

Representations with Weighted Frames and Framed Parabolic Bundles

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Abstract. There is a well-known correspondence (due to Mehta and Seshadri in the unitary case, and extended by Bhosle and Ramanathan to other groups), between the symplectic variety M_h of representations of the fundamental group of a punctured Riemann surface into a compact connected Lie group G , with fixed conjugacy classes h at the punctures, and a complex variety \mathcal{M}_h of holomorphic bundles on the unpunctured surface with a parabolic structure at the puncture points. For $G = \mathrm{SU}(2)$, we build a symplectic variety P of pairs (representations of the fundamental group into G , “weighted frame” at the puncture points), and a corresponding complex variety \mathcal{P} of moduli of “framed parabolic bundles”, which encompass respectively all of the spaces M_h, \mathcal{M}_h , in the sense that one can obtain M_h from P by symplectic reduction, and \mathcal{M}_h from \mathcal{P} by a complex quotient. This allows us to explain certain features of the toric geometry of the $\mathrm{SU}(2)$ moduli spaces discussed by Jeffrey and Weitsman, by giving the actual toric variety associated with their integrable system.

1 Introduction

The geometry of moduli spaces of vector bundles over a Riemann surface exhibits in a beautiful way the interplay between two types of structure. On one hand, one can think of these spaces as classifying holomorphic structures, and the emphasis is on complex geometry. On the other, via the Narasimhan-Seshadri theorem [NS], one can also represent the moduli space as a space of flat unitary connections on the surface, or, integrating, in terms of representations of the fundamental group into unitary groups; here the emphasis is on the symplectic geometry. The key link between the two is provided by the fact that on both sides one can view the moduli space as being obtained by a quotient construction, starting from the same infinite dimensional space of unitary connections [AB]. On the holomorphic side, one quotients by the action of the group of complex gauge transformations; on the symplectic side, one performs a symplectic reduction, first setting the curvature (moment map) to zero, then quotienting out by the group of unitary gauge transformations. The general equivalence between geometric invariant theory quotients and symplectic quotients tells us that we should obtain the same result.

Following Mehta and Seshadri, one can extend this picture to the case of a Riemann surface with punctures; on the symplectic side, one considers a moduli space of flat connections over the surface, such that the conjugacy class of the holonomy of the connection around the punctures is fixed; on the holomorphic side, the corresponding object is a moduli space of vector bundles over the filled-in Riemann surface, with the extra data of a parabolic structure at the marked points, that is, a flag of subspaces in the fiber of the bun-

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dle at each of the marked points. The conjugacy class determines the choice of polarisation that one uses to define the quotient.

From the symplectic point of view, it is tempting to try and obtain the moduli spaces of flat connections on a surface by “glueing” moduli spaces of connections over the elements of a decomposition of the surface into punctured surfaces. This requires moduli spaces over the punctured surface which are larger than the parabolic moduli; one needs, at the very least, to have all possible conjugacy classes of the holonomy around the puncture, as well as some sort of framing at the punctures to remove the ambiguity in glueing. The smallest moduli spaces with which one could have hoped to achieve this are the T -extended moduli spaces of Jeffrey [J1], where T is the maximal torus for our group G . These are spaces of equivalence classes of flat connections on the surface, along with a trivialisation at each puncture in which the holonomy at the puncture lies in the maximal torus T . The holonomy variables (eigenvalues of the holonomy) are the “associated momenta” of the framing variables, so that if one decomposes a surface Σ into two punctured surfaces Σ_1, Σ_2 , the glueing operation on the moduli spaces of matching the holonomies and then quotienting out the equivalent framings amounts to a symplectic quotient of the product of the two moduli spaces associated to Σ_1, Σ_2 by the diagonal T -action on the framings. This glueing works well at the points for which the holonomy is regular, but fails when it is not; for one, the extended moduli are no longer necessarily symplectic at these points [J1]; also, the quotient space of the moduli associated to the two surfaces maps to the moduli of the glued surface, but is no longer an isomorphism over the connections with non-regular holonomy along the circle of glueing. Various schemes can be employed to repair this problem, for example the q -Hamiltonian spaces of Alekseev, Malkin and Meinrenken [AMM], or the loop group valued framings of Meinrenken and Woodward [MW].

Our aim is different. For $G = \text{SU}(2)$, $T = S^1$, we obtain a moduli space P which is symplectic, and which encompasses all of the parabolic moduli, in some sense, “repairing” the T -extended moduli space. This new space will have a Hamiltonian T -action associated to each puncture, whose moment map is the holonomy, and whose symplectic T -quotients will be the parabolic moduli spaces. The moduli space gets interpreted as a moduli space of bundles with “weighted frames” at the punctures; see Section 2. In Section 3 we will see that the spaces associated to two punctured surfaces can indeed be glued, though one does not obtain the moduli space of bundles over the glueing of the two surfaces.

The space P allows one to clear up a question concerning the Goldman flows [G1] on the moduli space. When one is dealing with the product of the two T -extended moduli spaces associated to two punctured surfaces Σ_1 and Σ_2 , one has an action of $T \times T$, given by the action of T on the trivialisations over the two punctures. Let Σ be the surface obtained by glueing Σ_1 and Σ_2 , identifying curves around the punctures to obtain a single curve C in Σ . In the corresponding glueing of the T -extended moduli spaces, we are just quotienting out the anti-diagonal T , and so the group $1 \times T$ acts on the quotient. This action only descends to the moduli of the glued surface over the set of connections with regular holonomy along C and gives a densely defined T -flow on the moduli space of the glued surface Σ , which is ill-defined precisely when the holonomy along C is not regular. In the case of $G = \text{SU}(2)$, $T = S^1$, doing this for the $3g - 3$ boundary circles of a decomposition of the genus g surface Σ , gives a Lagrangian foliation of the moduli space corresponding to a densely defined $(S^1)^{3g-3}$ action, with moment map onto a polytope Γ , and “almost” gives the moduli space the structure of a toric variety. This “almost” is tantalizing: if the variety

were toric, one could get a proof of the Verlinde formulae for the dimensions of the spaces of sections of line bundles over the moduli space, simply by counting lattice points in the moment polytope; indeed, Jeffrey and Weitsman, in [JW], count these points, and obtain the answers predicted by the Verlinde formulae. If the moduli space had only been toric, this would have been a proof, and indeed, understanding how the structure fails to be toric was one of our main motivations for this paper. Our spaces P provide the answer to this question. One can glue the spaces P_1, P_2 corresponding to Σ_1, Σ_2 by taking the anti-diagonal T -quotient: one obtains a moduli space \hat{P} associated to the glueing of Σ_1 and Σ_2 . If one does this for a family of trinions associated to a decomposition D of a Riemann surface, one obtains a toric variety P^D whose moment polytope is precisely the Γ of the Goldman flows; the moduli space \mathcal{M}^T maps to P^D in a generically bijective fashion, and the Goldman flows are the lifts of the flows on the toric variety.

One could also hope that the space P corresponds, as in the Narasimhan-Seshadri theorem, to a moduli space of holomorphic objects over the curve Σ , where one no longer has punctures but simply marked points. We will show that this is also the case: the objects one considers are framed parabolic bundles, parabolic bundles with the extra structure of a volume form on each of the successive quotients of the vector spaces in the flag. This space is not complete, and to complete it, one must allow the bundles to degenerate into certain sheaves with a small amount of torsion. Our construction, which we carry out here for the $SL(2, \mathbb{C})$ -case only, involves using the “master space” construction of Thaddeus in an essential way. One can again glue in the holomorphic category the moduli spaces P_1, P_2 associated to Σ_1, Σ_2 by what is now an anti-diagonal $T_{\mathbb{C}}$ geometric invariant theory quotient, and the holomorphic objects one obtains live over the singular curve obtained by identifying the marked points of Σ_1, Σ_2 .

In Section 2, we construct our moduli space P , from a symplectic point of view. Section 3 is devoted to the glueing construction and the example of the moduli space over the three-punctured sphere. In Section 4, we build the holomorphic versions of our spaces.

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2 Extended Moduli Spaces, and Implosions

Let G be a connected and simply connected compact Lie group, with maximal torus T , and let $\mathfrak{g}, \mathfrak{t}$ be their Lie algebras. We define the *open fundamental alcove* as the space

$$(2.1) \quad \Delta^0 = \{ \xi \in \mathfrak{t} : (\alpha_j, \xi) > 0, \alpha_{\max}(\xi) < 1 \}$$

where α_{\max} is the highest root and the α_j are the simple roots. The *fundamental alcove* is the closure of Δ^0 in \mathfrak{t} and will be denoted Δ . Every point in G is conjugate to a unique element in $\exp \Delta$. The regular elements of G correspond to the interior of Δ . For $G = SU(2)$, the fundamental alcove is an interval, representing the elements $\text{diag}(e^{i\pi\gamma}, e^{-i\pi\gamma})$, $\gamma \in [0, 1]$ in $SU(2)$.

In what follows, let Σ denote a Riemann surface of genus g with punctures at points $p_1, \dots, p_n, n \geq 0$.

We want to describe the surface as a polygon with sides identified. We can choose closed paths $a_1, \dots, a_g, b_1, \dots, b_g, c_1, \dots, c_n$ representing cycles of a homology basis of Σ . The a_i, b_i originate at a point \tilde{p}_1 near the puncture p_1 , and represent the standard homology

basis of the closure of the surface. The c_i originate at points \tilde{p}_i near p_i , and are small loops surrounding the punctures. As well, we can pick paths d_i , $i = 2, \dots, n$, joining p_1 to p_i . We can further assume that the paths do not intersect except at their end points, and are such that we can write Σ in the usual way as a polygon with sides (in order)

$$(2.2) \quad a_1, b_1, a_1^{-1}, b_1^{-1}, a_2, b_2, \dots, a_g^{-1}, b_g^{-1}, c_1, d_2, c_2, d_2^{-1}, d_3, c_3, d_3^{-1}, \dots, c_n, d_n^{-1}$$

with identifications, as given, along the edges. The fundamental group of Σ is then the free group on the generators $a_i, b_i, c_i, d_j; c_j d_j^{-1}$ with the relation that the product $a_1 b_1 a_1^{-1} \dots d_n^{-1}$ along the sides of the polygon is one.

We next define some moduli spaces. If

$$(A_1, \dots, A_g, B_1, \dots, B_g, D_2, \dots, D_n, C_1, \dots, C_n) \in G^{2g+2n-1},$$

define

$$(2.3) \quad \begin{aligned} F: G^{2g+2n-1} &\rightarrow G \\ (A_i, B_i, D_j, C_j) &\mapsto A_1 B_1 A_1^{-1} B_1^{-1} \dots A_g^{-1} B_g^{-1} C_1 D_2 C_2 D_2^{-1} \dots C_n D_n^{-1}. \end{aligned}$$

The \mathfrak{t} -extended moduli space $\mathcal{M}^{\mathfrak{t}}$ is defined by

$$(2.4) \quad \mathcal{M}^{\mathfrak{t}} = \{(A_i, B_i, D_j, \gamma_j) \in G^{2g+n-1} \times \mathfrak{t}^n : F(A_i, B_i, D_j, C_j) = 1\}$$

setting $C_i = \exp(\gamma_i)$. There is a corresponding \mathfrak{g} -extended moduli space where the γ_j are not restricted to lie in \mathfrak{t} but can lie anywhere in \mathfrak{g} :

$$(2.5) \quad \mathcal{M}^{\mathfrak{g}} = \{(A_i, B_i, D_j, \gamma_j) \in G^{2g+n-1} \times \mathfrak{g}^n : F(A_i, B_i, D_j, C_j) = 1\}$$

where as before $C_i = \exp(\gamma_i)$. One may also define the T, G - extended moduli spaces

$$(2.6) \quad \mathcal{M}^T = \{(A_i, B_i, D_j, C_j) \in G^{2g+n-1} \times (\exp \Delta)^n : F(A_i, B_i, D_j, C_j) = 1\}$$

and

$$(2.7) \quad \mathcal{M}^G = \{(A_i, B_i, D_j, C_j) \in G^{2g+n-1} \times G^n : F(A_i, B_i, D_j, C_j) = 1\}.$$

There is an obvious map from $\mathcal{M}^{\mathfrak{g}}$ to \mathcal{M}^G (sending the point $(A_j, B_j, D_j, \gamma_j)$ to the point (A_j, B_j, D_j, C_j)).

Finally, one may make a minor modification of these spaces: choosing an element $c \in Z(G)$ we may define

$$(2.8) \quad \mathcal{M}_c^{\mathfrak{g}} = \{(A_i, B_i, D_j, \gamma_j) \in G^{2g+n-1} \times \mathfrak{g}^n : A_1 B_1 A_1^{-1} \dots B_g^{-1} C_1 D_2 C_2 D_2^{-1} \dots C_n D_n^{-1} = c\}$$

and $\mathcal{M}_c^{\mathfrak{t}}, \mathcal{M}_c^G$ and \mathcal{M}_c^T are defined in the same way, replacing 1 by c in the definitions.

These spaces can be thought of as moduli spaces of flat connections on Σ , with some additional structure at the marked points. The space $\mathcal{M}^{\mathfrak{t}}$ is the quotient of a space of flat

connections on a Riemann surface Σ with n boundary components, where we have imposed some boundary conditions near the punctures, in particular fixing the connection form around each boundary component to lie in \mathfrak{t} . We divide only by those gauge transformations which are trivial on the boundary. The space $\mathcal{M}^{\mathfrak{g}}$ is defined similarly, removing the constraint that the form lie in \mathfrak{t} near the boundary. (See [J1].) The spaces $\mathcal{M}^T, \mathcal{M}^G$ can similarly be obtained from a space of flat connections by quotienting by a suitable space of gauge transformations which are trivial only at marked points near the boundaries. In other words, at a point near each of the boundary components we have fixed a framing or trivialization of the principal bundle on which the connection is defined. The space \mathcal{M}^T is a quotient of the space $\mathcal{M}^{\mathfrak{t}}$, by the n -fold product $(W \ltimes \mathbb{Z}^r)^n$ of the semidirect product of the Weyl group W and the \mathbb{Z}^r of translations such that $T = \mathfrak{t}/\mathbb{Z}^r$. Alternately, \mathcal{M}^T can be thought of as a subspace of $\mathcal{M}^{\mathfrak{t}}$, for which the γ_k are forced to lie in the fundamental alcove.

Group Actions

The group G^n acts on $\mathcal{M}^{\mathfrak{g}}$ in the following way, defined in (5.9) and (5.10) of [J1]: An element σ in the first copy G_1 of G acts on $\mathcal{M}^{\mathfrak{g}}$ by sending

$$\begin{aligned}
 A_j &\mapsto \sigma A_j \sigma^{-1} \text{ for } j = 1, \dots, g \\
 B_j &\mapsto \sigma B_j \sigma^{-1} \text{ for } j = 1, \dots, g \\
 D_j &\mapsto \sigma D_j \\
 \gamma_1 &\mapsto \sigma \gamma_1 \sigma^{-1} \\
 \gamma_j &\mapsto \gamma_j \text{ for } j = 2, \dots, n
 \end{aligned}
 \tag{2.9}$$

while an element σ in the l -th copy G_l of G (for $l = 2, \dots, n$) acts by

$$\begin{aligned}
 A_j &\mapsto A_j, \quad B_j \mapsto B_j \\
 D_j &\mapsto D_j \text{ for } j \neq l \\
 D_l &\mapsto D_l \sigma^{-1} \\
 \gamma_j &\mapsto \gamma_j \text{ for } j \neq l, \\
 \gamma_l &\mapsto \sigma \gamma_l \sigma^{-1}.
 \end{aligned}
 \tag{2.10}$$

The action on \mathcal{M}^G is defined in the same way (replacing the γ_j by the C_j). These actions restrict to define actions of T on $\mathcal{M}^{\mathfrak{t}}$ and \mathcal{M}^T .

In terms of flat connections, the group actions on $\mathcal{M}^{\mathfrak{g}}$ and $\mathcal{M}^{\mathfrak{t}}$ result from an action of the space of gauge transformations modulo those gauge transformations which are trivial on the boundary: in other words the action results from changing the trivialization of the bundle near the boundary components.

Smoothness

Proposition 2.11 *The spaces \mathcal{M}^G are smooth, and isomorphic to $G^{2(g+n-1)}$.*

Proof The relation $F = 1$ of (2.6) allows us to write C_1 as a function of the other variables A_i, B_i, C_j, D_j .

Proposition 2.12 *If $G = \text{SU}(2)$, the spaces \mathcal{M}^g are smooth near any point $(\{A_j, B_j\}, \{\gamma_j, D_j\})$ where either at least one of the γ_j is such that the exponential map is a local diffeomorphism in a neighbourhood of γ_j (this is the case when γ_j is regular, or $\gamma_j = 0$), or the stabilizer of the action of G by conjugation at the point $(\{A_j, B_j\})$ is the centre $Z(G)$.*

Proof The proof of the previous proposition shows that the differential of the map $F \circ E: G^{2g+n-1} \times \mathfrak{g}^n \rightarrow G$ is surjective if the exponential map is a local diffeomorphism in a neighbourhood of at least one of the γ_j . Here, F was defined by equation (2.3) and

$$E: (\{A_j, B_j, \gamma_j, D_j\}) \mapsto (\{A_j, B_j, \exp \gamma_j, D_j\}).$$

If γ_j is not regular then $\exp \gamma_j = \pm 1$; thus, if none of the γ_j are regular, all the C_j are in the centre $Z(G)$ and so $C_1 \prod_{k=2}^n D_k C_k D_k^{-1} \in Z(G)$. Also, the image of the differential of the map

$$R: G^{2g} \rightarrow G$$

given by

$$R(A_1, \dots, A_g, B_1, \dots, B_g) = \prod_{j=1}^g A_j B_j A_j^{-1} B_j^{-1}$$

is the orthocomplement of the Lie algebra of the stabilizer of $(A_1, \dots, A_g, B_1, \dots, B_g)$ under the conjugation action (see [G2, Proposition 3.7]). If the stabilizer at the point $\{A_j, B_j\}$ is $Z(G)$, Goldman's result implies the differential of R is surjective. This completes the proof.

The above proof extends to give the following result.

Proposition 2.13 *If $G = \text{SU}(2)$ and any of the C_j is regular for $j \geq 1$, or the stabilizer of the conjugation action of G at the point $\{A_j, B_j\} \in G^{2g}$ is $Z(G)$, then \mathcal{M}^t is smooth at the point $\{A_j, B_j, \gamma_j, D_j\}$ and \mathcal{M}^T is smooth at the point $\{A_j, B_j, C_j, D_j\}$.*

Symplectic Structures and Imploded Cross-Sections

In [J1] and [J2] a closed 2-form on \mathcal{M}^g was constructed, for which the map

$$\mu_k: (\{A_j, B_j\}, \{\gamma_j, D_j\}) \mapsto -\gamma_k$$

is the moment map for the action of the k -th copy of G . Similarly, for \mathcal{M}^t , there is again a two-form for which γ_k is the moment map for the T -action. This form is invariant under $(W \times \mathbb{Z}^r)^n$, and so descends to \mathcal{M}^T :

Proposition 2.14 *The space \mathcal{M}^g is equipped with a closed 2-form, whose restriction to $\bigcap_{j=1}^g \mu_j^{-1}(B)$ is nondegenerate. Here, B is the subset of \mathfrak{g} where the exponential map is locally a diffeomorphism.*

The space \mathcal{M}^t is equipped with a closed 2-form, whose restriction to $\bigcap_{j=1}^g \mu_j^{-1}(\text{Reg})$ is nondegenerate. Here Reg is the subset of \mathfrak{t} consisting of elements which exponentiate to regular elements of the torus; it consists of all translates under $(W \times \mathbb{Z}^r)^n$ of the interior of the fundamental alcove.

Quotienting, \mathcal{M}^T in turn is equipped with a closed 2-form, whose restriction to $\bigcap_{j=1}^g \mu_j^{-1}(\Delta^0)$ is nondegenerate.

Proof See [J1], and Huebschmann [H, Theorem 2.15)]. Huebschmann in fact proves this result for a smooth extended moduli space with a Hamiltonian action of $H \times G^n$, where $H \cong G$ and the action of H is free (and hence 0 is a regular value for its moment map): the appropriate subset of our space \mathcal{M}^g is recovered by taking the symplectic quotient by the action of H .

The space \mathcal{M}^T is the cross-section $\bigcap_{j=1}^n (\mu_j)^{-1}(\Delta)$ of \mathcal{M}^g . We shall need to construct auxiliary spaces (*imploded cross-sections*) where the degenerate directions of the two-form have been collapsed. The definition of these spaces is due to Sjamaar [GJS].

Definition 2.16 Let M be a symplectic manifold equipped with the Hamiltonian action of a compact Lie group G with maximal torus T and moment map $\Phi: M \rightarrow \mathfrak{g}$. Let σ index the faces of the fundamental Weyl chamber. Let G_σ be the stabilizer of σ under the adjoint action on \mathfrak{g} . The imploded cross-section of M is then

$$M_{\text{impl}} = \coprod_{\sigma} \frac{\Phi^{-1}(\sigma)}{[G_\sigma, G_\sigma]}$$

where $[G_\sigma, G_\sigma]$ is the commutator subgroup of G_σ .

The imploded cross-section admits an action of T .

Theorem 2.17 [GJS] *The space M_{impl} inherits from M the structure of a stratified symplectic space and a Hamiltonian T -action. The symplectic strata are the quotients $\Phi^{-1}(\sigma)/[G_\sigma, G_\sigma]$.*

A special case of the imploded cross-section is the imploded cross-section of the cotangent bundle $T^*G \cong G \times \mathfrak{g}$. The cotangent bundle admits an action of $G \times G$ as follows, corresponding to left and right multiplication. If $(g_1, 1)$ denotes a point in the first copy of G in $G \times G$, it sends a point (h, X_h) (where X_h denotes the value at h of the left invariant vector field generated by $X \in \mathfrak{g}$) to $(g_1 h, X_{g_1 h})$. Likewise, the element $(1, g_2)$ of $G \times G$ sends the point (h, X_h) to $(hg_2, (\text{Ad}(g_2^{-1})X)_{hg_2})$. The imploded cross-section of T^*G (corresponding to the action of the first copy of G) is

$$\mathcal{E}(T^*G) = \coprod_{\sigma} \frac{G}{[G_\sigma, G_\sigma]} \times \sigma,$$

where σ ranges over the (interiors of the) faces of the fundamental Weyl chamber, and G_σ is the stabilizer of σ under the conjugation action. The imploded cross section of T^*G inherits an action of $T \times G$.

When $G = \text{SU}(2)$, one obtains, for example, that $\mathcal{E}(T^*G) = \mathbb{C}^2$, with the standard two-form. In general, one has [GJS] that the implosion $\mathcal{E}(T^*G)$ is only smooth over the locus where $[G_\sigma, G_\sigma]$ is a product of $\text{SU}(2)$ s.

One finds [GJS] that the imploded cross section of any Hamiltonian G -manifold M is given by

$$(2.18) \quad \begin{aligned} M_{\text{impl}} &= (M \times \mathcal{E}(T^*G)) // G \\ &= \{(m, y) \in M \times \mathcal{E}(T^*G) : \Phi(m) = \Phi_G(y)\} / G. \end{aligned}$$

In this sense $\mathcal{E}(T^*G)$ may be thought of as a universal imploded cross-section. It gives the implosion of M as the symplectic quotient of $M \times \mathcal{E}(T^*G)$ under the diagonal action of G (where G acts on $\mathcal{E}(T^*G)$ via the action of $T \times G$, and we use minus the usual symplectic structure on $\mathcal{E}(T^*G)$): here we have written the moment map for the action of G on M as Φ and that for the action on $\mathcal{E}(T^*G)$ as Φ_G .

Note that this implies that when $G = \text{SU}(2)$, the implosion of M is smooth over the locus where G acts freely, or with a constant stabiliser over an open set.

We may construct the imploded cross-section of $\mathcal{M}^{\mathfrak{g}}$ with respect to the action of $\text{SU}(2)^n$ as

$$(2.19) \quad (\mathcal{M}^{\mathfrak{g}})_{\text{impl}} = \prod_{\sigma_1, \dots, \sigma_n} \frac{\bigcap_{i=1}^n (\mu_i)^{-1}(\sigma_i)}{[G_{\sigma_1}, G_{\sigma_1}] \times \cdots \times [G_{\sigma_n}, G_{\sigma_n}]}.$$

This object may also be exhibited as the symplectic quotient of $\mathcal{M}^{\mathfrak{g}} \times \mathcal{E}(T^*G) \times \cdots \times \mathcal{E}(T^*G)$ under the diagonal action of G^n .

Quasi-Hamiltonian G -Spaces

We note that we have not discussed any symplectic structure for \mathcal{M}^G ; indeed, the appropriate structure on \mathcal{M}^G is that of a *quasi-Hamiltonian G -space*. Following Alekseev, Malkin and Meinrenken [AMM], we will give a definition of such spaces, which are equipped with group actions and group-valued moment maps.

Definition 2.20 [AMM] A quasi-Hamiltonian (or q-Hamiltonian) G -space is a manifold M with a G -action together with an invariant 2-form ω and an equivariant map $\Phi: M \rightarrow G$ (the G -valued moment map) for which

1.

$$d\omega = -\Phi^* \chi$$

where χ is the 3-form given in terms of the left-invariant Maurer-Cartan form $\theta \in \Omega^1(G) \otimes \mathfrak{g}$ and the bi-invariant inner product (\cdot, \cdot) on \mathfrak{g} by

$$\chi = (1/12)(\theta, [\theta, \theta]).$$

2. The map Φ satisfies

$$\iota(\nu_\xi)\omega = (1/2)\Phi^*(\theta + \bar{\theta}, \xi)$$

where $\bar{\theta}$ is the right-invariant Maurer-Cartan form and ν_ξ is the vector field corresponding to ξ .

3. For $x \in M$ the kernel of ω_x is given by

$$\text{Ker}(\omega_x) = \{v_\xi, \xi \in \text{Ker}(\text{Ad } \Phi_x + 1)\}.$$

Remarks

- (1) We note that if G is Abelian, then M is symplectic; in general, of course, it is not.
- (2) If a space is a quasi-Hamiltonian G -space, it is not necessarily a quasi-Hamiltonian H -space for a subgroup H of G .

An example of a quasi-Hamiltonian space is the double $D(G) = G \times G$, which is a quasi-Hamiltonian $G \times G$ -space with the action

$$(2.21) \quad (g_1, g_2) : (a, b) \mapsto (g_1 a g_2^{-1}, g_2 b g_1^{-1})$$

and the $G \times G$ -valued moment map $\Phi: (a, b) \mapsto (ab, a^{-1}b^{-1})$. (We refer to $G \times G$ as $G_1 \times G_2$, where $G_j = G$.) The two-form ω is given by

$$(2.22) \quad \omega = (1/2)(\pi_1^* \theta, \pi_2^* \bar{\theta}) + (1/2)(\pi_1^* \bar{\theta}, \pi_2^* \theta).$$

It is shown in [AMM, Proposition 3.2] that $D(G)$ is a quasi-Hamiltonian $G \times G$ -space.

It will be convenient to introduce an alternative system of coordinates on $D(G)$: these are given by (u, v) where $u = a$ and $v = ba$. In these coordinates the action of $(g_1, g_2) \in G \times G$ is

$$(2.23) \quad (g_1, g_2) : (u, v) \mapsto (g_1 u g_2^{-1}, \text{Ad}_{g_2} v)$$

and the moment maps are

$$(2.24) \quad \Phi_1(u, v) = \text{Ad}_u v, \quad \Phi_2(u, v) = v^{-1}.$$

A second example of a quasi-Hamiltonian G -space is the space M^G : there is a quasi-Hamiltonian structure on the spaces \mathcal{M}^G and \mathcal{M}^T , for which the maps

$$(2.25) \quad \Phi_k : (\{A_j, B_j\}, \{C_j, D_j\}) \mapsto C_k^{-1}$$

are G -valued moment maps for the action of the k -th copy of G . (See [AMM, equation (39), Section 9.2].)

One may define reduction for quasi-Hamiltonian G -spaces: it is shown in [AMM, Theorem 5.1] that if M is a q-Hamiltonian $G_1 \times G_2$ -space and $f \in G_1$ is a regular value of the moment map $\Phi_1 : M \rightarrow G_1$ then the reduced space

$$(2.26) \quad M_f = (\Phi_1)^{-1}(f)/Z_f$$

is equipped with the structure of a quasi-Hamiltonian G_2 -space. (Here, Z_f is the stabilizer of f under the action of G_1 by conjugation.)

There is also a cross-section theorem for quasi-Hamiltonian G -spaces. If M is a quasi-Hamiltonian G -space, and $f \in G$, and Z_f is the centralizer of f in G , there is an open subset $U \subset Z_f$ for which the map

$$(2.27) \quad G \times_{Z_f} U \rightarrow G, \quad [g, u] \mapsto gu$$

is a diffeomorphism onto an open subset of G . It then follows [AMM, Proposition 7.1] that $\Phi^{-1}(U)$ is a smooth Z_f -invariant submanifold and is a quasi-Hamiltonian Z_f -space with the restriction of Φ as a moment map. Again, if Z_f is Abelian, the cross-section is symplectic.

Imploded Cross Sections of Quasi-Hamiltonian G -Spaces

Let H, G be two Lie groups. We may formulate a definition of imploded cross-sections of quasi-Hamiltonian $H \times G$ -spaces analogous to the definition for Hamiltonian G -spaces. Let the faces of $\exp \Delta \subset T \subset G$ (which are the same as the faces of Δ) be denoted by σ , and let G_σ be the stabilizer of the action of G by conjugation at a point in the interior of σ . Then we define the imploded cross-section of a quasi-Hamiltonian G -space M as

$$(2.28) \quad M_{\text{impl}} = \coprod_{\sigma} \frac{\Phi^{-1}(\sigma)}{[G_\sigma, G_\sigma]}.$$

We note that the generic locus of this implosion, that is, the stratum corresponding to the interior of the alcove, is a quasi-Hamiltonian $H \times T$ -space. In general, one would have to go through the analysis in [GJS] to show that one has a suitable structure of a stratified $H \times T$ -space on the full M_{impl} . For the case $G = \text{SU}(2)$, things can be somewhat simplified: the possible σ are $\sigma^0 = \{\text{diag}(e^{i\gamma}, e^{-i\gamma}), 0 < \gamma < \pi\}$ (with $G_\sigma = T$, and $[G_\sigma, G_\sigma]$ trivial) and $\sigma^+ = 1$ and $\sigma^- = -1$ (with $G_\sigma = [G_\sigma, G_\sigma] = G$); here ± 1 refer to ± 1 times the identity matrix in $\text{SU}(2)$. The stratum of M_{impl} corresponding to each σ is a quasi-Hamiltonian $H \times T$ -space, since according to the reduction theorem, Theorem 5.1 of [AMM], the reduced spaces $\Phi^{-1}(1)/G$ and $\Phi^{-1}(-1)/G$ inherit the structure of a quasi-Hamiltonian $H \times T$ -space (with T acting trivially), and according to the cross-section theorem, Proposition 7.1 of [AMM], the cross-section $\Phi^{-1}(\sigma^0)$ also inherits the structure of a quasi-Hamiltonian $H \times T$ -space. To see that the strata fit together correctly, we again, as above, construct a universal implosion, not from T^*G , but from the double $G \times G$, for $G = \text{SU}(2)$.

Proposition 2.29 For $G = \text{SU}(2)$, $T = S^1$, the implosion $D(G)_{\text{impl}}$ is smooth, isomorphic to S^4 , and is a quasi-Hamiltonian $\text{SU}(2) \times S^1$ -space.

Proof (see also [AMM2, Prop. 2.3.3]) We have $D(G)$ as a quasi-Hamiltonian $G \times G$ -space and (under the action of $G = G_2$ in equation (2.23)) we obtain the cross-section $\Phi^{-1}(\sigma^0)$ as $\{(u, v) \in G \times G : v^{-1} \in \exp \Delta\}$. The action of $g_2 \in G_\sigma$ on $G \times \sigma$ is by right multiplication by g_2^{-1} on G and the adjoint action on σ (the latter being trivial by definition of G_σ). By the above discussion, the strata $\Phi^{-1}(\sigma)/[G_\sigma, G_\sigma]$ are quasi-Hamiltonian $G \times T$ -spaces. The stratum σ^0 has stabilizer T , and $[T, T]$ is trivial, so that $\phi^{-1}(\sigma^0)/[G_\sigma, G_\sigma] = G \times \sigma^0 = S^3 \times \sigma^0$. On the other hand the strata σ^+ and σ^- have stabilizers $G_\sigma = [G_\sigma, G_\sigma] = G$,

so for these strata we have $\Phi^{-1}(\sigma)/[G_\sigma, G_\sigma] = G/G$ which is a point (denoted ± 1). The imploded cross-section is thus

$$D(G)_{\text{impl}} = \coprod_{\sigma} \frac{G}{[G_\sigma, G_\sigma]} \times \sigma = (G \times \sigma^0) \coprod \{1\} \coprod \{-1\}.$$

We may identify this with S^4 by identifying σ^+ with the north pole N and σ^- with the south pole S , and projecting the cylinder $G \times (0, \pi) \simeq S^3 \times (-1, 1)$ in the standard fashion to the sphere $S^4 - \{N, S\}$. By the cross-section theorem, $S^4 - \{N, S\}$ inherits a $G \times T$ quasi-Hamiltonian structure; one checks by explicit computation that it extends analytically to the whole of S^4 , and that it is non-degenerate at the poles. (Of course, this form is not closed.) The residual $G \times T$ action on $G \times \sigma^0$ is given by

$$(2.30) \quad (g, t) \cdot (u, v) = (gut^{-1}, v),$$

and corresponding moment maps:

$$(2.31) \quad \Phi_G: (u, v) \mapsto (uvu^{-1}), \quad \Phi_T: (u, v) \mapsto v^{-1}.$$

This action extends to S^4 , fixing the poles.

We can use this universal implosion to give an alternate definition of the implosion of any quasi-Hamiltonian G -space, for $G = \text{SU}(2)$.

Proposition 2.32 *Let M be a quasi-Hamiltonian $H \times G$ -space (where $G = \text{SU}(2)$) and let $D(G)_{\text{impl}}$ be the imploded cross-section of $D(G)$. Then*

$$(2.33) \quad \begin{aligned} M_{\text{impl}} &= M \times D(G)_{\text{impl}} / G \\ &= \{(m, \xi) \in M \times D(G)_{\text{impl}} : \Phi(m) = \Phi_G(\xi)\} / G. \end{aligned}$$

The space M_{impl} is a quasi-Hamiltonian $H \times T$ -space, and it is smooth over the locus of points $(m, \xi) \in M \times D(G)_{\text{impl}}$ where the stabiliser of the G -action is trivial.

Proof We want to identify the stratum of M_{impl} corresponding to σ with the set of points in the quotient $M \times D(G)_{\text{impl}} / G$ with $\Phi(m) = \Phi_G(\xi)$ lying in the orbit of σ under G . If (m, ξ) , $\xi = (u, v)$ is such a point, we can normalise so that $\Phi(m) = \Phi_G(\xi) = uvu^{-1} \in \sigma$, in other words, $\Phi^{-1}(G\sigma) \times_{G_\sigma} \Phi_G^{-1}(G\sigma) / G \simeq \Phi^{-1}(\sigma) \times_\sigma \Phi_G^{-1}(\sigma) / G_\sigma$. One then has, for $(u, v) \in \Phi_G^{-1}(\sigma)$, that $uvu^{-1} = v$ (since v lies in σ), and so u lies in G_σ , so that $\Phi_G^{-1}(\sigma) = (G_\sigma \times \sigma) / [G_\sigma, G_\sigma]$. One wants to compare $\Phi^{-1}(\sigma) \times_\sigma \Phi_G^{-1}(\sigma) / G_\sigma \simeq \Phi^{-1}(\sigma) \times_\sigma ((G_\sigma \times \sigma) / [G_\sigma, G_\sigma]) / G_\sigma$ with $\Phi^{-1}(\sigma) / [G_\sigma, G_\sigma]$. This equivalence is straightforward. On the open stratum, corresponding to the interior of the fundamental alcove, we note that our comparison is between $\Phi^{-1}(\sigma) \times_\sigma \Phi_G^{-1}(\sigma) / T$ and $\Phi^{-1}(\sigma)$.

The smoothness of the quotient follows from the smoothness of $D(G)_{\text{impl}} = S^4$. As the product $M \times D(G)_{\text{impl}}$ is a quasi-Hamiltonian $H \times G \times T$ -space, the reduction is a Hamiltonian $H \times T$ -space, by the reduction theorem of [AMM].

Imploded Cross-Section of \mathcal{M}^G

We would now like to apply the implosion construction above to \mathcal{M}^G , which, as we saw, is a quasi-Hamiltonian G^n -space, where the moment map associated to the action of G at the i -th puncture is just C_i^{-1} . Performing the implosion for each puncture, we will denote the resulting $\mathcal{M}_{\text{impl}}^G$ by P ; it is a Hamiltonian T^n space.

For $j = 2, \dots, n$, we find that we can first identify the elements $\{(D_j, C_j)\}$ of (2.6) with elements of $D(G)$: in the coordinate system (u, v) on $D(G)$, D_j corresponds to u and C_j to v . (See [AMM, Section 9.2].) It is clear that the imploded cross-section of \mathcal{M}^G (with respect to the action of the j -th copy of G , where $j = 2, \dots, n$) is $M^0 \cup M^+ \cup M^-$, where M^0 consists of the points in \mathcal{M}^G for which $(C_j)^{-1} \in \exp \Delta^0$, while M^\pm are the quotients by G of the sets of points for which $C_j = \pm 1$. If $C_j = \pm 1$, the action of $g \in G_j$ sends D_j to $D_j g^{-1}$ and preserves the values of A_j, B_j, C_j . Notice that for $j \geq 2$ the j -th copy of G acts on \mathcal{M}^G only through its action on the variables C_j and D_j .

It follows that the imploded cross-section of \mathcal{M}^G (after imploding in turn the actions of the $n - 1$ copies of G corresponding to $j = 2, \dots, n$) is

$$\left\{ (A_j, B_j, C_1, W_2, \dots, W_n) : A_k, B_k, C_1 \in G, W_j \in (D(G))_{\text{impl}}, \prod_{j=1}^g [A_j, B_j] C_1 \prod_{j=2}^n (\Phi_G(W_j)) = 1 \right\}.$$

Here, $\Phi_G : D(G)_{\text{impl}} \rightarrow G$ is the map $(D, C) \mapsto DCD^{-1}$.

Finally, to complete the description of the imploded cross-section, we need to take the cross-section for the first copy of G (denoted G_1 in the paragraph preceding (2.9)) and collapse it appropriately. The imploded cross-section is the union of strata given by

$$\frac{\{ \{A_k, B_k\}, C_1, W_2, \dots, W_n : \prod_{j=1}^g [A_j, B_j] C_1 \prod_{j=2}^n (\Phi_G(W_j)) = 1, C_1 \in \sigma \}}{[G_\sigma, G_\sigma]}.$$

Setting $W_1 = (D_1, C_1)$, with an arbitrary D_1 , the imploded cross-section becomes

$$(2.34) \quad P = M_{\text{impl}}^G = \left\{ \{A_k, B_k\}, W_1, \dots, W_n : \prod_{j=1}^g [A_j, B_j] \Phi_G(W_1) \prod_{j=2}^n \Phi_G(W_j) = 1 \right\} / G,$$

where the $W_j \in D(G)_{\text{impl}}$ and the action of $g \in G$ is by

$$(2.35) \quad (A_k, B_k, W_j = (u_j, v_j)) \mapsto (gA_k g^{-1}, gB_k g^{-1}, (gu_j, gv_j)).$$

One may also understand this in terms of an extended moduli space $\tilde{M}^G \cong G^{2g+2n}$ equipped with an action of $G^{n+1} = G_0 \times \dots \times G_n$, $G_i \simeq \text{SU}(2)$ (this corresponds to putting in an extra puncture): our space M^G is the quasi-Hamiltonian quotient of \tilde{M}^G with respect to the action of G_0 . The space \tilde{M}^G is defined by

$$\tilde{M}^G = \{A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_n, D_1, \dots, D_n\}$$

and the action of $\sigma \in G_0$ is given by

$$\begin{aligned} A_j &\mapsto \sigma A_j \sigma^{-1}, & B_j &\mapsto \sigma B_j \sigma^{-1} \text{ for } j = 1, \dots, g \\ D_j &\mapsto \sigma D_j \\ C_j &\mapsto C_j \end{aligned}$$

while the actions of the other G_j (for $j = 1, \dots, n$) are given by the formulas (2.10). The moment map for the action of G_0 is then given by

$$\Phi_0 : (A_1, \dots, A_g, B_1, \dots, B_g, C_1, \dots, C_n, D_1, \dots, D_n) \mapsto \prod_{j=1}^g A_j B_j A_j^{-1} B_j^{-1} \prod_{k=1}^n D_k C_k D_k^{-1}.$$

The space M_{impl}^G is obtained by imploding the actions of G_1, \dots, G_n and finally taking the quasi-Hamiltonian quotient by the action of G_0 .

Weighted Frames and the Relation to \mathcal{M}^T

One obtains the imploded cross-section by taking the inverse image under the moment map of the fundamental alcove, and then quotienting by a group which depends on which face of the alcove one is mapping to. In the case at hand, this inverse image is \mathcal{M}^T , and we have a projection $\mathcal{M}^T \rightarrow P$, which is an isomorphism over the interior $(\Delta^0)^n$ of the alcove.

Proposition 2.37 *Over $(\Delta^0)^n$, $\mathcal{M}^T \rightarrow P$ is a symplectomorphism.*

Proof For both \mathcal{M}^T and \mathcal{M}^G , the symplectic, or quasi-Hamiltonian structures are best defined in terms of their description on spaces of flat connections. For \mathcal{M}^T , one can define the relevant space of flat connections in various ways, but the most convenient is that given in [J1], where one considers the space \mathcal{A}^T of connections which are flat and which are, in a neighbourhood of the punctures, constant, of the form $a d\theta$, with $a \in \mathfrak{t}$ and θ an angular variable around the puncture. If α, β are two \mathfrak{g} -valued 1-forms representing the variation of such connections, one defines the symplectic form as

$$(2.38) \quad \sigma(\alpha, \beta) = \int_{\Sigma} \alpha \wedge \beta.$$

For \mathcal{M}^G , following [AMM, Section 9], one uses a larger space of connections \mathcal{A}^G , which has well defined values on the boundary circle surrounding the puncture. One then has restriction maps to the boundary circles

$$(2.39) \quad R_i : \mathcal{A}^G \rightarrow L\mathfrak{g},$$

where $L\mathfrak{g}$ is the loop algebra of \mathfrak{g} . One then defines the two-form on \mathcal{A}^G by

$$(2.40) \quad \hat{\sigma}(\alpha, \beta) = \sigma(\alpha, \beta) + \sum_i R_i^* \omega,$$

where ω is a form on Lg , given in [AMM, equation (37)]. The key point for us is that restricting to \mathcal{A}^T , we have that the map R_i takes its values in a one-dimensional space of constant loops, and so $R_i^*\omega$ is zero on this space. This then means that if $I: \mathcal{A}^T \rightarrow \mathcal{A}^G$ is the inclusion, inducing the map $\mathcal{M}^T \rightarrow P$, one has that $I^*\hat{\sigma} = \sigma$, giving us our symplectomorphism.

An element of \mathcal{M}^T is a flat connection, with trivialisations at the punctures such that the holonomy lies in the fundamental alcove. One obtains P from \mathcal{M}^T by collapsing some subvarieties, and one could ask what class of geometric objects our space P corresponds to. The answer can be seen as follows. Over the interior of the alcove, one has $\mathcal{M}^T \simeq P$, and so one again has a framing at each puncture, as before. As one moves to the boundary of the alcove, *i.e.*, to a holonomy of ± 1 , one finds in T that any trivialisation gives a holonomy in the fundamental alcove. In P , all of these trivialisations are identified; the trivialisation is “collapsed”. One can think of it in these terms: an element $(h, u) \in S^3 \times [0, 1]$ gives a framing (h), and a holonomy class u in the fundamental alcove; as one moves u to the boundary (to 0 or 1), one collapses the framing (which can be thought of as a unit vector in \mathbb{C}^2), scaling it by a “weight” to zero, and obtains S^4 ; S^4 is a space of pairs (“weighted” frame, holonomy angle), where the weight depends on the holonomy angle and goes to zero as the holonomy angle moves to 0 or 1. The space P then appears as a space of flat connections with these weighted frames at the punctures.

3 Glueing, and Connections on the Trinion

i) Glueing

Now suppose that we have a punctured surface Σ_0 , possibly disconnected, with punctures $p_+, p_-, p_1, \dots, p_n$ and corresponding moduli spaces \mathcal{M}_0^T, P_0 . Choosing parametrised boundary curves c_+ around p_+ , c_- around p_- , we can build a surface Σ by identifying these curves to one curve c_{\pm} ; let \mathcal{M}^T, P be the moduli spaces corresponding to Σ . The group $T = S^1$ embeds antidiagonally into $S^1 \times S^1$; this group acts naturally on the framings at p_+, p_- of elements of \mathcal{M}_0^T , with moment map the difference $t_+ - t_-$ of the holonomy angles. We can take the symplectic quotient

$$(3.1) \quad \mathcal{M}_{0,\text{red}}^T = \mathcal{M}_0^T // T.$$

(The symplectic form is of course degenerate when $t_{\pm} = \pm 1$.) Geometrically, this amounts to asking that the difference of the holonomy angles $t_+ - t_-$ vanish; one can then use the trivialisations to glue together the flat connections and obtain a flat connection on Σ . Along a T -orbit, the glueings give the same flat connection, so that there is a well-defined map

$$(3.2) \quad \mathcal{M}_{0,\text{red}}^T \rightarrow \mathcal{M}^T.$$

The fiber of this map is a point if $t_+ \neq 0, 1$. If $t_+ = 0$ or 1 , the fiber is S^3/S^1 if $n \geq 1$; the reason is that the holonomy then lies in the maximal torus for any trivialisation, but one is only quotienting out a torus' worth of such choices. If $t_+ = 0$ or 1 , and $n = 0$, the fiber is the quotient of S^3 by S^1 and the action of the group of automorphisms of the bundle.

The action of the subgroup $1 \times S^1$ of $S^1 \times S^1$ commutes with that of T , and so descends to the quotient; using the map (3.2), one has an action of S^1 on the moduli \mathcal{M}^T which is only defined when $t_+ \neq 0, 1$. These are the flows of Goldman [G1].

In parallel to this, one can also take the symplectic quotient on the space P^0 , obtaining

$$(3.3) \quad P_{0,\text{red}} = P_0 // S^1.$$

This space is symplectic. The action of the subgroup $1 \times S^1$ descends to $P_{0,\text{red}}$. Also, the map $\mathcal{M}_0^T \rightarrow P_0$ descends to

$$(3.4) \quad \mathcal{M}_{0,\text{red}}^T \rightarrow P_{0,\text{red}}$$

which commutes with the $1 \times S^1$ -action.

This construction can obviously be iterated; in particular, if Σ is a surface with moduli space \mathcal{M}^T , we can cut Σ along k circles into pieces $\Sigma_1, \dots, \Sigma_\ell$. For any such circle, defining, say, the boundary between Σ_i and Σ_j , we have an action of $S^1 \times S^1$ on the trivialisations of the bundles at the marked points of Σ_i and Σ_j . If we take the symplectic quotients by the anti-diagonal S^1 's, we have maps

$$(3.5) \quad (\mathcal{M}_1^T \times \mathcal{M}_2^T \times \dots \times \mathcal{M}_n^T) // (S^1)^k \rightarrow \mathcal{M}^T$$

$$(3.6) \quad (\mathcal{M}_1^T \times \mathcal{M}_2^T \times \dots \times \mathcal{M}_n^T) // (S^1)^k \rightarrow (P_1 \times P_2 \times \dots \times P_n) // (S^1)^k$$

The map (3.6) factors through \mathcal{M}^T , giving a map

$$(3.7) \quad \Psi: \mathcal{M}^T \rightarrow (P_1 \times P_2 \times \dots \times P_n) // (S^1)^k.$$

Over the generic locus of points where all the holonomies at the punctures are not central, this map is an isomorphism; over a connection E in \mathcal{M}^T where some, say ℓ , of the holonomies are central, the fiber is the quotient of $(S^3)^\ell$ by the action of the automorphisms of E .

All of these maps commute with the $1 \times S^1$ -actions defined above; on \mathcal{M} , these gave the Goldman flows, which are only defined when the holonomy is non-central; on the right, we have globally defined Hamiltonian S^1 -flows.

One case which is of prime interest is when we have decomposed the surface with no punctures into a sum of $2g - 2$ trinions. In this case, as we saw, we almost had on \mathcal{M}^T the structure of a toric variety. We will see that the right hand side of (3.7) will then simply be the toric variety corresponding to \mathcal{M}^T . We first want to understand the structure of P when Σ is a trinion.

ii) Bundles on a Trinion

We now specialise to the case of $SU(2)$ bundles on the three-punctured sphere or trinion. The first question is to analyse the space of representations of the fundamental group. This group is the free group on two letters, but, as in [JW], we want a description which is symmetric in the three punctures. Let k_1, k_2, k_3 then be three generators, such that $k_1 k_2 k_3 = 1$. In terms of the preceding notation, we have $k_1 = c_1, k_2 = d_2 c_2 d_2^{-1}, k_3 = d_3 c_3 d_3^{-1}$. We let K_1, K_2, K_3 denote the images of k_1, k_2, k_3 under a representation into $SU(2)$. If the K_i are not equal to ± 1 , they each determine in turn a unique maximal torus, and indeed a unique translate in $\mathbb{P}_1 = SU(2)/T$ of the fundamental chamber $\Delta = [0, \pi]$; these three translates

are distinct for an irreducible representation. More concretely, for each element, say K_1 , there is a unique point \hat{K}_1 in S^2 such that the action of conjugation by K_1 on $S^2 = \text{SU}(2)/S^1$ is given by an anti-clockwise rotation $R_{2\gamma_1}^{\hat{K}_1}$ by an angle $2\gamma_1$, $\gamma_1 \in (0, \pi)$ around \hat{K}_1 . From the relation

$$R_{2\gamma_1}^{\hat{K}_1} R_{2\gamma_2}^{\hat{K}_2} R_{2\gamma_3}^{\hat{K}_3} = 1,$$

one finds that the vertices $\hat{K}_1, \hat{K}_2, \hat{K}_3$ form a spherical triangle with angles $\pi - \gamma_1, \pi - \gamma_2, \pi - \gamma_3$ respectively, and that \hat{K}_3 can belong to only one of the two hemispheres bounded by the great circle determined by \hat{K}_1, \hat{K}_2 (there is an orientation). From these facts, or referring to [JW], one has the relations:

$$(3.8) \quad \begin{aligned} \gamma_1 + \gamma_2 + \gamma_3 &\leq 2\pi \\ \gamma_1 + \gamma_2 - \gamma_3 &\geq 0 \\ \gamma_1 - \gamma_2 + \gamma_3 &\geq 0 \\ -\gamma_1 + \gamma_2 + \gamma_3 &\geq 0. \end{aligned}$$

This defines a tetrahedron Γ in \mathbb{R}^3 , and one has

Lemma 3.9 *The equivalence classes of representations of the fundamental group of the trinion into $\text{SU}(2)$ are in bijective correspondence with elements of Γ .*

The proof is given in [JW]. We note here that the action of conjugation on a representation is simply that of rotating the triangle in S^2 . One can normalise an irreducible representation so that \hat{K}_1 is the north pole (conjugating K_1 to Δ), and then rotate around the north pole so that \hat{K}_2 lies on a fixed meridian.

Several other remarks are in order.

1. The interior of Γ corresponds to irreducible representations, or again to non-degenerate triangles in S^2 .
2. The interior of the faces of the tetrahedron correspond to reducible representations, with none of the K_1, K_2, K_3 equal to ± 1 . In terms of triangles, they correspond to degenerate triangles, with either all three vertices at the same point, or two of the vertices at one point and the other at the antipodal point.
3. The interiors of the edges correspond to representations at which one and only one of the K_1, K_2, K_3 is central. There are six edges, corresponding to $K_1 = 1, K_1 = -1, K_2 = 1, K_2 = -1, K_3 = 1, K_3 = -1$ respectively.
4. The vertices correspond to representations for which all of the K_1, K_2, K_3 are central. There are four vertices, corresponding to $(K_1, K_2, K_3) = (1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)$ respectively.

We now consider the spaces \mathcal{M}^T, P for the trinion, and group $\text{SU}(2)$. We have

$$\mathcal{M}^T = \{(C_1, C_2, C_3, D_2, D_3) \in \Delta^3 \times \text{SU}(2)^2 \mid C_1 D_2 C_2 D_2^{-1} D_3 C_3 D_3^{-1} = 1\}.$$

In terms of representations, we have normalised K_1 to $C_1 \in T$, and $K_2 = D_2 C_2 D_2^{-1}, K_3 = D_3 C_3 D_3^{-1}$. The fiber of this space over the representation space R , for an irreducible element

of R , is given by $T^3 / \pm (1, 1, 1) \simeq (S^1)^3$. On the other hand, for the space P , one has fibers over the interior of Γ of the form $S^1 \times S^1 \times S^1 / \pm 1 = (S^1)^3$. For representations in the interior of one of the faces (reducible representations with none of the K_1, K_2, K_3 central), one has for each of K_1, K_2, K_3 a frame in S^1 , and one then quotients by the diagonal S^1 to obtain as fiber an $S^1 \times S^1$. When one goes to the edges, one loses one of the framings as one of K_1, K_2, K_3 is then central; the fiber is then a circle. Finally, at the vertices, there is no extra information apart from K_1, K_2, K_3 and so the fiber is a point.

In short, this is compatible with P being the toric variety corresponding to Γ ; this variety is \mathbb{P}^3 . To see that this is the case, we first note

Proposition 3.10 *The map $\mu: P \rightarrow \Gamma$ which associates to each representation the angles $\gamma_1, \gamma_2, \gamma_3$ is the moment map for the action of $(S^1)^3$ on P .*

The map μ has a Lagrangian section, given as the fixed point set of an anti-symplectic involution.

Proof The first statement is found in [J1, Proposition 4.1]. For the second part, it is easiest to give the involution on \mathcal{M}^T first. To do this, we suppose that the three punctures on our Riemann sphere all lie on the real equator $\mathbb{R}\mathbb{P}^1 \subset \mathbb{C}\mathbb{P}^1$. The involution $z \mapsto \bar{z}$ induces a transformation on the space of representations, by pull-back. As the map reverses orientations, it is anti-symplectic. Choosing our base-points correctly, we can give the action on \mathcal{M}^T , up to conjugation, by

$$(C_1, C_2, C_3, D_2, D_3) \mapsto (C_1^{-1}, C_2^{-1}, C_3^{-1}, C_1 D_2, D_3).$$

This does not leave the C 's in Δ , so we compose with conjugation by an element J representing the non-zero Weyl group element, which gives $J C_i^{-1} J^{-1} = C_i, i = 1, 2, 3$. The result of the composition is then

$$(C_1, C_2, C_3, D_2, D_3) \mapsto (C_1, C_2, C_3, J C_1 D_2 J^{-1}, J D_3 J^{-1}).$$

The induced action on K_1, K_2, K_3 is $(K_1, K_2, K_3) \mapsto (K_1, K_1^{-1} J K_2^{-1} J^{-1} K_1, J K_3^{-1} J^{-1})$. Let us consider this action for the irreducible representations in the interior of Γ . Choosing polar coordinates $(\theta, \phi) \in [0, \pi] \times [0, 2\pi]$ on the sphere so that J corresponds to the point $\hat{J} = (\pi/2, 0)$, the elements K_1, K_2, K_3 are fixed if \hat{K}_3 lies on the great circle $\phi = \pm\pi/2$, and \hat{K}_2 lies on the great circle $\phi = \pm(\gamma_1 + \pi/2)$. Taking orientations into account, this gives two possibilities for (K_1, K_2, K_3) once the representation has been fixed. Once K_1, K_2, K_3 are fixed, one finds that there are two possibilities each for D_2, D_3 , giving eight possibilities in all in the fiber $(S^1)^3$ of P over Γ . The real points in the fiber are an orbit of $\mathbb{Z}/2 \times \mathbb{Z}/2 \times \mathbb{Z}/2$ in $S^1 \times S^1 \times S^1$. Over the interior of Γ , we can normalise our choices so as to get a Lagrangian section. This section extends in \mathcal{M}_Δ to the interior of the faces of Γ . As the involution preserves conjugacy classes, one checks that the involution descends to P , and here, it extends to all of Γ . From all of this, we obtain:

Proposition 3.11 *The variety P is isomorphic to $\mathbb{P}^3(\mathbb{C})$.*

Proof This is in essence a case of Delzant's theorem [D]. Indeed, Γ is the moment polytope for $\mathbb{P}^3(\mathbb{C})$ with the action of $(a, b, c) \in (\mathbb{C}^*)^3$ given in inhomogeneous coordinates by

$(a, b, c) \cdot (x, y, z) = (abx, bcy, acz)$. We note that this action is not generically free, as the action of $\pm(1, 1, 1)$ is trivial, and because of this, strictly speaking, we do not have a Delzant polytope. The action factors through $(\mathbb{C}^*)^3 / \pm(1, 1, 1) = (\mathbb{C}^*)^3$, however.

We first note that the variety P is smooth, except possibly at four points corresponding to the vertices. This follows from the fact, referring to (2.34), that P is defined as the quotient of a subvariety of $(D(G)_{\text{impl}})^3 = (S^4)^3$, by the action of $SU(2)$, and the action of the group is free away from these points. Leaving these four points aside for the moment, both P and $\mathbb{P}^3(\mathbb{C})$ have Lagrangian sections for their respective moment maps to Γ , and we can identify these sections. This identifies our section of P above with the positive octant in $\mathbb{P}^3(\mathbb{R}) \subset \mathbb{P}^3(\mathbb{C})$. The group actions are however compatible, and so we can use the action to identify P and $\mathbb{P}^3(\mathbb{C})$ globally.

To see that the quotient space P is smooth even at the four vertex points, we first map a neighbourhood of the origin in \mathbb{C}^3 to $(S^4)^3$, in a way that is generically bijective on the quotient. Let us do this, for the point corresponding to the trivial flat connection. We first extend the real involution to $(S^4)^3$. For this, we write S^4 as the result of collapsing the ends of $\Delta \times SU(2)$. The real involution on $(S^4)^3$ is then obtained from the involution on $(\Delta \times SU(2))^3$:

$$I: (C_1, C_2, C_3, D_1, D_2, D_3) \mapsto (C_1, C_2, C_3, JD_1J^{-1}, JD_1C_1D_1^{-1}D_2J^{-1}, JD_3J^{-1}).$$

The action of $g \in SU(2)$ maps D_i to gD_i , and fixes the C_i ; the action of $h \in S^1$, on the j -th factor, is given by $(C_j, D_j) \mapsto (C_j, D_jh)$. Note that $I \circ g = JgJ^{-1} \circ I$. Recall also that in S^4 , the D_i 's get collapsed if $C_i = \pm 1$. Now lift the real octant in P to $(S^4)^3$ by setting $D_1 = 1$, and then extend this via the $(S^1)^3$ actions to a map of \mathbb{C}^3 . This gives the slice, and so expresses the $SU(2)$ -invariant functions on $(S^4)^3$ as a subring of the functions on \mathbb{C}^3 .

One can then build invariant coordinates on P : set $C_j = \exp(\text{diag}(i\theta_j, -i\theta_j))$, and let $r_1 = \theta_2 + \theta_3 - \theta_1$, $r_2 = \theta_1 + \theta_3 - \theta_2$, $r_3 = \theta_1 + \theta_2 - \theta_3$. One considers the functions

$$\begin{aligned} V_1 &= r_1 C_1^{1/2} D_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge D_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ V_2 &= -r_2 D_3 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge D_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \\ V_3 &= r_1 C_1^{1/2} D_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix} \wedge D_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \end{aligned}$$

with values in $\Lambda^2(\mathbb{C}^2) = \mathbb{C}$. These functions are invariant, and provide a $(S^1)^3$ -equivariant identification of P with an open set of \mathbb{C}^3 , showing that the subring is in fact the whole ring of functions on \mathbb{C}^3 .

Let us summarise the situation of the Goldman flows for a Riemann surface Σ of genus g , with n punctures p_1, \dots, p_n . The moduli space \mathcal{M}^T is then of dimension $6g + 4n - 6$. We can cut the Riemann surface along $3g - 3 + n$ disjoint circles, decomposing the surface into trinions, with at most one of the original punctures in each trinion; the ends are either the cut circles or circles around the punctures, that is $3g - 3 + 2n$ circles in all; let D denote this decomposition. Quotienting by the antidiagonal S^1 -actions at each of the cut circles, there

is then a map, as in (3.7)

$$(3.12) \quad \Psi: \mathcal{M}^T \rightarrow P^D = (\mathbb{P}^3(\mathbb{C}) \times \mathbb{P}^3(\mathbb{C}) \times \dots \times \mathbb{P}^3(\mathbb{C})) // (S^1)^{3g-3+n}$$

which is generically an isomorphism. Here there are $2g - 2 + n$ copies of $\mathbb{P}^3(\mathbb{C})$ in the product, corresponding to the same number of trinions in the decomposition. On the right hand side, one has an effective action of $(S^1)^{3g-3+2n}$, and so the variety is toric. Let $\Gamma \subset P^D$ denote the set of elements in P^D corresponding to flat connections whose holonomy is central along one of the $3g-3+n$ cut circles, and Γ' the locus corresponding to connections which are central along all of the n puncture circles. We can choose the decomposition D in such a way that the flat connections which are reducible all lie in $\Psi^{-1}(\Gamma)$, for example letting one of the cut circles be null-homologous; let us call these good decompositions. One can reduce both \mathcal{M}^T and P^D under the action of the S^1 's associated to the punctures. On \mathcal{M}^T , the resulting $\mathcal{M}_{\text{red}}^T$, for values of the moment map (holonomy) $\mathcal{M}^t \rightarrow \exp(\Delta)$ different from ± 1 at each puncture, is a parabolic moduli space. From P^D , one obtains a variety P_{red}^D which is again toric, and one has a map

$$(3.13) \quad \Psi_{\text{red}}: \mathcal{M}_{\text{red}}^T \rightarrow P_{\text{red}}^D.$$

Finally, we note that we can restrict the holonomy at one of the punctures, say p_1 , to be -1 . This corresponds to reducing at the singular value -1 of the moment map; doing this for P^D , one obtains a space P_{-}^D . If we then reduce at regular values of the moment map at the other punctures, we have a space $P_{-, \text{red}}^D$. One could do the same for \mathcal{M}^T ; it is preferable however to use the space $\mathcal{M}_{-}^T = \mathcal{M}_{-1}^T$ of (2.8) associated to the surface punctured at p_2, \dots, p_n . There is a map

$$(3.14) \quad \Psi_{-}: \mathcal{M}_{-}^T \rightarrow P_{-}^D,$$

which is an isomorphism away from the varieties Γ_{-}, Γ'_{-} of elements whose holonomy is central at one of the cut circles, or all of puncture circles, respectively. Reducing at the other punctures, at regular elements, one has spaces and a map:

$$(3.15) \quad \Psi_{-, \text{red}}: \mathcal{M}_{-, \text{red}}^T \rightarrow P_{-, \text{red}}^D.$$

Summarising, one has:

Theorem 3.13

- a) For $n > 0$, the variety \mathcal{M}^T is smooth away from $\Psi^{-1}(\Gamma')$. For any n and a good decomposition it is smooth away from $\Psi^{-1}(\Gamma \cap \Gamma')$. The variety P^D is smooth away from Γ . The map Ψ of (3.12), away from $\Psi^{-1}(\Gamma \cup \Gamma')$, is a symplectomorphism between \mathcal{M}^T and

the toric variety P^D . Over an element E in $\Gamma \cup \Gamma'$ where some, say j , of the holonomies are central, the fiber is the quotient of $(S^3)^j$ by the action of the automorphisms of E .

The real codimension of Γ in P^D is six; in \mathcal{M}^T , the codimension of $\Psi^{-1}(\Gamma)$ or of $\Psi^{-1}(\Gamma')$ is three.

- b) For $n > 0$, reducing by the $(S^1)^n$'s acting on the framings at the punctures, at holonomies different from ± 1 , the resulting $\mathcal{M}_{\text{red}}^T$ is smooth, and is symplectomorphic to P^D away from the reduction of $\Psi^{-1}(\Gamma)$. P_{red}^D is smooth away from Γ .

The map Ψ_{red} is a symplectomorphism, away from the reduction of $\Psi^{-1}(\Gamma)$. The real codimension of the reduction of Γ in P_{red}^D is six, and that of the reduction of $\Psi^{-1}(\Gamma)$ in $\mathcal{M}_{\text{red}}^T$ is three.

- c) The spaces \mathcal{M}_-^T are smooth away from $\Psi_-^{-1}(\Gamma'_-)$, and the $\mathcal{M}_{-, \text{red}}^T$ are smooth, again reducing at holonomies different from ± 1 at the punctures p_2, \dots, p_n . The map Ψ_- of (3.14), away from $(\Psi_-)^{-1}(\Gamma_- \cup \Gamma'_-)$, is a symplectomorphism between \mathcal{M}_-^T and the toric variety P_-^D . The map $\Psi_{-, \text{red}}$ of (3.15) is a symplectomorphism, away from the reduction of $(\Psi_-)^{-1}(\Gamma_-)$.

The real codimension of Γ_- in P_-^D is six; in \mathcal{M}^T , the codimension of $\Psi_-^{-1}(\Gamma_-)$ or of $\Psi_-^{-1}(\Gamma'_-)$ is three.

The real codimension of the reduction of Γ_- in $P_{-, \text{red}}^D$ is six, and that of the reduction of $\Psi_-^{-1}(\Gamma_-)$ in $\mathcal{M}_{-, \text{red}}^T$ is three.

4 Moduli of Framed Parabolic Sheaves

If one fixes the conjugacy class of the holonomy at the punctures, there is a well-established correspondence, due to Mehta and Seshadri, between the moduli spaces of representations of $\pi_1(\Sigma)$ into $SU(2)$ on one hand, and, on the other, rank-2 holomorphic bundles with trivial determinant and with a parabolic structure at the punctures (a choice of a line ℓ_p in the fiber above the puncture p , along with some weights). The eigenvalues $\exp(\pi i \delta)$, $\exp(\pi i(1 - \delta))$, $\delta \in [0, 1]$ of the holonomy get translated into weights of the parabolic structure, and so to different choices of polarization. In our case, we are dealing with a space in which all of the possible holonomies appear: we will thus want a space which contains all of these parabolic moduli spaces. We add in the extra "conjugate" variables of (partial) framings at the punctures, and so consider a space of framed parabolic bundles, that is bundles equipped with a trivialisation of the lines ℓ_p . As our bundles E have an $SL(2, \mathbb{C})$ structure, this is the same as a trivialisation of E_p/ℓ_p , and so can be thought of as a map $E \rightarrow \mathbb{C}_p$, whose kernel at p is the line ℓ_p . Moduli of such pairs (bundles, maps to a fixed sheaf) have been studied by Huybrechts and Lehn [HL] and we want to adapt their work. Our moduli spaces will have an extra parameter, corresponding to an $SL(2, \mathbb{C})$ structure on the bundle.

We first recapitulate and make some comments on the moduli with fixed holonomy. It will turn out that for $\theta = 1$, our moduli space, if it is to be compact, will contain sheaves with torsion; we will want to adapt a notion of stability of [HL] for pairs consisting of a coherent sheaf and a map of the sheaf into a fixed sheaf, which extends in our case the stability used for parabolic bundles. Let p_1, \dots, p_n denote the punctures and let S be the sheaf $\bigoplus_i \mathbb{C}_{p_i}$. A (parabolic) framing of a sheaf E is then a map of sheaves $\alpha: E \rightarrow S$, which we can write as $\alpha_i: E \rightarrow \mathbb{C}_{p_i}$. Let $\gamma_1, \dots, \gamma_n \in [0, 1]$ be a set of weights. For a subsheaf F

of E , we set $\mu_i(F) = 1$ if F lies in the kernel of α_i at p_i , and $\mu_i = 0$ otherwise. The cases considered in [HL] had $\alpha_i \neq 0$; we extend here to $\alpha_i = 0$. Let $\sigma_i(F) = 1/\text{rk}(E)$ if $F = E$, and 0 otherwise. We will say that a pair (E, α) is (semi) stable if for all non-trivial coherent subsheaves F (including E),

$$(4.1) \quad \text{rk}(E) \deg(F) < (\leq) \text{rk}(F) \left(\deg(E) - \sum_i \gamma_i \right) + \text{rk}(E) \sum_i (1 - \mu_i(F) + \sigma_i(F)) \gamma_i.$$

In our case, $\text{rk}(E) = 2$, $\deg(E) = 0$, and so this becomes

$$(4.2) \quad 2 \deg(F) < (\leq) \sum_i \gamma_i (2 - \text{rk}(F) - 2\mu_i(F) + 2\sigma_i(F)).$$

Lemma 4.3 *If (E, α) is semi-stable, then*

- i) *The kernel of α is torsion free, and the torsion subsheaf of E is concentrated over the p_i , and is either 0 or \mathbb{C} at each p_i ,*
- ii) *If $\gamma_i > 0$, then the map α is non-zero at p_i ,*
- iii) *If $\gamma_i < 1$, then E is torsion free at p_i ,*
- iv) *For $\gamma_i \in (0, 1)$, one has a parabolic structure at p_i , and if all the weights γ_i lie in $(0, 1)$, the stability condition (4.2) is identical to that for parabolic bundles with weights $(1 - \gamma_i)/2$, $(1 + \gamma_i)/2$,*
- v) *Let (E, α) be a semistable pair, which is locally free at p_i , with a non-zero α . For $\gamma_i = 0$, there is a family (E_t, α_t) , $t \in \mathbb{C}$ of semistable pairs such that $(E_t, \alpha_t) \simeq (E, \alpha)$, $t \neq 0$, and $\alpha_0 = 0$. For $\gamma_i = 1$, there is a family (E_t, α_t) , $t \in \mathbb{C}$ of semistable pairs such that $(E_t, \alpha_t) \simeq (E, \alpha)$, $t \neq 0$, and E_0 has torsion at p_i .*

Proof For i), the torsion subsheaf of $\ker(\alpha)$ has rank 0, so its degree must be zero, and so it must be zero [HL]. For ii), one simply considers the stability condition for E as a subsheaf of E . For iii), one considers the torsion subsheaf of E at p_i . The existence of a parabolic structure in iv) follows from ii) and iii); the stability condition for parabolic bundles is $\deg(G) + \sum_i (\mu_i(1 + \gamma_i)/2 + (1 - \mu_i)(1 - \gamma_i)/2) < (\leq) n/2$, which is equivalent to (4.2) for line bundles, and the stability conditions for other subsheaves is automatically satisfied. For v), when $\gamma_i = 0$, one simply sets $(E_t, \alpha_t) = (E, t\alpha)$. For $\gamma_i = 1$, we choose a local coordinate z near p_i , with p_i corresponding to $z = 0$. Over, say, $|z| < 1$, we then choose a local isomorphism $E \simeq \mathcal{O} \oplus \mathcal{O}$, such that α is represented by the vector $(1, 0)$. We can write $\mathcal{O} \oplus \mathcal{O}$ as a quotient, for $t \neq 0$:

$$(4.4) \quad \begin{array}{ccc} \begin{pmatrix} -z & t & 0 \end{pmatrix} & & A = \begin{pmatrix} t & z & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \mathcal{O} & \longrightarrow & \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O} & \longrightarrow & \mathcal{O} \oplus \mathcal{O}. \end{array}$$

The map $t^{-1}\alpha = (t^{-1}, 0)$ can be represented by the map $(1, 0, 0)$ on $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ at $z = 0$. For $t = 0$, the quotient in (4.4) is not $\mathcal{O} \oplus \mathcal{O}$, but $\mathbb{C}_{p_i} \oplus \mathcal{O} \oplus \mathcal{O}$. One builds the family E_t by using the matrix A in (4.4) to map $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ to E over $z \neq 0$; as this annihilates the kernel \mathcal{O} , it glues the quotient sheaf. The map α_t is then represented by $(1, 0, 0)$. One checks that E_0 is still semistable, using a natural map $E_0/\text{torsion} \rightarrow E_t$, $t \neq 0$; this maps a subsheaf L

of E_0 to a subsheaf \tilde{L} of E_r , killing the torsion, but leaving the difference of the degree and μ_i unchanged.

Remark 4.5 As a consequence of v), any point in the moduli space will have a representative with torsion at p_i for $\gamma_i = 1$. At such points one can write the sheaf as a sum of its torsion piece and the kernel of α_i . If one quotients out the torsion at the k points with $\gamma_i = 1$, one obtains a bundle with $c_1(E) = -k$, and the parabolic structure has disappeared. Similarly, for $\gamma_i = 0$, we can choose a representative with $\alpha_i = 0$, so again the parabolic structure has disappeared. If one does this for both the points for which $\gamma_i = 0$, and for which $\gamma_i = 1$, one obtains a bundle of degree $-k$ with parabolic structure at the k_i remaining points, and one checks that our stability condition becomes the stability condition for such bundles.

We now recall some of the construction of Huybrechts and Lehn, modified to correspond to our case, and in particular taking into account the existence of several punctures; the weights, initially, are in $(0, 1)$, so that one is dealing with vector bundles, and the α_i are all non-zero. The construction also gives the moduli of parabolic bundles; see also [Th], [Bh], and [Gi]. As with any moduli space involving vector bundles, the starting point is the Grothendieck *Quot*-scheme parametrising quotients of the trivial bundle

$$(4.6) \quad \mathcal{O}^{\oplus N} \rightarrow \tilde{E}.$$

(We twist E by a fixed line bundle L so that the resulting \tilde{E} is generated by global sections and so that the maps (4.6) induce isomorphisms on global sections; as we were dealing originally with $SL(2, \mathbb{C})$ -bundles, our bundles now have an induced isomorphism $\gamma: \Lambda^2(\tilde{E}) \simeq L^2$.)

In other words, we are not only considering (E, α_i) , but have added in a basis of sections of E ; one has a good scheme parametrising such objects, but we now must quotient by the group $GL(N, \mathbb{C})$.

It is convenient, again following [HL], [Th], [Bh], [Gi], to transform the datum encoding E somewhat. One takes the induced map on second exterior powers, and then the map that this induces on sections; using the map $\gamma: \Lambda^2(E) \rightarrow L^2$ given by the $SL(2, \mathbb{C})$ structure, one gets an element $\hat{\beta}$ of

$$(4.7) \quad V_1 = \text{Hom}(\Lambda^2 \mathcal{O}^{\oplus N}, H^0(L^2)).$$

This map is a finite injective morphism [Th, Section 7].

Next, one adds in the parabolic data: each non-zero framing, acting on the global sections, yields an element $\hat{\alpha}_i$ in copy $V_{2,i}$ of

$$(4.8) \quad V_2 = H^0(\mathcal{O}^{\oplus N})^*,$$

for each puncture p_i .

If, as in the case of parabolic bundles, one is only interested in the α_i up to (independent) scale, one then obtains a point representing the equivalence class of $(E$ with its parabolic structure) in a closed subvariety X in

$$(4.9) \quad Z_1 \times Z_{2,1} \times Z_{2,2} \times \cdots \times Z_{2,n},$$

where $Z_1 = \mathbb{P}(V_1)$, $Z_{2,i} = \mathbb{P}(V_{2,i})$, under the natural action of $SL(N, \mathbb{C})$ on these spaces. The space X is independent of the choice of weights; what varies is the polarisation, that is the choice of line bundles on which the action linearises. Let $\pi_1, \pi_{2,i}$ denote the projections onto $Z_1, Z_{2,i}$ respectively. Set $L_0 = \pi_1^*(\mathcal{O}(N))$, $L_{1,i} = \pi_1^*(\mathcal{O}(N - 1)) \otimes \pi_{2,i}^*(\mathcal{O}(2))$; then the linearisation corresponding to the (rational) weights γ_i is $(L_0)^{s_0} \otimes (\bigotimes_i (L_{1,i})^{s_{1,i}})$, where $s_0(\gamma_i) = s_{1,i}(1 - \gamma_i)$.

We now pass to the problem of fitting all of these moduli together, and adding in the cases $\gamma_i = 0, 1$. Fortunately, there is a technique due to Thaddeus [Th] which allows us to consider all of these quotients at once. One builds over the space X the $(\mathbb{P}^1)^n$ -bundle

$$(4.10) \quad Y = \mathbb{P}(L_0 \oplus L_{1,1}) \oplus \mathbb{P}(L_0 \oplus L_{1,2}) \oplus \cdots \oplus \mathbb{P}(L_0 \oplus L_{1,n}),$$

endowing it with its natural polarisation $\mathcal{O}(1, 1, \dots, 1)$. This space has embedded in it all the stable points for the various γ_i , and one can obtain the moduli spaces for each γ_i by taking an appropriate $SL(N, \mathbb{C}) \times (\mathbb{C}^*)^n$ -quotient; the γ_i now appear as the parameters for the appropriate $(\mathbb{C}^*)^n$ polarization [Th]. We will be interested, however, in the $SL(N, \mathbb{C})$ quotient of the whole “master” space Y ; it will turn out that this is not quite the moduli space which we want, but it is almost the correct one. One must modify the definition when $\gamma_i = 0$.

Indeed, we consider the space of quadruples $(E, \alpha_i, \langle \alpha_i \rangle, \gamma)$, where E is a rank 2 sheaf, α_i is the map $E \rightarrow \bigoplus_i \mathbb{C}_{p_i}$, $\langle \alpha_i \rangle$ is a subspace of $E|_{p_i}$ which is the kernel of α when α is non-zero, one-dimensional when E has no torsion at p_i . This adds a parabolic structure when $\gamma_i = 0$; one has a projective class $\langle \hat{\alpha}_i \rangle$ of the corresponding map $\hat{\alpha}_i$ even when α_i vanishes. Our quadruples then give elements $(\langle \hat{\beta} \rangle, \langle \hat{\alpha}_1 \rangle, \dots, \langle \alpha_n \rangle)$ in $X \subset Z_1 \times Z_{2,1} \times Z_{2,2} \times \cdots \times Z_{2,n}$.

There are additional questions to be dealt with when γ_i goes to 1; the bundles are then not necessarily locally free, and so can acquire torsion. We examine the $SL(2, \mathbb{C})$ -structure of such sheaves E .

Lemma 4.11 *Let $E_t, t \in \mathbb{C}$ be a coherent family of rank 2 sheaves over the curve Σ , with E_t locally free at p for $t \neq 0$, and E_0 with \mathbb{C}_p as torsion subsheaf near p . Let $\phi_t \in H^0(\Sigma, \Lambda^2(E)^*)$ be a family of $SL(2, \mathbb{C})$ -structures on E_t . Then ϕ_0 vanishes at p .*

Proof The question is local. If z is a coordinate on Σ with $z = 0$ corresponding to p , one can obtain E_t locally (changing the parameter t if necessary), as the cokernel of

$$\mathcal{O} \xrightarrow{(0, t^k, z)} \mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O},$$

for some integer k . The forms ϕ_t are then given as multiples of $e_1^* \wedge (-ze_2^* + t^k e_3^*)$, which vanishes at $z = t = 0$.

What this tells us is that there is up to scale a fiducial two-form ϕ , which is non-zero away from the torsion points and vanishes at the torsion points. The choice of an $SL(2, \mathbb{C})$ structure is the choice of the (non-zero) scale for such a form.

One must see how our quadruples correspond to orbits in Y , living over the orbits of $\hat{E}_X = (\langle \hat{\beta} \rangle, \langle \hat{\alpha}_1 \rangle, \dots, \langle \hat{\alpha}_n \rangle)$ in X . We note that as we are dealing with projective bundles,

we can tensor with line bundles without changing the structure, so that Y can be written as the $(\mathbb{P}^1)^n$ -bundle

$$(4.12) \quad Y = \mathbb{P}\left(\pi_1^*(\mathcal{O}(-1)) \oplus \pi_{2,1}^*(\mathcal{O}(-2))\right) \oplus \mathbb{P}\left(\pi_1^*(\mathcal{O}(-1)) \oplus \pi_{2,2}^*(\mathcal{O}(-2))\right) \\ \oplus \cdots \oplus \mathbb{P}\left(\pi_1^*(\mathcal{O}(-1)) \oplus \pi_{2,n}^*(\mathcal{O}(-2))\right)$$

so that one is considering projectivisations of the sum of a tautological bundle and a second power of a tautological bundle. We then have a natural lift of \hat{E}_X to Y , given by

$$\hat{E}'_X = ((\hat{\beta}, \hat{\alpha}_1^2), \dots, (\hat{\beta}, \hat{\alpha}_n^2)).$$

This must be extended to the cases when the sheaf ceases to be locally free and acquires torsion at the p_i . Indeed, when the bundle acquires torsion, at p_i , say, one finds that by rescaling the torsion piece of the bundle we can modify the scale of $\hat{\alpha}_i^2$ to $c\hat{\alpha}_i^2$, say. These should be represented in the same orbit, and if the lift $\hat{\beta}$ were non zero, this would not be the case. One would want the i -th pair in \hat{E}'_X to be of the form $(0, \hat{\alpha}_i^2)$ which would then absorb the scale. This can be achieved as follows. Recall that $\hat{\beta}$ is defined as the composition.

$$\Lambda^2(\mathbb{C}^N) \xrightarrow{\sigma} H^0(\Sigma, \Lambda^2(E)) \xrightarrow{\xi} H^0(\Sigma, L^2).$$

The second map ξ is obtained as follows. Recall that we were originally dealing with bundles E_0 with $SL(2, \mathbb{C})$ -structure, and so a map $\xi_0: H^0(\Sigma, \Lambda^2(E_0)) \rightarrow H^0(\Sigma, \mathcal{O}) = \mathbb{C}$. The bundle E is obtained as $E_0 \otimes L$, giving the induced map ξ . We have a commutative diagram

$$\begin{array}{ccc} H^0(\Sigma, \Lambda^2(E_0)) & \xrightarrow{\xi_0} & H^0(\Sigma, \mathcal{O}) \\ \downarrow ev_{\Lambda^2(E_0)} & & \downarrow ev_{\mathcal{O}} \\ \Lambda^2(E_0)|_{p_i} & \xrightarrow{\xi_0^{p_i}} & \mathbb{C}. \end{array}$$

One then has that $\xi_0 = ev_{\mathcal{O}}^{-1} \circ \xi_0^{p_i} \circ ev_{\Lambda^2(E_0)}$; and it is this composition that one uses to extend the definition of $\hat{\beta}$ to the cases when the bundle acquires torsion at the p_i . One obtains then maps $\hat{\beta}_i$, and one checks (by a calculation in local coordinates, say for a one-parameter family; see Lemma 4.11 above) that $\hat{\beta}_i$ vanishes when the bundle acquires torsion at the p_i ; when there is no torsion, one simply has the preceding definition. One then sets

$$\hat{E}'_X = ((\hat{\beta}_1, \hat{\alpha}_1^2), \dots, (\hat{\beta}_n, \hat{\alpha}_n^2)).$$

We analyse the stability of elements of Y . We had that the presence of torsion in the kernel of α_i destabilised an element of X ; the same is true of elements of Y :

Lemma 4.13 *A semi-stable element y of Y corresponds to a bundle E without torsion in the kernel of α .*

Proof Let $U = H^0(\mathcal{O}^N)$, and split U as $U_0 \oplus U_1$, with U_0 the sections of the torsion subsheaf of E , and U_1 the sections of $E/\text{torsion}$. The element of Z_1 corresponding to E is represented by an element of V_1 ; if in addition there is a non-trivial subspace W of U_0 lying in the kernel of α , one can destabilise y by a one parameter multiplicative subgroup acting with positive weight on W and negative weight on the complement.

It follows then that the semi-stable y have their torsion at the p_i , and that it is either 0 or \mathbb{C}_{p_i} . From now on we restrict ourselves to sheaves whose torsion is of this type.

Now we consider pairs $((E, \alpha) \in X, \phi)$. As noted above, the sheaf E corresponds to an element of Z_1 , and the forms α_i gives us non-zero elements $\tilde{\alpha}_i$ of V_2^* . One can then map our pairs into Y by

$$(E, \alpha, \phi) \mapsto \left((\phi(p_1)^N, \phi(p_1)^{N-1} \tilde{\alpha}_1^2), (\phi(p_2)^N, \phi(p_2)^{N-1} \tilde{\alpha}_2^2), \dots, (\phi(p_n)^N, \phi(p_n)^{N-1} \tilde{\alpha}_n^2) \right).$$

Now we take the closure \hat{Y} of the image of this map. This closure allows the α_i to go to zero (while $\phi(p_i) \neq 0$), while preserving the information of a subspace of E at p_i . Note that E is then torsion free at p_i , since $\phi(p_i) \neq 0$.

Proposition 4.14 *Let*

$$y = ((a_{0,1}, a_{1,1}), (a_{0,2}, a_{1,2}), \dots, (a_{0,n}, a_{1,n}))$$

be an element of \hat{Y} . Let $\Gamma(y)$ be the set of γ_i such that $\gamma_i \in [0, 1]$, $\gamma_i = 0$ if $a_{1,i} = 0$, and $\gamma_i = 1$ if $a_{0,i} = 0$. Then y is semi-stable iff for one element γ of $\Gamma(y)$, $\pi(y) \in X$ is γ -semi-stable.

Proof If $\pi(y)$ is (semi)-stable for one choice of weights in $\Gamma(y)$, then y is (semi)-stable. This follows from the fact that the $SL(N, \mathbb{C})$ quotient of X is just the $SL(N, \mathbb{C}) \times (\mathbb{C}^*)^n$ quotient of Y , with the γ_i defining the weights of the $(\mathbb{C}^*)^n$ action (Thaddeus [Th]).

On the other hand, let $\pi(y)$ be unstable for all γ_i in $\Gamma(y)$. We first note that for $a_{1,i} = 0$ (hence $\gamma_i = 0$), one can simply remove the point p_i , as it does not affect the definition of stability either in X or in \hat{Y} . Similarly, if $a_{0,i} = 0$, (hence $\gamma_i = 1$) one has torsion. Stability for E with $\gamma_i = 1$ is then equivalent to stability for $E/\text{torsion}$ with $\gamma_i = 0$, and by the preceding argument, we can drop the point, at the price of considering $E/\text{torsion}$ instead.

We are then in the situation of having a bundle E of degree $k < 0$ which can be destabilised by any choice of $\gamma_i \in [0, 1]$. Let $\gamma_i = 0$ to start. There is then a subline bundle of maximal degree j , with $2j > k$. If $L \subset \ker(\alpha_i)$, increasing γ_i only makes L more unstable, so we drop those points, and assume that L does not lie in any of the kernels. Now L will destabilise for a choice of γ_i if $2j - \sum_i \gamma_i > k$; assume that one can increase the γ_i so that $2j - \sum_i \gamma_i = k$, so that L no longer destabilises. By hypothesis there is another L' which now destabilises. L' cannot be a subsheaf of L , since then L would still destabilise. This tells us that the degree j' of L' is less than the degree of E/L , that is, $k - j$. This gives

$$(4.15) \quad k < 2j' + \sum_i (\pm \gamma_i) \leq 2k - 2j + \sum_i \gamma_i = 2k,$$

a contradiction since $k < 0$.

The same line bundle L then destabilises uniformly in the γ_i , and so one can construct a one parameter subgroup destabilising the corresponding element of \hat{Y} .

Semi-stable elements of \hat{Y} thus all live above semi-stable elements of X for some choice of weights, and the quotient $\hat{Y} // \text{SL}(N, \mathbb{C})$ corresponds to (S-equivalence classes of) quadruples $(E, \alpha_i, \hat{\alpha}_i, \phi)$, where E is a rank 2 sheaf, α is the map $E \rightarrow \bigoplus_i \mathbb{C}_{p_i}$, $\hat{\alpha}_i$ is a subspace of $E|_{p_i}$ which is the kernel of α when α is non-zero, one-dimensional when E has no torsion at p_i ; ϕ is an $\text{SL}(2, \mathbb{C})$ -structure. The pair (E, α) will be semi-stable for some choice of weights satisfying the constraint that $\gamma_i = 0$ when $\alpha_i = 0$, and $\gamma_i = 1$ when E has torsion at p_i .

This is not quite the moduli space that we want, as the $\hat{\alpha}_i$ constitute extra information, but only when $\alpha_i = 0$. At these points, the $\hat{\alpha}_i$ lie in $\mathbb{P}_1 = \mathbb{P}(E|_{p_i})$. Embedding V_1 into $W_{1,i} = V_1^{\otimes p}$, a non-zero element of $L_{0,i}$ can be thought of as an element of $W_{1,i}^*$; similarly, a non-zero element of $L_{1,i}$ corresponds to an element of $W_{2,i} = V_1^{\otimes p-1} \otimes V_2^{\otimes 2}$; the subvariety \hat{Y} maps to a subvariety \tilde{Y} in $\mathbb{P}(W_{1,1} \oplus W_{2,1}) \times \mathbb{P}(W_{1,2} \oplus W_{2,2}) \times \dots \times \mathbb{P}(W_{1,n} \oplus W_{2,n})$. This map collapses the unwanted \mathbb{P}_1 's, and otherwise is an embedding. The stability analysis goes through unchanged, since for $\gamma_i = 0$, the group action does not “see” the extra \mathbb{P}_1 , which lives in the Z_2 factor.

Let $\mathcal{P} = \tilde{Y} // \text{SL}(N, \mathbb{C})$ be the geometric quotient; we will call it the moduli space of framed parabolic bundles.

4.1 The Map from P to \mathcal{P}

Here again, we only give the map for the case $\text{SU}(2)$.

Let Σ be a closed Riemann surface, and let $\Sigma_0 = \Sigma - \{p_1, \dots, p_n\}$. We define a map from the space $P(\Sigma_0)$ to the space of framed parabolic sheaves over Σ with parabolic structure at the p_i .

We have fixed a rank two C^∞ bundle E over Σ with $c_1(E) = 0$, which can be taken to be the trivial bundle. Let ρ be an element of P . First of all, it defines a representation of the fundamental group of Σ_0 into $\text{SU}(2)$, and so a local system over Σ_0 , which then defines a holomorphic structure over Σ_0 . The local system has non trivial holonomy at the puncture, but any two bases of flat sections define the same holomorphic structure as they are related by constant matrices.

There remains to fill in the holomorphic structure at the punctures. Recall that we had points \tilde{p}_i near the p_i , and choose parametrisations of neighbourhoods D_i of these points in Σ so that p_i corresponds to $z = 0$ and \tilde{p}_i to $z = 1$. We now extend the holomorphic structure to the D_i as follows. One has that the monodromy of the local system at the i -th puncture is given by

$$(4.16) \quad \begin{pmatrix} e^{\pi\gamma_i} & 0 \\ 0 & e^{-\pi\gamma_i} \end{pmatrix}, \quad \gamma_i \in [0, 1].$$

Let us first consider the case of $\gamma_i \in (0, 1)$. One then has associated to each p_i a volume form on the eigenspaces, or, what is equivalent, a trivialisation t in which the monodromy of the puncture is diagonal. We consider a flat trivialisation on the complement of $z = 0$ which is equal to t at \tilde{p}_i . This trivialisation is of course multi-valued as we move around

the puncture. We glue the bundle over Σ_0 in this trivialisation to the trivial bundle $\mathcal{O} \oplus \mathcal{O}$ over D_i over the intersection $\Sigma_0 \cap D_i = D_i^*$ (or, more properly, over the universal cover of the intersection) by the transition matrix:

$$(4.17) \quad \begin{pmatrix} z^{\gamma_i/2} & 0 \\ 0 & z^{-\gamma_i/2} \end{pmatrix}.$$

The multi-valuedness of this transition matrix then compensates for the multi-valuedness of the flat trivialisation, and one has a well-defined holomorphic bundle on Σ . We take as extra structure on the resulting bundle E at p_i the map $v_i: E_{p_i} \rightarrow \mathbb{C}$ given in these trivialisations by $v_i = (\sin(\pi\gamma_i/2), 0)$.

When γ_i moves to zero, the extra data in P which gave the trivialisation at p_i disappears; on the other hand, the glueing matrix becomes the identity matrix, and v_i becomes zero, so the framing is no longer required, and the framed parabolic sheaf one obtains is simply the trivial one.

The case when γ_i moves to 1 is more interesting. What we glue in over D_i is not a bundle, but the sheaf $\mathcal{O} \oplus \mathcal{O} \oplus \mathbb{C}_0$ which has some torsion at the origin. Away from the origin this is just $\mathcal{O} \oplus \mathcal{O}$, and we have the map

$$(4.18) \quad \mathcal{O} \oplus \mathcal{O} \xrightarrow{\begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix}} \mathcal{O} \oplus \mathcal{O}.$$

If on the left hand side of (4.18), we have over the disk a volume form 1 and a map at the origin from the fiber to \mathbb{C} given by $(0, 1)$, we have correspondingly on the right hand side the volume form z , and a form $V = (0, 0)$. When one composes this with the glueing matrix (4.17) for $\gamma_i = 1$, one obtains:

$$(4.19) \quad \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix},$$

which is central for all values of z . Choosing a trivialisation at \tilde{p}_i , we use (4.19) to glue in the sheaf $\mathcal{O} \oplus \mathcal{O} \oplus \mathbb{C}_0$, with its degenerating $SL(2, \mathbb{C})$ structure to our bundle over the punctured surface, and a v at $z = 0$ given by $(0, 0, 1)$. Because (4.19) is central, the result is independent of the framing chosen, and so, even though the framing in P disappears at $\gamma_i = 1$, there is still a well defined map $\rho: P \rightarrow \mathcal{P}$.

The representation of stable bundles of degree -1 in terms of flat $SU(2)$ connections with holonomy -1 at an extra puncture point is, of course, not new; see e.g. [AB] or [Do].

Theorem 4.20 *The map $P \rightarrow \mathcal{P}$ is an isomorphism.*

Proof We have the natural map:

$$\Gamma: P \rightarrow [0, 1]^n$$

which associates to each representation the $(\gamma_1, \dots, \gamma_n)$ of holonomies at the punctures. For each value of the γ_i , we have the result, due to Mehta and Seshadri [MS] and Narasimhan and Seshadri [NS], that the moduli space of representations with holonomy

given by the γ_i is equivalent to the moduli space of parabolic bundles with polarisations again determined by the γ_i . This tells us, in essence, that “fiberwise”, our theorem is already proven. We note, following Remark 4.5 above that this holds even when some of the γ_i are 0 or 1; for $\gamma_i = 0$, it is as if the puncture were simply not there, and when some of the $\gamma_i = 1$, say k of them, on the holomorphic side one is considering a moduli space of sheaves of degree zero of the form (vector bundle of degree $-k$ plus torsion piece of the form $\bigoplus_j C_{p_j}$), where the p_j 's are the points for which $\gamma_j = 1$. One simply ignores the torsion piece, which doesn't contribute to the moduli, and has a moduli space of parabolic bundles (but with parabolic structure only at the points for which $\gamma_j \neq 0, 1$), and with the first Chern class lowered by k ; the analysis of this moduli space is the same as in degree zero.

We construct a map $\Delta: \mathcal{P} \rightarrow [0, 1]^n$ with $\Delta \circ \rho = \Gamma$. Recall that \mathcal{P} is obtained by taking a geometric $SL(N, \mathbb{C})$ quotient of the $(\mathbb{P}^1)^n$ -bundle over $X: Y = \mathbb{P}(L_{0,1} \oplus L_{1,1}) \oplus \mathbb{P}(L_{0,2} \oplus L_{1,2}) \oplus \dots \oplus \mathbb{P}(L_{0,n} \oplus L_{1,n})$, and then collapsing some \mathbb{P}^1 's. This collapsing is essentially irrelevant to the present proof, and so we omit it from now on. We now recall from [Th] that the moduli spaces for each set of δ_i is obtained as a $(\mathbb{C}^*)^n \times SL(N, \mathbb{C})$ -quotient of Y , where the δ_i are weights of the $(\mathbb{C}^*)^n$ -action. Now recall (see e.g. [MFK]) the equivalence due to Mumford, Guillemin-Sternberg, *etc.*, between symplectic quotients and geometric invariant theory quotients. We have, for the Kahler structure under the natural embedding of Y into projective space given by the $\mathcal{O}(1, 1, \dots, 1)$ polarisation, that the action of $(S^1)^n \subset (\mathbb{C}^*)^n$ has a moment map

$$(4.21) \quad \hat{\Delta}: Y \rightarrow [0, 1]^n.$$

This moment map restricts on each fiber $(\mathbb{P}^1)^n$ to the standard moment map. Since the action of $(S^1)^n$ commutes with that of $SL(N, \mathbb{C})$, this map is $SL(N, \mathbb{C})$ -invariant, and so $\hat{\Delta}$ descends to P , and gives a map Δ . To see that it has the right properties, note that

$$(4.22) \quad \begin{aligned} Y // (\mathbb{C}^*)^n \times SL(N, \mathbb{C}) &\simeq [\hat{\Delta}^{-1}(\delta_1, \dots, \delta_n) / (S^1)^n] // SL(N, \mathbb{C}) \\ &\simeq \Delta^{-1}(\delta_1, \dots, \delta_n) / (S^1)^n \end{aligned}$$

for each parabolic moduli space. (We note that this result again extends to the cases when δ_i takes values 0 or 1. For such values, on the holomorphic side, the weights of the action are such that there is a point with value 0 ($\delta_i = 0$) or ∞ ($\delta_i = 1$) in the relevant \mathbb{P}^1 -factor of the fiber of Y over X which lies in the closure of any semi-stable orbit of the $(\mathbb{C}^*)^n$ -action. On these points the action of the corresponding \mathbb{C}^* -factor is trivial. Similarly, on the symplectic side, the S^1 -action is trivial. One is then reduced to Thaddeus's result for $n - k$ instead of n .)

From this, one obtains that $\Gamma = \Delta \circ \rho$. As the map from P to \mathcal{P} commutes with the $(S^1)^n$ -action, and we have isomorphisms $\Gamma^{-1}(\gamma) / (S^1)^n \simeq \Delta^{-1}(\gamma) / (S^1)^n$, we are done.

4.2 The Case of a Trinion

We now consider the special case of \mathbb{P}^1 with three marked points. We know from the results above that the moduli space is $\mathbb{P}^3(\mathbb{C})$, but it is instructive to obtain this as a complex quotient, though the quotient here is not the standard one. We first examine which sheaves E on \mathbb{P}_1 arise from semi-stable framed parabolic bundles:

When E is a bundle, the semi-stability condition tells that either $E \simeq \mathcal{O} \oplus \mathcal{O}$ or $\mathcal{O}(1) \oplus \mathcal{O}(-1)$, for if $E \simeq \mathcal{O}(j) \oplus \mathcal{O}(-j)$, $j \geq 2$, one finds that $\mathcal{O}(j)$ is a destabilising subbundle, no matter what the map α is. When the bundle is trivial, one finds that the framed bundle is stable when none of the α_i are non-zero and any two of the subspaces $\ker(\alpha_i)$, $i = 1, 2, 3$ span \mathbb{C}^2 in the global trivialisation.

When E has torsion at one point p , the bundle $E^{**} = E/\text{torsion}$ must be of the form $\mathcal{O}(-1) \oplus \mathcal{O}$. The other possibilities would be $\mathcal{O}(-j) \oplus \mathcal{O}(j-1)$, $j \geq 2$, and then the sheaf $\mathcal{O}(j-1) \oplus \mathbb{C}_p$ would be destabilising.

Similarly, when E has torsion at two points, one finds that $E \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathbb{C}_p \oplus \mathbb{C}_q$. For E to be semistable, E cannot have three torsion points.

This discussion shows that the sheaves corresponding to semi-stable points are all such that $E(1)$ is generated by four global sections, from which it follows:

Proposition 4.23 E fits into an exact sequence:

$$(4.24) \quad \mathcal{O}(-2)^{\oplus 2} \xrightarrow{A+Bz} \mathcal{O}(-1)^{\oplus 4} \longrightarrow E \longrightarrow 0,$$

and the map $\mathcal{O}(-2)^{\oplus 2} \rightarrow \mathcal{O}(-1)^{\oplus 4}$ is injective (as a map of bundles) away from the torsion points of E .

The bundle is then determined by two 4×2 matrices. The v_i , $i = 1, 2, 3$ can then be realised as maps $\mathcal{O}^{\oplus 4}(-1)|_{p_i} \rightarrow \mathbb{C}$ which annihilate the image of $\mathcal{O}(-2)^{\oplus 2}$; letting the points p_i correspond to $z = 0, 1, \infty$, this gives us row vectors V_i in \mathbb{C}^4 satisfying the conditions

$$(4.25) \quad V_1A = 0, \quad V_2(A+B) = 0, \quad V_3B = 0.$$

The isomorphism $\Lambda^2(E) = (\Lambda^2\mathcal{O}(-2)^{\oplus 2})^* \otimes \Lambda^4(\mathcal{O}(-1)^{\oplus 4})$ defines a volume form on E , and so we have a framed parabolic $\text{SL}(2, \mathbb{C})$ -bundle.

The group $S(\text{GL}(2, \mathbb{C}) \times \text{GL}(4, \mathbb{C}))$ acts on this data, preserving the isomorphism class, by:

$$(4.26) \quad (g, G)(A, B, V_i) = (GA g^{-1}, GB g^{-1}, V_i G^{-1}),$$

and so we are really interested in the orbit space of this action. There is a four-tuple of functions which are projectively invariant under the action:

$$(4.27) \quad \det(A, B), \quad \det \begin{pmatrix} V_1B \\ V_2A \end{pmatrix}, \quad \det \begin{pmatrix} V_1B \\ V_3A \end{pmatrix}, \quad \det \begin{pmatrix} V_2A \\ V_3B \end{pmatrix}.$$

We also note that if D is a multiple of the 2×2 identity matrix, then the action of $(D^2, \text{diag}(D, D))$ simply rescales the data (A, B, V_i) . We can then simply consider the orbits in the projectivisation of the space of (A, B, V_i) under the action of the smaller group $\text{SL}(2, \mathbb{C}) \times \text{SL}(4, \mathbb{C})$.

Proposition 4.28 *The moduli space of semi-stable framed parabolic sheaves is the geometric quotient of the subset V of $\mathbb{P}(M(2, 4)^2 \times (\mathbb{C}^4)^3)$ cut out by conditions (4.25), by the group $\text{SL}(2, \mathbb{C}) \times \text{SL}(4, \mathbb{C})$.*

Lemma 4.29 *A point (A, B, V_i) is semi-stable for this action if and only if one of the invariants of (2) is non-zero.*

Proof For an element whose invariants do not vanish, the orbit is necessarily semi-stable. Conversely, for an orbit whose invariants are zero it suffices to find a destabilising \mathbb{C}^* in $SL(2, \mathbb{C}) \times SL(4, \mathbb{C})$, that is an whose action on (A, B, V_i) gives an orbit with zero in its closure. The fact that the invariants all vanish tells us that we can find a basis in \mathbb{C}^2 such that $V_i A, V_i B$ are all of the form $(*, 0)$. We subdivide into cases, according to the size of the span in \mathbb{C}^4 of the V_i . For the case when the span is three-dimensional, one sets $V_i = (*, *, *, 0)$, and one then has that both A and B are of the form

$$(4.30) \quad \begin{pmatrix} * & 0 \\ * & 0 \\ * & 0 \\ * & * \end{pmatrix}.$$

Now choosing integers n, m such that $0 < n < m < 3n$, we find that the subgroup of elements $(\text{diag}(z^m, z^{-m}), \text{diag}(z^{-n}, z^{-n}, z^{-n}, z^{3n}))$, $z \in \mathbb{C}$ destabilises. The other cases are similar, if slightly more elaborate.

We then have a well-defined map Φ from the moduli of semi-stable framed parabolics to $\mathbb{P}^3(\mathbb{C})$.

Theorem 4.31 *The map Φ is an isomorphism.*

Proof We must check that the closed orbits map bijectively into $\mathbb{P}^3(\mathbb{C})$. Again, there is a case by case analysis.

Case (i) $\text{Det}(A, B) \neq 0$. One can choose bases so that

$$(4.32) \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The stabiliser of this form is $SL(2, \mathbb{C})$. Conditions (4.25) tell us that

$$(4.33) \quad V_1 = (0, 0, a, b), \quad V_2 = (c, d, -c, -d), \quad V_3 = (e, f, 0, 0).$$

One is left with the problem of quotienting out the action of $SL(2, \mathbb{C})$ on the three-tuple $(a, b), (c, d), (e, f)$ of vectors in \mathbb{C}^2 . The stable points with finite stabiliser are those for which the three vectors span \mathbb{C}^2 . The three functions

$$(4.34) \quad \det \begin{pmatrix} V_1 B \\ V_2 A \end{pmatrix}, \quad \det \begin{pmatrix} V_1 B \\ V_3 A \end{pmatrix}, \quad \det \begin{pmatrix} V_2 A \\ V_3 B \end{pmatrix}$$

map this set bijectively to $\mathbb{C}^3 \setminus \{0\}$. All the others are just semi-stable, with the single closed orbit $((0, 0), (0, 0), (0, 0))$, which corresponds to the origin in \mathbb{C}^3 . All of these orbits have coordinate $(1, 0, 0, 0)$ in \mathbb{P}^3 .

Case (ii) $\text{Det}(A, B) = 0$, $\text{Im}(A) + \text{Im}(B)$ three dimensional, and $(Az_0 + Bz_1)$ injective for all $(z_0, z_1) \neq (0, 0)$: One can then normalise (A, B) to

$$(4.35) \quad A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

One then finds that

$$(4.36) \quad V_1 = (0, 0, a, b), \quad V_2 = (c, -c, c, d), \quad V_3 = (e, 0, 0, f).$$

The projective coordinate functions of (4.27) are $(0, -ac, -ae, ec)$. When these are non-zero, the stabiliser is finite and the points are stable. When one of a, c, e is zero, then the orbit is not closed, and its coordinates correspond to one of the points $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ or $(0, 0, 0, 1)$ in \mathbb{P}^3 .

Case (iii) $\text{Det}(A, B) = 0$, $\text{Im}(A) + \text{Im}(B)$ three dimensional, but $(Az_0 + Bz_1)$ fails to be injective for some $(z_0, z_1) \neq (0, 0)$: One checks that the stability condition forces z_0/z_1 to be either 0, 1 or ∞ , and that injectivity fails at only one point. We will suppose that the point is zero; the other cases are similar. We can then normalise A, B to:

$$(4.37) \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

One then has that

$$(4.38) \quad V_1 = (0, a, b, g), \quad V_2 = (c, 0, -c, d), \quad V_3 = (e, 0, 0, f)$$

and the projective coordinate functions of (1) are $(0, ac, ae, 0)$. From this, one has that a must be non-zero, and that either c or e is non-vanishing. When both of them are non-zero, the point is stable, and when they are not, the orbit is not closed.

Case (iv) $\text{Det}(A, B) = 0$, $\text{Im}(A) + \text{Im}(B)$ two dimensional: Here one finds that $(Az_0 + Bz_1)$ fails to be injective for two elements z_0/z_1 of \mathbb{P}^1 , and that these points must lie in $0, 1, \infty$. Again, we do one case, that of the points $0, \infty$, the others being similar. One can normalise to

$$(4.39) \quad A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

with

$$(4.40) \quad V_1 = (0, a, b, c), \quad V_2 = (0, 0, d, e), \quad V_3 = (f, 0, g, h)$$

and projective coordinates $(0, 0, -ag, 0)$. The unique closed orbit is that of

$$(4.41) \quad V_1 = (0, a, 0, 0), \quad V_2 = (0, 0, 0, 0), \quad V_3 = (f, 0, 0, 0).$$

This in fact exhausts all cases of semi-stable orbits. Recapitulating, we see that all the points of \mathbb{P}^3 apart from $(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$ correspond to stable points, with the four exceptional points each corresponding to a unique closed semi-stable orbit.

The four cases given above each correspond to different types of sheaf. In Case (i), the condition $\text{Det}(A, B) \neq 0$ corresponds to the sheaf being $\mathcal{O} \oplus \mathcal{O}$, and one has a \mathbb{C}^3 of framed parabolics with such a structure. In Case (ii), one has a bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$, and the space of framed parabolics with such a structure is the plane \mathbb{P}^2 at infinity, minus the three coordinate lines. In Case (iii), there is one point p amongst $0, 1, \infty$ at which there is torsion, and the sheaf is of the form $\mathcal{O} \oplus \mathcal{O}(-1) \oplus \mathcal{C}_p$. The parabolics corresponding to these correspond to the three coordinate lines in the plane at infinity, minus their intersections. Finally, the points in Case (iv) correspond to sheaves $\mathcal{O}(-1) \oplus \mathcal{O}(-1) \oplus \mathcal{C}_q \oplus \mathcal{C}_p$, with $\{p, q\} \in \{0, 1, \infty\}$: there are three choices and a corresponding three points in the plane at infinity, given by the intersection of the coordinate axes.

4.3 A Glueing Map and a Toric Variety

Given a pair of Riemann surfaces Σ_0, Σ_1 with marked points p_0, p_1 , one can define a singular nodal curve Σ by identifying the points p_0 and p_1 . There is also on the level of the moduli spaces $\mathcal{P}_0, \mathcal{P}_1$ of framed parabolic sheaves a (partial) glueing map which one can define as follows: one has for a framed parabolic sheaf E_0 over Σ_0 a map $\alpha_0: (E_0)_{p_0} \rightarrow \mathbb{C}$, and similarly for Σ_1 . One can combine the two into a diagram:

$$(4.42) \quad (E_0)_{p_0} \rightarrow \mathbb{C} \leftarrow (E_1)_{p_1}.$$

If we consider two such diagrams to be equivalent if the framed parabolic bundles on both Σ_0 and Σ_1 are isomorphic and if the induced maps $(E_0)_{p_0}/\ker(\alpha_0) \rightarrow (E_1)_{p_1}/\ker(\alpha_1)$ are the same, then taking equivalence classes of such diagrams amounts to quotienting out the anti-diagonal action of \mathbb{C}^* on the framings at p_0, p_1 , and we define the moduli space of framed parabolic sheaves over Σ to be the geometric quotient:

$$(4.43) \quad \mathcal{P}_\Sigma = \mathcal{P}_0 \times \mathcal{P}_1 // \mathbb{C}^*.$$

This is the complex analogue of the symplectic glueing defined above. From the symplectic point of view, the space \mathcal{P} which was very close to the moduli space \mathcal{M}_Σ^T of the glueing of Σ_0 and Σ_1 into a smooth surface $\tilde{\Sigma}$. In this picture, the geometry of what the glued moduli corresponds to is more evident: one is dealing with a moduli space of a degenerate curve, and one should be able to obtain \mathcal{P}_Σ as a degeneration of \mathcal{M}_Σ^T .

We note that there still remains the diagonal action of \mathbb{C}^* at the puncture. Now let us consider a decomposition of a smooth surface $\tilde{\Sigma}$ into trinions, and let us pinch the boundaries of the trinions to points, so that one has a singular surface Σ , consisting of $2g - 2$ spheres touching at $3g - 3$ points. The moduli space \mathcal{P}_Σ is $3g - 3$ dimensional and

is obtained from the glueing of $2g - 2$ trinion moduli spaces “at” $3g - 3$ points. It has an action of $(C^*)^{3g-3}$, and is a toric variety.

For \mathbb{P}^3 , the moduli space of the trinion, the action of $(a, b, c) \in (C^*)^3$ is $(x, y, z, w) \mapsto (x, aby, acz, bcw)$ in the coordinates given in the previous section. Its moment map is given by the holonomies, and the moment polytope is the tetrahedron of Section 3. The moment polytope PT_Σ of \mathcal{P}_Σ is given by taking the intersection with some hyperplanes of the product of $2g - 2$ tetrahedra.

The main purpose of [JW] is to provide a justification for the Verlinde formulae in terms of real quantisation. To do this, as we saw, they exhibited a toric structure associated to a trinion decomposition of the surface, which was however ill defined when the holonomy of the flat connections along the circles associated to the decomposition was central. There is still, however, a well defined moment map to a polytope. This polytope, as one can check in [JW], is exactly PT_Σ .

Referring to the results of Section 3, one therefore has a commuting diagram

$$(4.44) \quad \begin{array}{ccc} \mathcal{M}_\Sigma^T & & \mathcal{P}_\Sigma \\ & \searrow & \downarrow \\ & & PT_\Sigma \end{array} .$$

On one hand, for M_Σ , the dimension of the spaces of sections of line bundles over M_Σ (which is what the Verlinde formula computes) is given by the heuristic of Bohr-Sommerfeld quantisation in terms of the number of (fractional) integer points in the polytope PT_Σ . On the other, for \mathcal{P}_Σ , the theory of toric varieties tells us that the space of sections of the corresponding line bundle over \mathcal{P}_Σ , is given in terms of the same count of points. Furthermore, the correspondence between $\tilde{\Sigma}$ and Σ is that of a degeneration of the surface; this leads us to believe that a symplectic cobordism can be built between M_Σ and \mathcal{P}_Σ , which preserves the dimension of the space of sections, giving (yet another) proof of the Verlinde formula.

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