DIRECT THEOREMS ON METHODS OF SUMMABILITY

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§ 1. Introduction

1.1. A regular Toeplitz method of summability is given by a transformation

$$\sigma_m = \sum_{n=0}^{\infty} a_{mn} s_n, \qquad m = 0, 1, 2, \ldots \tag{1.1(1)}$$

of the sequence s_n into the sequence σ_m. According to the definition of regularity, every such method sums a convergent sequence s_n to the value $\lim s_n$. The question naturally arises, whether there are more extensive classes of sequences summable by all regular methods 1.1(1) or at least by all such methods subject to some simple additional conditions. Questions of this kind have been treated by the author (Lorentz [2], [5]) and, from another point of view, by R. P. Agnew [2] [3] ; in this paper we wish to discuss the problem systematically.

We first define classes of sequences considered in the sequel. The *characteristic function* $\omega(n)$ of a (finite or infinite) sequence $n_1 < n_2 < \ldots$ of positive integers is defined for all $n \geqq 0$ as the number of n_ν satisfying the inequality $n_\nu \leqq n$. *Throughout the paper*, $\Omega(n)$ *denotes a non-decreasing positive function defined for* $n \geqq 0$ *and tending to* $+\infty$ *for* $n \to +\infty$. For any such function, the class $\mathfrak{C}_1(\Omega)$ consists of all real bounded sequences s_n for which the set of indices $n_1 < n_2 < \ldots$ with non-vanishing s_n has a characteristic function $\omega(n) \leqq \Omega(n)$. Again, the class $\mathfrak{C}_2(\Omega)$ is constituted of all real sequences s_n such that the sums $S_n = s_0 + \ldots + s_n$ have the property $S_n = O(\Omega(n))$.

$\Omega(n)$ is a *summability function of the first kind or of the second kind for the method* A, if all sequences of $\mathfrak{C}_1(\Omega)$ or $\mathfrak{C}_2(\Omega)$, respectively, are A-summable. Summability functions of the first kind (for brevity, we shall sometimes simply call them summability functions) have been introduced by the author (Lorentz [5]).

In § 2, we state some properties of summability functions, and give necessary and sufficient conditions for a function $\Omega(n)$ to be a summability function for a Toeplitz method A. § 3 deals with relations between summability functions and Tauberian conditions. § 4 introduces simple sufficient and simple necessary conditions used later. In § 5 we determine summability functions for several special methods; and Hausdorff methods are treated in § 6.

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§ 2. Summability Functions of the First and the Second Kind

2.1. First some remarks on characteristic functions. If the sequences n_ν, n'_ν have characteristic functions $\omega(n)$, $\omega'(n)$ respectively, then their sum has a characteristic function $\leqq \omega(n) + \omega'(n)$. Further, $n_\nu \leqq n'_\nu$ for all ν is equivalent to $\omega(n) \geqq \omega'(n)$, $n = 0, 1, \ldots$.

For every sequence n_ν with the characteristic function $\omega(n)$ and every positive integer k, a decomposition of n_ν into a sum of $k+1$ subsequences exists such that one sequence is finite and the others have characteristic functions $\leqq \omega(n)/k$. Indeed, the subsequences $n_1, n_2, \ldots, n_{k-1}$ and $\{n_{sk+i}\}$ $(s = 1, 2, \ldots,$ $i = 0, 1, \ldots, k-1)$ provide the required decomposition.

2.2. A summability function of the second kind for a method A is also a summability function of the first kind for A. If $\Omega(n)$ is a summability function and $\Omega'(n) \leqq C\Omega(n)$ for all $n \geqq 0$, C being a constant, then $\Omega'(n)$ is also a summability function of the same kind. The proof depends upon the decomposition of the sequence n_ν mentioned in **2.1.**

If the method A is contained in the method B, $A \subset B$, all summability functions of A are also summability functions of B.

For a summability function of the first kind $\Omega(n)$ we always may assume that

2.2(1) $$\Omega(n+1) - \Omega(n) \leqq 1, \qquad n \geqq 0.$$

For if $\Omega'(n)$ is defined by $\Omega'(x) = \Omega(0)$, $0 \leqq x < 1$ and, inductively, by $\Omega'(x) = \min\,[\Omega'(n-1) + 1, \Omega(n)]$, $n \leqq x < n+1$, $n = 1, 2, \ldots$, then Ω' has the property 2.2(1) and is or is not a summability function together with Ω.

2.3. We proceed to formulate necessary and sufficient conditions in order that a given function $\Omega(n)$ be a summability function for a Toeplitz method 1.1(1).

Theorem 1. *$\Omega(n)$ is a summability function of the first kind for the method* 1.1(1), *if and only if*

2.3(1) $$\lim_{m\to\infty} \sum_{\nu=1}^{\infty} |a_{mn_\nu}| = 0$$

for every sequence n_ν with a characteristic function $\omega(n) \leqq \Omega(n)$, or if and only if

2.3(2) $$\lim_{m\to\infty} A(m;\Omega) = 0$$

where, for $m = 0, 1, \ldots$, $A(m;\Omega)$ is the least upper bound of $\sum_{\nu=1}^{\infty} |a_{mn_\nu}|$ for all sequences n_ν with $\omega(n) \leqq \Omega(n)$.

These conditions being fulfilled, every sequence $s_n \epsilon \mathfrak{C}_1(\Omega)$ is A-summable to 0.

For the proof, see Lorentz [5, theorems 9 and 10].

Theorem 2. *A function $\Omega(n)$ is a summability function of the second kind for a method* 1.1(1), *if and only if*

2.3(3) $$\lim_{m\to\infty} \sum_{n=0}^{\infty} \Omega(n)\, |a_{mn} - a_{m,n+1}| = 0.$$

If this condition is satisfied, every sequence $s_n \epsilon \mathfrak{C}_2(\Omega)$ is A-summable to 0.

Proof. First suppose that $\Omega(n)$ is a summability function of the second kind. Let ξ_n be any bounded sequence; put $S_n = \Omega(n)\xi_n$, $n = 0, 1, \ldots$; $S_{-1} = 0$ and $s_n = S_n - S_{n-1}$. Then

$$\sigma_m = \sum_{n=0}^{\infty} a_{mn} s_n = \sum_{n=0}^{\infty} a_{mn} [\Omega(n)\xi_n - \Omega(n-1)\xi_{n-1}]$$

is convergent and has a limit for $m \to \infty$. We choose $\xi_{2n} = \zeta_n$, $\xi_{2n+1} = 0$; σ_m becomes

$$\sigma_m = \sum_{n=0}^{\infty} \Omega(2n)\,(a_{m,\,2n} - a_{m,\,2n+1})\,\zeta_n = \Sigma b_{mn}\zeta_n.$$

This is a summability method applicable to all bounded sequences ζ_n. Since $b_{mn} \to 0$ for $m \to \infty$, $n = 0, 1, \ldots$, by a theorem of I. Schur [1] we have

$$\lim_{m\to\infty} \sum_{n=0}^{\infty} \Omega(2n)\,\left|a_{m,\,2n} - a_{m,\,2n+1}\right| = 0.$$

Odd indices $2n+1$ can be treated similarly; we thus obtain 2.3(3).

Conversely, if 2.3(3) holds, and if s_n is a sequence such that $S_n = O(\Omega(n))$, we have $a_{m,\,n+1}\Omega(n) \to 0$ for $n \to \infty$ and every fixed $m = 0, 1, \ldots$ and therefore $a_{m,\,n+1}S_n \to 0$. Letting n become infinite in the finite Abel transformation we have

$$\sigma_m = \sum_{n=0}^{\infty} a_{mn}s_n = \sum_{n=0}^{\infty} (a_{mn} - a_{m,\,n+1})\,S_n.$$

2.3(3) and $\lim_n a_{mn} = 0$ imply $\sigma_m \to 0$. This completes the proof.

2.4. A method A is called *strongly regular*, if every almost convergent sequence s_m is A-summable; the necessary and sufficient condition is (Lorentz [5]):

2.4(1) $$\lim_{m\to\infty} \sum_{n=0}^{\infty} \left| a_{mn} - a_{m,\,n+1} \right| = 0.$$

From this, one easily obtains:

THEOREM 3. *A regular Toeplitz method A is strongly regular if and only if A possesses a summability function of the second kind.*

The assertion follows at once from the criterion 2.4(1) and the following lemma:

LEMMA 1. *If $a_{mn} \geqq 0$ and if $\eta_m = \sum_{n=0}^{\infty} a_{mn} \to 0$, a function $\Omega(n)$ exists such that $\Sigma_n \Omega(n) a_{mn} \to 0$.*

Proof. For every $m = 0, 1, 2, \ldots$ there is clearly a function $\Omega_m(n)$ of the required kind for which $\Sigma_n \Omega_m(n) a_{mn} \leqq 2\eta_m$. Moreover, we can choose a non-decreasing sequence of positive integers $k_m \to +\infty$, for which $k_m\eta_m \to 0$. Let

$$\Omega(n) = \min_{m=0,1,\ldots} \left\{k_m + \Omega_m(n)\right\}, \qquad n = 0, 1, \ldots.$$

This function is non-decreasing, positive, and it is easy to see that $\Omega(n) \uparrow +\infty$. Finally, $\Sigma_n \Omega(n) a_{mn} \leqq k_m\eta_m + 2\eta_m \to 0$, which proves the lemma.

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The analogue to Theorem 3 for summability functions of the first kind is this (Lorentz [5]): A Toeplitz method A possesses summability functions of the first kind, if and only if, for $m \to \infty$,

2.4(2) $$\max_{n=0,1,\ldots} |a_{mn}| \to 0.$$

2.5. Another consequence of the lemma is

THEOREM 4. *For every summability function of the second kind $\Omega(n)$ for a method A there is another such function $\Omega_1(n)$ with $\Omega_1(n)/\Omega(n) \uparrow +\infty$.*

That generally not all summability functions of the second kind are of the form $\Omega(n) = o(\phi(n))$ with an appropriate $\phi(n)$ is shown by the example of Riemann's method R_2 in **5.4.**

A similar theorem is:

THEOREM 5. *For every summability function of the first kind $\Omega(n)$ for a method A there is another such function $\Omega_1(n)$ for which $\Omega_1(n)/\Omega(n) \uparrow +\infty$.*

Proof. Using the notation of Theorem 1 we have $A(m; \Omega) \to 0$. Moreover, by the theorem referred to above,

$$\delta_m = \max_n |a_{mn}| \to 0.$$

We choose a non-decreasing sequence of positive integers $k_m \uparrow +\infty$ such that

$$k_m A(m; \Omega) \to 0, \quad k_m \delta_m \to 0.$$

Then

2.5(1) $$A(m; k_m \Omega) \to 0.$$

For, according to **2.1**, every sequence of integers n_ν with the characteristic function $\leqq k_m \Omega(n)$ is a sum of a finite sequence consisting of k_m elements and of k_m infinite sequences whose characteristic functions are $\leqq \Omega(n)$. Therefore

$$\sum_\nu |a_{mn_\nu}| \leqq k_m \delta_m + k_m A(m; \Omega)$$

and 2.5(1) follows from the definition of $A(m; k_m \Omega)$.

We may choose integers $N_m \uparrow +\infty$ such that

$$\epsilon_m = \sum_{n > N_m} |a_{mn}| \to 0.$$

Now let

$$\Omega_1(n) = k_m \Omega(n), \qquad N_m \leqq n < N_{m+1}, \, n = 1, 2, \ldots,$$

then $\Omega_1(n)/\Omega(n) \uparrow +\infty$ and

$$A(m; \Omega_1) \leqq \epsilon_m + \sup_{\{n_\nu\}} \sum_{n_\nu \leqq N_m} |a_{mn_\nu}|,$$

where $\{n_\nu\}$ stands for all sequences whose characteristic functions are $\leqq \Omega_1(n)$. Since $\Omega_1(n) \leqq k_m \Omega(n)$ for $n \leqq N_m$,

$$A(m; \Omega_1) \leqq \epsilon_m + A(m; k_m \Omega).$$

This completes the proof.

§ 3. Relations between Summability Functions and Tauberian Conditions

3.1. A condition on a sequence s_n is called a *Tauberian condition* for a method A, if every sequence s_n, which is A-summable and satisfies this condition, is convergent. We now show, that the knowledge of summability functions of a method enables one to draw some conclusions about its Tauberian conditions.

Theorem 6. *If $\Omega(n)$ is a summability function (of the first kind) for a regular Toeplitz method A, then*

3.1(1) $$u_n = s_n - s_{n-1} = o(\Omega(n)^{-1})$$

is not a Tauberian condition for this method.

This theorem gives a precise limit from below for Tauberian conditions of the form $u_n = o(\phi(n))$ for all summability methods treated in §§ 5-6, except for the rather pathological method 5.3(3).

For the proof of Theorem 6 we need Lemma 2, interesting in itself. The following remarks are intended to elucidate its meaning. Both the conditions

3.1(2) $$u_n = O(n^{-\alpha}),\ \sigma_n = S_n/n = (s_0 + \ldots + s_n)/n = o(n^{\alpha-1})$$

constitute for every $0 \leqq \alpha \leqq 1$ a Tauberian condition for the method C_1 (Ananda Rau [1], Karamata [1], Boas, Jr. [1]). We show that the conditions 3.1(2) may not be relaxed: for every function $\Omega(n)$ for which $\Sigma\Omega(n)^{-1} = +\infty$ there is a divergent series Σu_n such that $u_n = O(\Omega(n)^{-1})$, $\sigma_n = O(\Omega(n)/n)$. Even more is true:

Lemma 2. *For every function $\Omega(n)$ for which $\Sigma\Omega(n)^{-1} = +\infty$ a bounded divergent sequence s_n exists such that $u_n = s_n - s_{n-1} = O(\Omega(n)^{-1})$ and that the characteristic function of the indices n which have the property $s_n \neq 0$ does not exceed $\Omega(n)$.*

Proof. We define two sequences of positive integers $m_1 < n_1 < \ldots < m_\nu < n_\nu < \ldots$ such that $\Omega(m_1)^{-1} < 1/9$, that the characteristic function of the sum of the intervals $m_\nu \leqq n \leqq n_\nu$, $\nu = 1, 2, \ldots$ is $\leqq \Omega(n)$ and that furthermore

3.1(3) $$\sum_{m_\nu \leqq n \leqq n_\nu} \Omega(n)^{-1} \geqq 1/3, \qquad \nu = 1, 2, \ldots.$$

We proceed by induction. Let $m_1, n_1, \ldots, m_{\nu-1}, n_{\nu-1}$ be already defined. Then m_ν is chosen such that

$$\sum_{\mu=1}^{\nu-1} (n_\mu - m_\mu + 1) \leqq \Omega(m_\nu)/2 \leqq \Omega(n)/2, \qquad n \geqq m_\nu.$$

Since $\Omega(m_\nu) \geqq 4$, the integer $N = m_\nu + 1$ has the property that the characteristic function of the interval $m_\nu \leqq n \leqq N$ does not exceed $\Omega(n)/2$ for $n \geqq m_\nu$. If all integers $N \geqq m_\nu + 1$ have this property, n_ν is chosen arbitrarily to satisfy 3.1(3). On the other hand, if N_0 is the first integer $> m_\nu + 1$ lacking the above property, we put $n_\nu = N_0 - 1$. Then $N_0 - m_\nu + 1 \geqq \Omega(N_0)/2$. Hence

$$\sum_{m_\nu \leqq n \leqq N_0} \Omega(n)^{-1} \geqq (N_0 - m_\nu + 1)\Omega(N_0)^{-1} \geqq \frac{1}{2}.$$

In this case 3.1(3) also holds.—The sum of the intervals $m_\nu \leqq n \leqq n_\nu$, $\nu = 1, 2, \ldots$ thus defined has a characteristic function $\leqq \Omega(n)$.

We now choose integers l_ν, $\nu = 1, 2, \ldots$ for which $m_\nu \leqq l_\nu < n_\nu$ and

$$\sum_{m_\nu \leqq n \leqq l_\nu} \Omega(n)^{-1} \geqq \frac{1}{9}, \quad \sum_{l_\nu < n \leqq n_\nu} \Omega(n)^{-1} \geqq \frac{1}{9}, \qquad \nu = 1, 2, \ldots$$

Then for $m_\nu \leqq n \leqq n_\nu$, $\nu = 1, 2, \ldots$ real u_n exist such that $|u_n| \leqq \Omega(n)^{-1}$, $u_n \geqq 0$ for $m_\nu \leqq n \leqq l_\nu$, $u_n \leqq 0$ for $l_\nu < n \leqq n_\nu$ and

$$\sum_{m_\nu \leqq n \leqq l_\nu} u_n = \frac{1}{9}, \quad \sum_{l_\nu < n \leqq n_\nu} u_n = -\frac{1}{9}.$$

Let $u_n = 0$ for all remaining n. It is easily seen that the sequence $s_n = \sum_0^n u_\nu$ has all the required properties.

Proof of Theorem 6. For a summability function $\Omega(n)$ of a regular Toeplitz method we have $\Sigma\Omega(n)^{-1} = +\infty$. For otherwise $n\Omega(n)^{-1} \to 0$, and so $n = o(\Omega(n))$ would be a summability function. Thus all bounded sequences are A-summable to 0, which contradicts the theorem of I. Schur referred to in **2.3**. Thus by Lemma 2, we obtain the assertion of Theorem 6 with the condition $u_n = O(\Omega(n)^{-1})$ instead of 3.1(1). The general statement now follows from Theorem 5.

§ 4. Summability Functions of the Form $\Omega(n) = o(n)$

4.1. Before treating special methods we insert a few simple sufficient and simple necessary conditions for summability functions. In many cases they are all contained in the formula $\Omega(n) = o(n)$.

Theorem 7. *Suppose that there is a monotone majorant* (α_{mn}) *of the matrix* (a_{mn}) *having the properties*: $\alpha_{mn} \geqq 0$, α_{mn} *is non-increasing for every fixed* $m = 0, 1, \ldots$, $\alpha_{m0} \to 0$, $|a_{mn}| \leqq \alpha_{mn}$ *and finally*

4.1(1) $$\sum_{n=0}^{\infty} \alpha_{mn} \leqq M, \qquad m = 0, 1, \ldots$$

with a constant M. *Then all functions* $\Omega(n) = o(n)$ *are summability functions for the method* 1.1(1).

Proof. It will be sufficient to prove

4.1(2) $$\lim_{m\to\infty} \sum_{\nu=1}^{\infty} \alpha_{mn_\nu} = 0$$

for every sequence n_ν with a characteristic function $\omega(n) = o(n)$. Let p be an arbitrary positive integer; we may assume (by omitting, if necessary, a finite number of n_ν), that $\omega(n) \leqq n/p$ holds for all $n = 0, 1, \ldots$. Since $n_\nu \geqq p\omega(n_\nu)$

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$= \nu p$, it is possible, by induction in k, to determine integers $n_\nu{}^k$, $\nu = 1, 2, \ldots$, $k = 1, 2, \ldots, p$, all of which are different, such that $n_\nu{}^k \leqq n_\nu$. Then

$$p \sum_{\mu=1}^{\nu} a_{m,n_\mu} \leqq \sum_{\mu \leqq \nu} a_{m,n_\mu^k} \leqq M.$$

Hence $\Sigma_\nu a_{mn_\nu} \leqq M/p$. Since p was arbitrary, our result is established.

Summability functions other than those of the form $o(n)$ may occur in abnormal cases only, as is shown by the following theorem:

THEOREM 8. *If a Toeplitz method A possesses a summability function $\Omega(n) \neq o(n)$, then A is a gap method, that is to every $\epsilon > 0$ and every $C > 0$ there is a corresponding n' as large as we please such that*

4.1(3) $$\sum_{n' \leqq n \leqq Cn'} |a_{mn}| < \epsilon, \qquad m = 0, 1, \ldots.$$

Proof. Suppose that $\overline{\lim}\, \Omega(n)/n > 2c > 0$. Let $\epsilon > 0$ and $C > 0$ be chosen arbitrarily; we may assume that C/c is an integer. According to 2.3(2), an m_0 exists such that

4.1(4) $$A(m; \Omega) < \epsilon c/C, \qquad m \geqq m_0.$$

For an n_0 sufficiently large we have

4.1(5) $$\sum_{n=n_0}^{\infty} |a_{mn}| < \epsilon, \qquad m \leqq m_0.$$

Let an integer n' be chosen for which $n' \geqq n_0$, $cn' \geqq 1$ and $\Omega(n')/n' \geqq 2c$. Each of the closed intervals

$$[n', n' + cn'],\ [n' + cn', n' + 2cn'], \ldots, \left[n' + \left(\frac{C}{c} - 1\right)cn', n' + Cn'\right]$$

contains no more than

$$cn' + 1 \leqq 2cn' \leqq \Omega(n')$$

integers; characteristic functions of each interval are therefore $\leqq \Omega(n)$ for $n = 0, 1, \ldots$. By 4.1(4),

$$\sum_{n' \leqq n \leqq Cn'} |a_{mn}| \leqq \frac{C}{c} A(m; \Omega) < \epsilon, \qquad m \geqq m_0.$$

The same inequality for $m \leqq m_0$ follows from 4.1(5).

The following remarks are useful for the determination of summability functions of the second kind. Let σ_n signify the Cesàro mean $\sigma_n = (s_0 + \ldots + s_n)/(n+1)$ of a sequence s_n. From the definition **1.1** it follows that all functions $\Omega(n) = o(\phi(n))$ are summability functions of the second kind of a method A if and only if $\sigma_n - s = o(\phi(n)n^{-1})$ implies the A-summability of the sequence s_n to 0. Thus, the assertion, that all functions $\Omega(n) = o(n)$ have this property for A is equivalent to $A \supset C_1$. Suppose, for instance, that $a_{mn} \geqq 0$ for all m, n and that for every fixed m the a_{mn} are first increasing up to their maximal value for $n = n_0 = n_0(m)$ and then decreasing to 0. Then $A \supset C_1$ holds if and only if

4.1(6) $$n_0 a_{mn_0} \leqq M$$

is true with an M independent of m. For 4.1(6) is clearly equivalent to

4.1(7) $$\sum_{n=0}^{\infty} n \left| a_{mn} - a_{m,\,n+1} \right| \leqq M'$$

which is necessary and sufficient for the inclusion $A \supset C_1$ for any regular Toeplitz method A (Orlicz [1]).

§ 5. Some Special Methods

5.1. The methods of Cesàro and Abel. All summability functions of the method C_1 are clearly given by the formula $\Omega(n) = o(n)$. The same is true for $C_\alpha(\alpha > 0)$ and the Abel method A, since all these methods are equivalent for bounded sequences. All functions $\Omega(n) = o(n)$ are also summability functions of the second kind for the methods $C_\alpha(\alpha \geqq 1)$ and A, since these methods contain C_1. There are no other functions, for A is not a gap method as is seen from

$$(1 - r) \sum_{\nu = n}^{2n-1} r^\nu = r^n(1 - r^n) \to e^{-1} - e^{-2}$$

for $r = 1 - n^{-1}$, $n \to \infty$.

Finally, by a simple computation it may be deduced from Theorem 2 that all summability functions of the second kind for $C_\alpha(0 < \alpha < 1)$ are furnished by the formula $\Omega(n) = o(n^\alpha)$.

5.2. The methods of Euler and Borel. The Euler method $E_t(0 < t < 1)$ is defined by the transformation

5.2(1) $$\sigma_n = \sum_{\nu=0}^{n} p_\nu(t) s_\nu,$$

$$p_\nu(t) = p_{\nu n}(t) = \binom{n}{\nu} t^\nu (1 - t)^{n-\nu},\ 0 \leqq \nu \leqq n,\ p_{n+1,\,n}(t) = 0,$$

and the Borel method by

5.2(2) $$\sigma(x) = e^{-x} \sum_{n=0}^{\infty} \frac{x^n}{n!} s_n, \qquad x \to +\infty.$$

For these methods a function $\Omega(n)$ is a summability function of the first or of the second kind if and only if $\Omega(n) = o(\sqrt{n})$. Since $E_t \subset B$, it will be sufficient to prove: (i) Every function $\Omega(n) = o(\sqrt{n})$ is a summability function of the second kind for $E_t(0 < t < 1)$; (ii) If $\Omega(n) \neq o(\sqrt{n})$, $\Omega(n)$ is not a summability function for B.

To prove (i) observe that the sum 2.3(3) takes for the transformation 5.2(1) the form

$$\begin{aligned}\Delta(n;\Omega) &= \sum_{\nu=0}^{n} \Omega(\nu) \left| p_\nu(t) - p_{\nu+1}(t) \right| \leqq \Omega(n) \sum_{\nu=0}^{n} \left| p_\nu(t) - p_{\nu+1}(t) \right| \\ &\leqq 2\Omega(n) \max_{0 \leqq \nu \leqq n} p_\nu(t) \leqq 2Cn^{-\frac{1}{2}}\, \Omega(n) \to 0,\end{aligned}$$

if $\Omega(n) = o(\sqrt{n})$. As a matter of fact the "Newtonian probability" $p_{\nu n}(t)$ for fixed n, t and ν varying in the interval $0 \leqq \nu \leqq n$ increases first and then decreases; and the maximal value of $p_\nu(t)$ does not exceed $Cn^{-\frac{1}{2}}$, where C depends on t only.

(ii) Let $\Omega(n) \neq o(\sqrt{n})$, then a $\delta > 0$ exists for which $\Omega(n) \geqq \delta\sqrt{n}$ holds for an infinity of n. For these n and the Borel transformation 5.2(2)

$$A(n;\Omega) \geqq e^{-n} \sum_{n<\nu\leqq n+\delta\sqrt{n}} n^\nu/\Gamma(\nu+1) \geqq C_1 e^{-n}\Sigma n^\nu \nu^{-(\nu+\frac{1}{2})}e^\nu$$

$$= C_1 \sum_{n<\nu\leqq n+\delta\sqrt{n}} \nu^{-\frac{1}{2}}\left(1-\frac{\nu-n}{\nu}\right)^\nu e^{\nu-n}.$$

(C_1, C_2, . . . are constants). Since $\nu \log\left(1-\frac{\nu-n}{\nu}\right) \geqq -(\nu-n)+C_2$ for $0 \leqq \nu - n \leqq \delta\sqrt{n}$,

$$A(n;\Omega) \geqq C_3 \sum_{n<\nu\leqq n+\delta\sqrt{n}} \nu^{-\frac{1}{2}} \geqq C_4 > 0.$$

Hence, by 2.3(2), $\Omega(n)$ is not a summability function for the method B.

5.3. Some other methods. For the Lambert method L

5.3(1)
$$\sigma(x) = \sum_{\nu=1}^{\infty} \frac{\nu(1-x)}{1-x^\nu} u_\nu = \sum_{\nu=1}^{\infty} a_\nu(x)s_\nu, \qquad x \to 1-,$$
$$a_\nu(x) = \frac{x^\nu(\nu - x - \cdots - x^\nu)}{(1+\cdots+x^{\nu-1})(1+\cdots+x^\nu)}$$

(both forms are equivalent, see for instance Lorentz [3, theorem 10]) and for the method

5.3(2)
$$\sigma_n = \sum_{\nu=0}^{n} \binom{n}{\nu} \frac{\nu!}{n^\nu} u_\nu = s_0 + \sum_{\nu=1}^{n} \left(1-\frac{1}{n}\right)\cdots\left(1-\frac{\nu-1}{n}\right)\frac{\nu}{n} s_\nu$$

(Rey-Pastor [1], S. Bernstein [1], Amerio [1]) all summability functions of the first and of the second kind and only these are furnished by $\Omega(n) = o(n)$. The positive part follows from 4.1(6) with $n_0 = 0$ for L and with $n_0 \cong \sqrt{n}$ for the method 5.3(2); and the negative part is a consequence of Theorem 8. We leave details to the reader.

The method

5.3(3)
$$\sigma_m = s_{2^m}$$

has no summability functions at all; but Tauberian theorems exist for this method. Thus, $u_n = o(n^{-1})$ is a Tauberian condition (o may not be replaced by O).

5.4. The Riemann methods R_k. R_k, $k = 2, 3, \ldots$ is related to the transformation

$$\sigma(x) = \sum_{n=0}^{\infty} \left(\frac{\sin nx}{nx}\right)^k u_n$$

5.4(1)
$$= \sum_{n=0}^{\infty} \left\{ \left(\frac{\sin nx}{nx}\right)^k - \left(\frac{\sin\,(n+1)x}{(n+1)x}\right)^k \right\} s_n, \qquad x \to 0+.$$

Both forms are equivalent for bounded s_n. We take the second as the definition.

The derivative of the function $\phi(u) = (\sin u/u)^k$ has the property $|\phi'(u)| \leqq C(u+1)^{-k}$, C depending only on k. Hence for $x > 0$

$$\left|\frac{d}{du}\,\phi(xu)\right| = |\,\phi'(xu)\,|\,x \leqq Cx(xu+1)^{-k} \leqq C_1 x s^{-k} \text{ for } u \geqq \frac{s\pi}{x},\ s = 1, 2, \ldots.$$

We put $a_n(x) = (\sin nx/nx)^k - (\sin\,(n+1)x/(n+1)x)^k$ and

$$\alpha_0(x) = C_1 x, \quad \alpha_n(x) = C_1 x s^{-k} \quad \text{for } \frac{s\pi}{x} \leqq n < \frac{(s+1)\,\pi}{x}.$$

The functions $\alpha_n(x)$ constitute a monotone majorant for $a_n(x)$ in the sense of Theorem 7. In particular,

$$\sum_{n=0}^{\infty} \alpha_n(x) \leqq \frac{C_2}{x}\, x \left(1 + \sum_{s=1}^{\infty} s^{-k}\right) = M < +\infty.$$

By Theorem 7 (or rather by its continuous analogue), all functions $\Omega(n) = o(n)$ are summability functions for the methods R_k, $k = 2, 3, \ldots$. There are no others by Theorem 8, since

$$\sum_{\frac{\pi}{x} \leqq n < \frac{2\pi}{x}} |\,a_n(x)\,|$$

converges for $x \to 0+$ to the positive number $\max |\phi(u)|$, $\pi \leqq u \leqq 2\pi$.

Summability functions of the second kind are also given by $\Omega(n) = o(n)$, if $k = 3, 4, \ldots$. As to R_2, they are defined by

5.4(2)
$$\sum_{n=1}^{\infty} \Omega(n) n^{-2} < +\infty.$$

For consider

$$\Delta(x;\Omega) = \sum_{n=0}^{\infty} \Omega(n)\,|\,a_n(x) - a_{n+1}(x)\,|$$

5.4(3)
$$= \sum_{n=0}^{\infty} \Omega(n)\,|\,\phi(nx) - 2\phi(nx+x) + \phi(nx+2x)\,|.$$

The derivative $\phi''(u)$ exists for $u \geqq 0$ and has the properties $|\phi''(u)| \leqq C_3 u^{-k}$, $|\phi''(u)| \leqq C_4$. Therefore $(0 < \theta < 1)$

$$|\Delta^2\phi(nx)| = x^2\,|\phi''(nx + 2\theta x)| \leqq C_4 x^2, \qquad 0 \leqq nx \leqq 1,$$
$$|\Delta^2\phi(nx)| \leqq C_3(nx)^{-k}x^2, \qquad nx \geqq 1.$$

Thus, 5.4(3) becomes

$$\Delta(x; \Omega) = \sum_{nx<1} + \sum_{nx\geqq 1}$$
$$\leqq C_4 x^2 \sum_{nx<1} \Omega(n) + C_3 x^{2-k} \sum_{nx\geqq 1} \Omega(n) n^{-k}.$$

Since $\Omega(n) = o(n)$ also holds in case of 5.4(2), the first term on the right is $= C_4 x^2 o(\Sigma_{n<x^{-1}} n) = C_4 x^2 o(x^{-2}) = o(1)$, while the second term $= C_3 x^{2-k}$ $o(\Sigma_{n\geqq x^{-1}} n^{1-k}) = C_3 x^{2-k} o(x^{k-2}) = o(1)$ for $k \geqq 3$ and $= C_3 o(1)$ for $k = 2$, if 5.4(2) is fulfilled. It remains to show that, if $\Sigma\Omega(n)n^{-2} = +\infty$, $\Omega(n)$ is not a summability function of the second kind for R_2.

For small $x > 0$, we wish to estimate $|\Delta^2\phi(nx)| = x^2|\phi''(u)|$, $u = nx + 2\theta x$, $0 < \theta < 1$ from below for $\left|n - \frac{s\pi}{x}\right| \leqq \pi/10x$. Since $\phi''(u) = 2u^{-2}\cos 2u + O(u^{-3})$, if $u \to \infty$, $|\phi''(u)| \geqq u^{-2}$ holds for $|u - s\pi| \leqq \pi/8$ and all sufficiently large integers s. Thus $|\Delta^2\phi(nx)| \geqq n^{-2}$ for $\left|n - \frac{s\pi}{x}\right| \leqq \pi/10x$. Since

$$\sum_{\left|n-\frac{s\pi}{x}\right| \leqq \frac{\pi}{10x}} \Omega(n)|\Delta^2\phi(nx)| \geqq \sum_{\left|n-\frac{s\pi}{x}\right| \leqq \frac{\pi}{10x}} \Omega(n)n^{-2}$$
$$\geqq C_5 \sum_{\frac{(s-1)\pi}{x}+\frac{\pi}{10x} \leqq n \leqq \frac{s\pi}{x}+\frac{\pi}{10x}} \Omega(n)n^{-2} = v_s$$

and since the series Σv_s is divergent, $\Delta(x;\Omega) = \sum_{n=0}^{\infty}\Omega(n)\,|\,\Delta^2\phi(nx)\,| = +\infty$, which proves our result.

§ 6. Hausdorff Methods

6.1. A Hausdorff method H_g is defined by the transformation

6.1(1) $$\sigma_n = \sum_{\nu=0}^{n} a_{n\nu}s_\nu = \int_0^1 \sum_{\nu=0}^{n} p_{\nu n}(x)s_\nu dg(x);$$

$g(x)$ is a function of bounded variation in [0, 1] and $p_{\nu n}(x) = p_\nu(x)$ is the "Newtonian probability" $\binom{n}{\nu}x^\nu(1-x)^{n-\nu}$. Without loss of generality we may assume, that $g(x)$ is normalized, that is, it has the properties $g(0) = 0$ and $g(x) = [g(x+) + g(x-)]/2$ for $0 < x < 1$.

H_g is regular if and only if $g(x)$ is continuous at $x = 0$ and if $g(1) = 1$. H_g is strongly regular if, and only if, H_g possesses summability functions of the *first* kind or, if and only if, $g(x)$ is continuous at $x = 1$ (Lorentz [5]). In this section we determine the summability functions for a method H_g in terms of the generating function $g(x)$. The following theorem yields summability functions for every method H_g with a discontinuous $g(x)$.

THEOREM 9. (i) *A regular Hausdorff method can not be a gap method.* (ii) *If H_g is strongly regular, all functions $\Omega(n) = o(\sqrt{n})$ are summability functions for H_g.* (iii) *If $g(x)$ has a jump at a point $0 < x_0 < 1$, then all summability functions of H_g are of the form $\Omega(n) = o(\sqrt{n})$.*

To prove (i), suppose that $g(x)$ is continuous at $x = 0$; let $0 < a < b < 1$ be two points of continuity of $g(x)$. We choose a q, $0 < q < 1$, arbitrarily. If H_g is a gap method, then, for every $\epsilon > 0$,

$$\sum_{N \leqq \nu \leqq bN/aq} |a_{n\nu}| < \epsilon, \qquad n = 0, 1, \ldots$$

is true for some N as large as we please. To every sufficiently large N there corresponds an integer $n = n(N)$ such that $nq < N/a < n$. Then $a < \nu/n < b$ implies $N \leqq \nu \leqq bN/aq$. Hence

$$\left| \int_0^1 \sum_{a \leqq \frac{\nu}{n} \leqq b} p_{\nu n}(x) dg(x) \right| \leqq \sum_{a \leqq \frac{\nu}{n} \leqq b} |a_{n\nu}| < \epsilon.$$

The sum under the integral converges to 1, if $a < x < b$ and to 0 outside of this interval. Letting n become infinite, we obtain

$$\left| \int_a^b dg(x) \right| \leqq \epsilon.$$

Since ϵ was arbitrary, $g(a) = g(b)$. Therefore, the normalized function $g(x)$ is $= 0$ in $0 \leqq x < 1$. As $g(1) = 1$, 6.1(1) becomes $\sigma_n = s_n$, which contradicts the assumption that H_g is a gap method.

(ii) Suppose that H_g is strongly regular, that $\Omega(n) = o(\sqrt{n})$ and that ν_k is a sequence with a characteristic function $\omega(n) \leqq \Omega(n)$. We put

6.1(2) $$f_n(x) = \sum_{\nu_k \leqq n} p_{\nu_k n}(x).$$

Then $f_n(x) \to 0$ uniformly in $\delta \leqq x \leqq 1 - \delta$ for any fixed $\delta > 0$. For in this interval the estimate $p_{\nu n}(x) \leqq Cn^{-\frac{1}{2}}$ holds with a constant C depending only on δ. Therefore $0 \leqq f_n(x) \leqq Cn^{-\frac{1}{2}} \omega(n) = o(1)$. We are now in a position to prove that 2.3(1) holds, that is that $\Omega(n)$ is a summability function. If $V(x) = \text{var}\, g(t)$ in $0 \leqq t \leqq x$,

$$\sum_{\nu_k \leqq n} |a_{n\nu_k}| \leqq \int_0^1 f_n(x) dV(x) = \int_0^\delta + \int_\delta^{1-\delta} + \int_{1-\delta}^1.$$

The second integral converges to zero as $n \to \infty$, and the remaining two are arbitrarily small with δ, since $0 \leqq f_n \leqq 1$ and $V(x)$ is continuous at $x = 0$ and $x = 1$.

It remains to prove (iii). Let α be the jump of $g(x)$ at $x = x_0$, and suppose $\Omega(n)n^{-\frac{1}{2}} \not\to 0$. We can choose a $\delta > 0$ such that for an infinity of n

6.1(3) $$\Omega(x_0 n/2)n^{-\frac{1}{2}} \geqq \delta.$$

We wish to show that $A(n; \Omega) \not\to 0$ (compare 2.3(2)). If n is an integer satisfying 6.1(3), let ν_k be the finite sequence consisting of all ν with the property $|\nu/n - x_0| < \delta/2n^{\frac{1}{2}}$. We need the following result concerning the probability $p_{\nu n}$: an absolute constant C_1 and a $C_2 = C_2(x_0, \delta) > 0$ exist such that

6.1(4) $$\begin{cases} \displaystyle\sum_{|\nu/n - x| \geqq n^{-\frac{1}{3}}} p_{\nu n}(x) \leqq C_1 n^{-2}, \\ \displaystyle\sum_{|\nu/n - x_0| < \delta/2n^{\frac{1}{2}}} p_{\nu n}(x) \geqq C_2 . \end{cases}$$

We have

$$A\ (n; \Omega) = \sum_k \left| \int_0^1 p_{\nu_k n}(x) dg(x) \right|$$
$$\geqq |\alpha| \sum_k p_{\nu_k n}(x_0) - \int_0^1 \sum_k p_{\nu_k n}(x) dV(x),$$

where $V(x)$ is the variation of the function $g^*(x)$, resulting from $g(x)$ by omission of the jump α. The latter integral is smaller than

$$\operatorname*{var}_{\left|\frac{\nu}{n} - x\right| \leqq n^{-\frac{1}{3}}} g^*(x) + \operatorname*{var}_{0 \leqq x \leqq 1} g^*(x) \cdot C_1 n^{-2} \to 0, \qquad n \to \infty.$$

Thus by 6.1(3) and the second part of 6.1(4), $A(n; \Omega) \not\to 0$, and the proof of the theorem is complete.

6.2. We now treat methods H_g for which the generating function $g(x)$ is absolutely continuous.

THEOREM 10. *If $g(x)$ is absolutely continuous, then the summability functions for the method H_g are furnished by the formula $\Omega(n) = o(n)$.*

Proof. That there are no summability functions which do not satisfy $\Omega(n) = o(n)$ is implied by Theorems 8 and 9(i). Suppose now that ν_k is any sequence of integers with a characteristic function $\omega(n) = o(n)$. If $f_n(x)$ is again defined by 6.1(2),

$$\int_0^1 f_n(x)dx = \sum_k \int_0^1 p_{\nu_k}(x)dx = \frac{1}{n+1}\omega(n) \to 0.$$

Therefore, for every $\epsilon > 0$ and every integer n sufficiently large, the interval $[0. 1]$ may be represented as the sum $e_n + e_n'$ of two disjoint measurable sets e_n, e_n' for which $me_n < \epsilon$ and $f_n(x) < \epsilon$ if $x \epsilon e_n'$. Hence

$$\sum_k |a_{n\nu_k}| \leqq \sum_k \int_0^1 p_{\nu_k}(x)\,|g'(x)|\,dx = \int_0^1 f_n(x)\,|g'(x)|\,dx$$
$$= \int_{e_n} + \int_{e_n'} \leqq \int_{e_n} |g'|\,dx + \epsilon \int_0^1 |g'|\,dx.$$

Since both the last terms are arbitrarily small with ϵ, the proof is complete.

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By combining a gap theorem of Agnew [1] with a result of the author (Lorentz [4, theorem 1]), one may deduce that

6.2(1) $$u_n = o\left(\frac{1}{n}\right)$$

is a Tauberian condition for any Hausdorff method. Theorems 6 and 10 show now that $u_n = o(\phi(n))$ is not a Tauberian condition for any Hausdorff method with an absolutely continuous $g(x)$ if $\phi(n) \downarrow 0$ and $n\phi(n) \to +\infty$.

As to the summability functions of the second kind, we are able to show that all functions $\Omega(n) = o(\sqrt{n})$ are summability functions of the second kind for the method H_g if $g(x) \epsilon$ Lip 1. We do not know whether this result can be improved.

Suppose $\Omega(n) = o(\sqrt{n})$; we shall show that

$$\Delta(n; \Omega) = \sum_{\nu=0}^{n} \Omega(\nu) \left| \int_0^1 \{ p_\nu(x) - p_{\nu+1}(x) \} \, dg(x) \right| \to 0.$$

Since $g'(x)$ is bounded, it will be sufficient to prove

6.2(2) $$\bar{\Delta}(n) = \Omega(n) \int_0^1 \sum_{\nu=0}^{n} | p_\nu(x) - p_{\nu+1}(x) | \, dx$$

$$= 2\Omega(n) \int_0^1 \max_{0 \leqq \nu \leqq n} p_\nu(x) dx \to 0.$$

For a fixed x, the above maximum is attained for a $\nu = \nu_0(x)$ such that $|\nu_0/n - x| < n^{-1}$. We use the estimate (Lorentz [1])

6.2(3) $p_{\nu n}(x) \leqq [x(1-x)n]^{-\frac{1}{2}}$, if $|\nu/n - x| < x/10$, $|\nu/n-x| < (1-x)/10$.

Hence,

$$p_{\nu_0 n}(x) \leqq [x(1-x)n]^{-\frac{1}{2}}, \qquad 10/n \leqq x \leqq (n-10)/n.$$

Therefore

$$\bar{\Delta}(n) = 2\Omega(n) \left\{ \int_{10/n}^{(n-10)/n} + \int_0^{10/n} + \int_{(n-10)/n}^1 \right\}$$

$$\leqq 2\Omega(n) \left\{ \int_0^1 \frac{dx}{\sqrt{x(1-x)n}} + \frac{20}{n} \right\} = o(1),$$

which proves 6.2(2).

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