in\operatorname{supp}(\lambda^k)} \lambda_i^k \nabla g_i(z_k) + \sum_{i\in\operatorname{supp}(\mu^k)} \mu_i^k \nabla h_i(z_k) - \sum_{i\in\operatorname{supp}(\alpha^k)} \alpha_i^k \nabla G_i(z_k)\\ &- \sum_{i\in\operatorname{supp}(\beta^k)} \beta_i^k \nabla H_i(z_k) + \sum_{i\in\operatorname{supp}(\gamma^k)} \gamma_i^k \eta_i^{\Phi_{\varepsilon_k}} \nabla G_i(z_k)\\ &+ \sum_{i\in\operatorname{supp}(\gamma^k)} \gamma_i^k \zeta_i^{\Phi_{\varepsilon_k}} \nabla H_i(z_k) = 0,\end{aligned}$$

and equation (4.1) yields

$$\operatorname{supp}(\alpha^k)\cap\operatorname{supp}(\gamma^k)=\emptyset,\quad \operatorname{supp}(\beta^k)\cap\operatorname{supp}(\gamma^k)=\emptyset. \tag{4.7}$$

Denote

$$\bar{\upsilon}_i^k = \begin{cases} \alpha_i^k, & i\in\operatorname{supp}(\alpha^k),\\ -\gamma_i^k\eta_i^{\Phi_{\varepsilon_k}}, & i\in\operatorname{supp}(\gamma^k)\backslash I_{+0}(\bar{z}),\\ 0, & \text{otherwise,}\end{cases}$$

and

$$\bar{\nu}_i^k = \begin{cases} \beta_i^k, & i\in\operatorname{supp}(\beta^k),\\ -\gamma_i^k\zeta_i^{\Phi_{\varepsilon_k}}, & i\in\operatorname{supp}(\gamma^k)\backslash I_{0+}(\bar{z}),\\ 0, & \text{otherwise.}\end{cases}$$

Then equation (4.5) reduces to

$$\begin{aligned}\nabla f(z_k) &+ \sum_{i=1}^m \lambda_i^k \nabla g_i(z_k) + \sum_{i=1}^p \mu_i^k \nabla h_i(z_k) - \sum_{i=1}^l \bar{\upsilon}_i^k \nabla G_i(z_k) - \sum_{i=1}^l \bar{\nu}_i^k \nabla H_i(z_k)\\ &+ \sum_{i\in I_{+0}(\bar{z})} \gamma_i^k \eta_i^{\Phi_{\varepsilon_k}} \nabla G_i(z_k) + \sum_{i\in I_{0+}(\bar{z})} \gamma_i^k \zeta_i^{\Phi_{\varepsilon_k}} \nabla H_i(z_k) = 0.\end{aligned} \tag{4.8}$$

We now prove that the sequence $\{(\lambda^k,\mu^k,\bar{\upsilon}^k,\bar{\nu}^k,\gamma^k_{I_{+0}(\bar{z})\cup I_{0+}(\bar{z})})\}$ is bounded. If it is unbounded, then there exists a subset K such that for $k\in K$, the normed sequence converges:

$$\frac{(\lambda^k,\mu^k,\bar{\upsilon}^k,\bar{\nu}^k,\gamma^k_{I_{+0}(\bar{z})\cup I_{0+}(\bar{z})})}{\|(\lambda^k,\mu^k,\bar{\upsilon}^k,\bar{\nu}^k,\gamma^k_{I_{+0}(\bar{z})\cup I_{0+}(\bar{z})})\|} \to (\lambda,\mu,\bar{\upsilon},\bar{\nu},\gamma_{I_{+0}(\bar{z})\cup I_{0+}(\bar{z})}) \neq 0.$$

Combined with (4.8), part (4) of Lemma 3.3 yields

$$\sum_{i=1}^m \lambda_i \nabla g_i(\bar{z}) + \sum_{i=1}^p \mu_i \nabla h_i(\bar{z}) - \sum_{i=1}^l \bar{\upsilon}_i \nabla G_i(\bar{z}) - \sum_{i=1}^l \bar{\nu}_i \nabla H_i(\bar{z}) = 0,$$

where $\lambda \geq 0$ and for all sufficiently large k,

$$\begin{aligned}&\operatorname{supp}(\lambda)\subseteq I_g(z_k)\subseteq I_g(\bar{z}),\\ &\operatorname{supp}(\bar{\upsilon})\subseteq I_G(z_k)\cup I_{\Phi_{\varepsilon_k}}(z_k)\backslash I_{+0}(\bar{z})\subseteq I_{00}(\bar{z})\cup I_{0+}(\bar{z}),\\ &\operatorname{supp}(\bar{\nu})\subseteq I_H(z_k)\cup I_{\Phi_{\varepsilon_k}}(z_k)\backslash I_{0+}(\bar{z})\subseteq I_{00}(\bar{z})\cup I_{+0}(\bar{z}).\end{aligned}$$

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We show that $(\lambda, \mu, \bar{\upsilon}, \bar{\nu}) \neq 0$. Actually, if $(\lambda, \mu, \bar{\upsilon}, \bar{\nu}) = 0$, then, at least for one $i \in I_{+0}(\bar{z}) \cup I_{0+}(\bar{z})$, we have $\gamma_i \neq 0$. Without loss of generality, assume that $\gamma_i \neq 0$ for an $i \in I_{+0}(\bar{z})$. Then, for all k sufficiently large, $\gamma_i^k \neq 0$. Consequently, $\bar{\nu}_i^k = -\gamma_i^k \zeta_i^{\Phi_{\varepsilon_k}} \neq 0$. Since $i \in I_{+0}(\bar{z})$, it follows from Lemma 3.3(4) that

$$\bar{\nu}_i = \lim_{k \in K} -\gamma_i^k \zeta_i^{\Phi_{\varepsilon_k}} \neq 0,$$

which contradicts the assumption that $\bar{\nu} = 0$.

By Definition 2.3, $(\lambda, \mu, \bar{\upsilon}, \bar{\nu}) \neq 0$ contradicts the fact that the MPEC–MFCQ holds at $\bar{z}$. Thus, we have proved that the sequence $\{(\lambda^k, \mu^k, \bar{\upsilon}^k, \bar{\nu}^k, \gamma^k_{I_{+0}(\bar{z}) \cup I_{0+}(\bar{z})})\}$ is bounded. Without loss of generality, we suppose that this sequence converges to a point $(\lambda^*, \mu^*, \bar{\upsilon}^*, \bar{\nu}^*, \gamma^*_{I_{+0}(\bar{z}) \cup I_{0+}(\bar{z})})$. It follows that $\lambda^* \geq 0$ and $\mathrm{supp}(\lambda^*) \subseteq I_g(\bar{z})$ and definitions of $\bar{\upsilon}$ and $\bar{\nu}$ yield

$$\mathrm{supp}(\bar{\upsilon}^*) \subseteq I_{00}(\bar{z}) \cup I_{0+}(\bar{z}), \quad \mathrm{supp}(\bar{\nu}^*) \subseteq I_{00}(\bar{z}) \cup I_{+0}(\bar{z}).$$

Since f, g, h, G and H are continuously differentiable, we have

$$\nabla f(\bar{z}) + \sum_{i=1}^{m} \lambda_i^* \nabla g_i(\bar{z}) + \sum_{i=1}^{p} \mu_i^* \nabla h_i(\bar{z}) - \sum_{i=1}^{l} \bar{\upsilon}_i^* \nabla G_i(\bar{z}) - \sum_{i=1}^{l} \bar{\nu}_i^* \nabla H_i(\bar{z}) = 0.$$

From the definitions of $\bar{\upsilon}_i^k$ and $\bar{\nu}_i^k$, we get $\bar{\upsilon}_i^* = 0,\ i \in I_{+0}(\bar{z})$, $\bar{\nu}_i^* = 0,\ i \in I_{0+}(\bar{z})$. In other words, $\bar{z}$ is weakly stationary.

In order to prove that $\bar{z}$ is C-stationary, we give the proof in the following three cases:

(1) For $i \in I_{00}(\bar{z})$, if $i \in \mathrm{supp}(\alpha^k)$ then from (4.6) we get $\bar{\upsilon}_i^k \geq 0$ and $\bar{\nu}_i^k \geq 0$, since $\mathrm{supp}(\alpha^k) \cap \mathrm{supp}(\gamma^k) = \emptyset$. Thus, $\bar{\upsilon}_i^* \bar{\nu}_i^* \geq 0$.
(2) For $i \in I_{00}(\bar{z})$, if $i \in \mathrm{supp}(\gamma^k) \backslash I_{+0}(\bar{z})$ then $\bar{\upsilon}_i^k < 0$ and $\bar{\nu}_i^k \leq 0$, since $\mathrm{supp}(\beta^k) \cap \mathrm{supp}(\gamma^k) = \emptyset$. Thus, $\bar{\upsilon}_i^* \bar{\nu}_i^* \geq 0$.
(3) For $i \in I_{00}(\bar{z})$, if $i \notin \mathrm{supp}(\alpha^k)$ and $i \notin \mathrm{supp}(\gamma^k) \backslash I_{+0}(\bar{z})$ then $\bar{\upsilon}_i^k = 0$. Therefore, $\bar{\upsilon}_i^* \bar{\nu}_i^* \geq 0$.

Based on the above discussion, we conclude that $\bar{\upsilon}_i^* \bar{\nu}_i^* \geq 0$ for all $i \in I_{00}(\bar{z})$. From Definition 2.2, it now follows that $\bar{z}$ is a C-stationary point of the original MPCC (1.1). This completes the proof. □

By using the definitions of $\overline{\upsilon}_i^k$ and $\overline{\nu}_i^k$ and the boundedness of the sequence $\{(\overline{\upsilon}^k, \overline{\nu}^k)\}$, we get the following result which is necessary for the subsequent analysis on M-stationary points of problem (1.1).

Corollary 4.1. *The sequence* $\{(\gamma^k_{\mathrm{supp}(\gamma^k) \cap I_{00}(\bar{z})})\}$ *is bounded under the conditions of Theorem* 4.2.

Next, we study the conditions for M-stationarity.

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THEOREM 4.3. *Let $\{\varepsilon_k\}$ be a positive sequence which converges to zero. Let z_k be a stationary point of problem (3.4) with $\varepsilon = \varepsilon_k$. If $\bar{z}$ is an accumulation point of the sequence $\{z_k\}$ as $k \to \infty$ such that the MPEC–MFCQ holds at $\bar{z}$, and for all $i \in supp(\gamma^k) \cap I_{00}(\bar{z})$,*

$$\lim_{k\to+\infty} \eta_i^{\Phi_{\varepsilon_k}} \zeta_i^{\Phi_{\varepsilon_k}} = 0,$$

then $\bar{z}$ is an M-stationary point of problem (1.1).

PROOF. By Theorem 4.2, we only need to show that for all $i \in I_{00}(\bar{z})$,

$$\bar{\upsilon}_i^* > 0, \quad \bar{\nu}_i^* > 0, \quad \text{or} \quad \bar{\upsilon}_i^* \bar{\nu}_i^* = 0. \tag{4.9}$$

Assume that conditions (4.9) do not hold. Then from (4.7) and the definitions of $\overline{\upsilon}_i^k$ and $\overline{\nu}_i^k$ we have $\bar{\upsilon}_i^* < 0$, $\bar{\nu}_i^* < 0$ for all $i \in I_{00}(\bar{z})$, and

$$\begin{cases} \bar{\upsilon}_i^* = -\lim\limits_{k\to+\infty} \gamma_i^k \eta_i^{\Phi_{\varepsilon_k}}, \\ \bar{\nu}_i^* = -\lim\limits_{k\to+\infty} \gamma_i^k \zeta_i^{\Phi_{\varepsilon_k}}. \end{cases}$$

Thus,

$$\bar{\upsilon}_i^* \bar{\nu}_i^* = \lim_{k\to+\infty} \gamma_i^k \eta_i^{\Phi_{\varepsilon_k}} \gamma_i^k \zeta_i^{\Phi_{\varepsilon_k}} > 0. \tag{4.10}$$

However, Corollary 4.1 and the conditions of Theorem 4.3 yield

$$\lim_{k\to+\infty} \gamma_i^k \eta_i^{\Phi_{\varepsilon_k}} \gamma_i^k \zeta_i^{\Phi_{\varepsilon_k}} = 0,$$

which contradicts (4.10). Therefore, $\bar{z}$ is an M-stationary point of problem (1.1), and this completes the proof. □

By using part (3) of Lemma 3.3, it is easy to see that the following result holds.

THEOREM 4.4. *Let $\{\varepsilon_k\}$ be a sequence of positive numbers which converges to zero, and z_k be a stationary point of problem (3.4) with $\varepsilon = \varepsilon_k$. If $\bar{z}$ is an accumulation point of the sequence $\{z_k\}$ as $k \to \infty$ such that the MPEC–MFCQ holds at $\bar{z}$ and $\lim_{k\to+\infty} \gamma_i^k = 0$ for all $i \in \mathrm{supp}(\gamma^k) \cap I_{00}(\bar{z})$, then $\bar{z}$ is a strongly stationary point of problem (1.1).*

REMARK 4.1. Since strong stationarity implies B-stationarity, the sufficient condition for the strongly stationary point of the MPCC given in Theorem 4.4 also ensures that $\bar{z}$ is a B-stationary point of problem (1.1). It is different from the results available in the literature [9, 21, 22], where the second-order necessary condition as well as the other additional conditions for strongly stationary convergence often need to be satisfied.

5. Numerical experiments

In this section, we test the numerical performance of Algorithm 1 by solving some test problems available in the literature.

By way of comparison, we implement both Algorithm 1 and the algorithm developed by Facchinei et al. [3] to solve the same test problems. In the numerical

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TABLE 1. A comparison between Algorithm 1 and the algorithm in [3].

Prob	(n, m, l)	f_F/f_L	k_F/k_L	$\text{time}_F/\text{time}_L$	$\text{maxvio}_F/\text{maxvio}_L$
1(a)	(1,6,4)	3.2077/3.2077	2/2	0.230290/<u>0.073070</u>	7.9945e−11/3.1834e−09
1(b)	(1,6,4)	3.2077/3.2077	2/2	0.159354/<u>0.072017</u>	7.8795e−11/5.3853e−09
2(a)	(1,6,4)	3.4494/3.4494	1/2	0.070082/0.093289	2.0535e−08/<u>3.3330e−09</u>
2(b)	(1,6,4)	3.4494/3.4494	1/2	0.048598/0.089176	1.3054e−08/<u>5.6526e−09</u>
3(a)	(1,6,4)	4.6043/4.6043	2/2	0.055994/0.132143	3.5256e−08/<u>3.1831e−09</u>
3(b)	(1,6,4)	4.6043/4.6043	1/2	0.057581/0.099780	3.5255e−08/<u>3.8902e−09</u>
4(a)	(1,6,4)	6.5927/6.5927	1/2	0.057295/0.071829	5.2768e−08/<u>3.9828e−09</u>
4(b)	(1,6,4)	6.5927/6.5927	1/2	0.057103/0.098933	5.2678e−08/<u>3.1831e−09</u>
5	(2,2,2)	−1.0000/−1.0000	2/1	0.100644/<u>0.039830</u>	5.8330e−11/<u>5.8010e−15</u>
6	(1,1,1)	−3266.7/−3266.7	1/2	0.922831/<u>0.034087</u>	3.7507e−10/2.1221e−09
7	(2,2,6)	4.9994/4.9994	1/2	0.169103/<u>0.053031</u>	3.0470e−09/4.4533e−09
8(a)	(1,4,8)	−343.3453/−343.3453	1/2	0.135239/0.175955	4.6388e−10/6.4219e−08
8(b)	(1,4,8)	−203.1551/−203.1551	1/2	0.129527/<u>0.109320</u>	5.1910e−10/4.7815e−09
8(c)	(1,4,8)	−68.1357/−68.1357	1/2	0.144460/<u>0.118080</u>	6.6346e−10/4.7736e−09
8(d)	(1,4,8)	−19.1541/−19.1541	1/2	0.072285/0.118738	8.6641e−10/4.7576e−09
8(e)	(1,4,8)	−3.1612/−3.1612	1/2	0.075084/0.304395	1.3583e−09/7.5678e−09
8(f)	(1,4,8)	−346.8932/−346.8932	1/2	0.273171/<u>0.155402</u>	4.4987e−08/<u>9.0198e−09</u>
8(g)	(1,4,8)	−224.0372/−224.0372	1/2	0.155166/<u>0.112821</u>	3.6486e−08/<u>4.7686e−09</u>
8(h)	(1,4,8)	−80.7860/−80.7860	1/2	0.120192/0.154580	1.8518e−08/<u>4.7504e−09</u>
8(i)	(1,4,8)	−22.8371/−22.8371	1/2	0.181496/<u>0.104794</u>	5.4911e−08/<u>4.9471e−09</u>
8(j)	(1,4,8)	−5.3491/−5.3159	1/3	0.167435/<u>0.150845</u>	1.7340e−08/<u>4.7456e−12</u>
9(a)	(2,2,2)	4.4816e−15/5.1049e−15	1/2	0.042437/<u>0.038047</u>	1.4140e−08/<u>3.1831e−09</u>
9(b)	(2,2,2)	5.3783e−15/2.4358e−11	1/2	0.039184/0.061679	1.4141e−08/<u>3.5027e−09</u>
9(c)	(2,2,2)	1.2633e−15/5.2939e−15	1/2	0.035922/0.066582	1.4142e−08/<u>3.1831e−09</u>
9(d)	(2,2,2)	1.5365e−07/2.5363e−09	1/2	0.098686/<u>0.054299</u>	1.4141e−08/<u>3.0011e−09</u>
9(e)	(2,2,2)	3.4774e−15/9.4146e−12	1/2	0.033581/0.050065	1.4140e−08/<u>3.1831e−09</u>
10	(4,4,12)	−6600.0/−6600.0	1/2	0.180838/0.227755	2.5000e−10/3.1830e−09
11	(2,6,4)	−12.6787/−12.6787	1/2	0.113889/<u>0.073480</u>	1.4832e−08/<u>6.4309e−09</u>

The underlined results show the numerical efficiency of Algorithm 1.

implementation, we use the built-in function fmincon in MATLAB 2012b to solve subproblem (3.4). The solution tolerance is set to 10^{-6} and the initial solutions for all the test problems are specified by the same values as those set by Facchinei et al. [3].

In the numerical experiments, we find that the change of the initial value of ε affects the obtained optimal value of the objective function (see Tables 1 and 2). To get the same values of the objective function [3], we choose the initial perturbation parameter $\varepsilon_1 = 0.001$ for the test problem Prob 7. For all other test problems, we take $\varepsilon_1 = 0.0001$.

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TABLE 2. Results for some selected test problems from [15] with dimension over 100.

Prob	dimension	f_L	k_L	time_L	maxvio_L
liswet1-050	152	0.01400	3	18.395330	5.8409e−09
liswet1-100	302	0.01376	5	20.849152	2.3579e−09
liswet1-200	602	0.01705	6	151.736173	7.6763e−09

The results in Table 2 show the efficiency of Algorithm 1 for some test problems from [15] with dimension over 100.

For each algorithm, the corresponding computer procedures are written in MATLAB, and run in the following environment: 1.50 GHz CPU, 1.47 GB memory based on Windows 7 operation system. The obtained numerical results are listed in Table 1.

In Table 1, 'Prob' denotes the test problem solved and (n, m, l) stand for the dimensions of the three types of the decision variables in the test problems. Also, f_F, k_F, time_F and maxvio_F denote the optimal objective function, the number of iterations, the computation time (in seconds), and the maxvio$(\bar{z})$, respectively, obtained by the algorithm of Facchinei et al. [3]. The symbols f_L, k_L, time_L and maxvio_L stand for the corresponding items obtained by Algorithm 1.

From Table 1, observe that we have obtained the same optimal value of the objective function by using Algorithm 1 as obtained by algorithm of Facchinei et al. [3]. For the 14 test problems, Algorithm 1 takes less machine time to find the optimal solution than their algorithm [3] (see the underlined results in Table 1), though the number of iterations of Algorithm 1 is slightly greater than their algorithm in some cases. However, as for the accuracy of the solution (measured by the value of maxvio$(\bar{z})$), there are 18 out of the 28 test problems whose optimal solutions have less violation degree than that by their algorithm. This indicates that our proposed smoothing method can generate a better approximation to the original MPCC.

Note that the dimensions of all the solved test problems of Facchinei et al. [3] are not more than 20. We attempt to implement our developed Algorithm 1 to solve some test problems with dimension over 100, selected from Leyffer's collection [15] (see Table 2).

6. Final remarks

In this paper, we have proposed a locally smoothing method for MPCC by which an efficient numerical algorithm has been developed. Theoretically, we have proved that the MFCQ holds for the approximate problem obtained from the smoothing method under certain conditions. These conditions have been presented to ensure that any accumulation point of a sequence of stationary points for the perturbed subproblems is a C-stationary point, an M-stationary point or a strongly stationary point. Preliminary numerical experiments show that the proposed method is more promising than the existing ones in the literature.

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