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ARITHMETIC SPECIAL CYCLES AND JACOBI FORMS

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Abstract We consider families of special cycles, as introduced by Kudla, on Shimura varieties attached to anisotropic quadratic spaces over totally real fields. By augmenting these cycles with Green currents, we obtain classes in the arithmetic Chow groups of the canonical models of these Shimura varieties (viewed as arithmetic varieties over their reflex fields). The main result of this paper asserts that generating series built from these cycles can be identified with the Fourier expansions of non-holomorphic Hilbert-Jacobi modular forms. This result provides evidence for an arithmetic analogue of Kudla's conjecture relating these cycles to Siegel modular forms.

Contents

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1. Introduction

The main result of this paper is a modularity result for certain generating series of 'special' cycles that live in the arithmetic Chow groups of Shimura varieties of orthogonal type.

We begin by introducing the main players. Let F be a totally real extension of $\mathbb{Q}$ with $d = [F:\mathbb{Q}]$, and let $\sigma_1,\dots,\sigma_d$ denote the archimedean places of F. Suppose V is a quadratic space over F that is of signature $((p,2),(p+2,0),(p+2,0),\cdots,(p+2,0))$ with $p>0$. In other words, we assume that $V\otimes_{F,\sigma_1}\mathbb{R}$ is a real quadratic space of signature $(p,2)$ and that V is positive definite at all other real places.

We assume throughout that V is anisotropic over F. Note that the signature condition guarantees that V is anisotropic whenever $d>1$.

Let $H = \mathrm{Res}_{F/\mathbb{Q}}\mathrm{GSpin}(V)$. The corresponding Hermitian symmetric space $\mathbb{D}$ has two connected components; fix one component $\mathbb{D}^+$ and let $H^+(\mathbb{R})$ denote its stabilizer in $H(\mathbb{R})$. For a neat compact open subgroup $K_f\subset H(\mathbb{A}_f)$, let $\Gamma := H^+(\mathbb{Q})\cap K_f$, where $H^+(\mathbb{Q}) = H(\mathbb{Q})\cap H^+(\mathbb{R})$, and consider the quotient

$$X(\mathbb{C}) \ := \ \Gamma\backslash\mathbb{D}^+. \tag{1.1}$$

This space is a (connected) Shimura variety; in particular, it admits a canonical model X over a number field $E\subset\mathbb{C}$ depending on K_f, see [13] for details. Moreover, as V is anisotropic, X is a projective variety.

Fix a Γ-invariant lattice $L\subset V$ such that the restriction of the bilinear form $\langle\cdot,\cdot\rangle$ to L is valued in $\mathcal{O}_F$, and consider the dual lattice

$$L' = \{\mathbf{x}\in L \mid \langle\mathbf{x},L\rangle\subset\partial_F^{-1}\}, \tag{1.2}$$

where ∂_F^{-1} is the inverse different.

For an integer n with $1\le n\le p$, let $S(V(\mathbb{A}_f)^n)$ denote the Schwartz space of compactly supported, locally constant functions on $V(\mathbb{A}_f)^n$, and consider the subspace

$$S(L^n) \ := \ \{\varphi\in S(V(\mathbb{A}_f)^n)^\Gamma \mid \mathrm{supp}(\varphi)\subset(\widehat{L'})^n \text{ and } \varphi(\mathbf{x}+l)=\varphi(\mathbf{x}) \text{ for all } l\in L^n\}. \tag{1.3}$$

Note that $S(L^n)$ is finite-dimensional, and is isomorphic to $\mathbb{C}[(L')^n/L^n]^\Gamma$. This isomorphism is induced by the following map: a basis function $e_\mu\in\mathbb{C}[(L')^n/L^n]$ attached to the coset $\mu\in(L')^n/L^n$ is associated to the characteristic function $\varphi_\mu\in S(V(\mathbb{A}_f)^n)$ of $\mu+L\otimes\widehat{\mathcal{O}_F}$.

For every $T\in\mathrm{Sym}_n(F)$ and Γ-invariant Schwartz function $\varphi\in S(L^n)$, there is an E-rational 'special' cycle

$$Z(T,\varphi) \tag{1.4}$$

of codimension n on X, defined originally by Kudla [13]; see Section 2.3 below.

It was conjectured by Kudla that these cycles are closely connected to automorphic forms; more precisely, he conjectured that upon passing to the Chow group of X, the

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generating series formed by the classes of these special cycles can be identified as the Fourier expansion of a Hilbert-Siegel modular form. When $F = \mathbb{Q}$, the codimension one case of this conjecture follows from results of Borcherds [2], and the conjecture for higher codimension was established by Zhang and Bruinier-Raum [4, 19]. When $F \neq \mathbb{Q}$ and in codimension one, the result follows from work of Bruinier [3], which we review in Section 3 below. For higher codimensional cycles with $F \neq \mathbb{Q}$, the result was established by Yuan-Zhang-Zhang [18], contingent on the convergence of the generating series in an appropriate sense. The same situation was investigated by Kudla [12] for more general signatures, who established the result (including convergence) contingent on the Beilinson-Bloch conjecture.

More recently, attention has shifted to the arithmetic analogues of this result, where one replaces the Chow groups with an 'arithmetic' counterpart, attached to a model $\mathcal{X}$ of X defined over a subring of the reflex field of E; these arithmetic Chow groups were introduced by Gillet-Soulé [9] and subsequently generalized by Burgos-Kramer-Kühn [5]. Roughly speaking, in this framework cycles are represented by pairs $(\mathcal{Z}, g_{\mathcal{Z}})$, where $\mathcal{Z}$ is a cycle on $\mathcal{X}$, and $g_{\mathcal{Z}}$ is a *Green object*, a purely differential-geometric datum that encodes cohomological information about the archimedean fibres of $\mathcal{Z}$.

In this paper, we consider the case where the model $\mathcal{X}$ is taken to be X itself. In order to promote the special cycles to the arithmetic setting, we need to choose the Green objects. For this, we employ the results of [8], where a family $\{\mathfrak{g}(T,\varphi;\mathbf{v})\}$ of Green forms was constructed. Note that these forms depend on an additional parameter $\mathbf{v} \in \mathrm{Sym}_n(F \otimes_{\mathbb{Q}} \mathbb{R})_{\gg 0}$, which should be regarded as the imaginary part of a variable in the Hilbert-Siegel upper half space.

With these Green objects in hand, we obtain classes

$$\widehat{Z}(T,\mathbf{v}) \in \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X) \otimes_{\mathbb{C}} S(L^n)^\vee, \tag{1.5}$$

where $\widehat{\mathrm{CH}}^n_{\mathbb{C}}(X)$ is the Gillet-Soulé arithmetic Chow group attached to X; these classes are defined by the formula

$$\widehat{Z}(T,\mathbf{v})(\varphi) \;=\; \big(Z(T,\varphi), \mathfrak{g}(T,\varphi;\mathbf{v})\big) \;\in\; \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X). \tag{1.6}$$

For reasons that will emerge in the course of the proof of our main theorem, we will also need to consider a larger arithmetic Chow group $\widehat{\mathrm{CH}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}})$, constructed by Burgos-Kramer-Kühn [5]. This group appears as an example of their general cohomological approach to the theory of Gillet-Soulé. There is a natural injective map $\widehat{\mathrm{CH}}^n_{\mathbb{C}}(X) \hookrightarrow \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}})$; abusing notation, we identify the special cycle $\widehat{Z}(T,\mathbf{v})$ with its image under this map.

Theorem 1.1. (i) *Suppose $1 < n \le p$. Fix $T_2 \in \mathrm{Sym}_{n-1}(F)$, and define the formal generating series*

$$\widehat{\mathrm{FJ}}_{T_2}(\boldsymbol{\tau}) \;=\; \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{Z}(T,\mathbf{v})\, q^T, \tag{1.7}$$

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where $\boldsymbol{\tau} \in \mathbb{H}_n^d$ *lies in the Hilbert-Siegel upper half space of genus* n*, and* $\mathbf{v} = \mathrm{Im}(\boldsymbol{\tau})$*. Then* $\widehat{\mathrm{FJ}}_{T_2}(\boldsymbol{\tau})$ *is the* q*-expansion of a (non-holomorphic) Hilbert-Jacobi modular form of weight* $p/2+1$ *and index* T_2*, taking values in* $\widehat{\mathrm{CH}}^n_{\mathbb{C}}(X, \mathcal{D}_{\mathrm{cur}}) \otimes S(L^n)^\vee$ *via the Weil representation.*

(ii) *When* $n = 1$*, the generating series*

$$\widehat{\phi}_1(\boldsymbol{\tau}) = \sum_{t \in F} \widehat{Z}(t, \mathbf{v}) q^t \tag{1.8}$$

is the q*-expansion of a (non-holomorphic) Hilbert modular form of weight* $p/2+1$*, valued in* $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(X) \otimes S(L)^\vee$*.*

Some clarification is warranted in the interpretation of this theorem. The issue is that there is no apparent topology on the arithmetic Chow groups for which the series (1.7) and (1.8) can be said to converge in a reasonable sense. In a similar vein, while the Green forms $\mathfrak{g}(T, \mathbf{v})$ vary smoothly in the parameter $\mathbf{v}$, upon passing to the arithmetic Chow group, they are regarded as elements of the quotient space of currents modulo exact currents; there does not appear to be a natural way in which the family of classes $\widehat{Z}(T, \mathbf{v})$ can be said to vary smoothly in the arithmetic Chow group.

To give a more precise account of the main theorem, what is being asserted is the existence of the following objects:

(i) finitely many classes $\widehat{Z}_1, \ldots \widehat{Z}_r \in \widehat{\mathrm{CH}_{\mathbb{C}}}^n(X, \mathcal{D}_{\mathrm{cur}})$ (or in $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(X)$ when $n = 1$),

(ii) finitely many $S(L^n)^\vee$-valued Jacobi modular forms (in the usual sense) $f_1, \ldots, f_r$,

(iii) and a Jacobi form $g(\boldsymbol{\tau})$ valued in $D^\bullet(X) \otimes S(L^n)^\vee$ (where $D^\bullet(X)$ is the space of currents on X) and is locally uniformly bounded in $\boldsymbol{\tau}$,

such that the T'th coefficient of the Jacobi form $\sum_i f_i(\boldsymbol{\tau}) \widehat{Z}_i + a(g(\boldsymbol{\tau}))$ coincides with $\widehat{Z}(T, \mathbf{v})$. Here. $a(g(\boldsymbol{\tau})) \in \widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}})$ is an 'archimedean class' associated to the current $g(\boldsymbol{\tau})$. A more detailed account may be found in Section 2.6.

To prove the theorem, we first prove the $n = 1$ case, using a modularity result due to Bruinier [3] that involves a different set of Green functions. The theorem in this case follows from a comparison between his Green functions and ours.

For $n > 1$, we exhibit a decomposition

$$\widehat{Z}(T, \mathbf{v}) = \widehat{A}(T, \mathbf{v}) + \widehat{B}(T, \mathbf{v}) \tag{1.9}$$

in $\widehat{\mathrm{CH}}^n(X, \mathcal{D}_{\mathrm{cur}}) \otimes S(L^n)^\vee$, which is based on a mild generalization of the star product formula [8, Theorem 4.10]. The main theorem then follows from the modularity of the series

$$\widehat{\phi}_A(\boldsymbol{\tau}) := \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{A}(T, \mathbf{v}) q^T \quad \text{and} \quad \widehat{\phi}_B(\boldsymbol{\tau}) = \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{B}(T, \mathbf{v}) q^T, \tag{1.10}$$

which are proved in Corollary 6.3 and Theorem 5.1, respectively. The classes $\widehat{A}(T, \mathbf{v})$ are expressed as linear combinations of pushforwards of special cycles along sub-Shimura varieties of X, weighted by the Fourier coefficients of classical theta series; the modularity

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of $\widehat{\phi}_A(\boldsymbol{\tau})$ follows from this description and the $n=1$ case. The classes $\widehat{B}(T,\mathbf{v})$ are purely archimedean, and the modularity of $\widehat{\phi}_B(\boldsymbol{\tau})$ follows from an explicit computation involving the Kudla-Millson Schwartz form [14].

This result provides evidence for the arithmetic version of Kudla's conjecture – namely, that the generating series

$$\widehat{\phi}_n(\boldsymbol{\tau}) \;=\; \sum_{T\in \mathrm{Sym}_n(F)} \widehat{Z}(T,\mathbf{v})\, q^T \tag{1.11}$$

is a Hilbert-Siegel modular form[1]; indeed, the series $\widehat{\mathrm{FJ}}_{T_2}(\boldsymbol{\tau})$ is a formal Fourier-Jacobi coefficient of $\widehat{\phi}_n(\boldsymbol{\tau})$. Unfortunately, there does not seem to be an obvious path by which one can infer the more general result from the results in this paper, as the decomposition of $\widehat{Z}(T,\mathbf{v})$ that we use depends on the lower-right matrix T_2, and it is not clear how to compare the decompositions for various T_2.

2. Preliminaries

2.1. Notation

- Throughout, we fix a totally real field F with $[F:\mathbb{Q}]=d$. Let $\sigma_1,\dots,\sigma_d$ denote the real embeddings. Using these embeddings, we identify $F\otimes_{\mathbb{Q}}\mathbb{R}$ with $\mathbb{R}^d$, and denote by $\sigma_i(\mathbf{t})$ the i'th component of $\mathbf{t}\in F\otimes_{\mathbb{Q}}\mathbb{R}$ under this identification.
- For any matrix A, we denote the transpose by A'.
- If $A\in \mathrm{Mat}_n(F\otimes_{\mathbb{Q}}\mathbb{R})$, we write

$$e(A) \;:=\; \prod_{i=1}^{d} \exp\big(2\pi i\,\mathrm{tr}\,(\sigma_i(A))\big). \tag{2.1}$$

- If (V,Q) is a quadratic space over F, let $\langle \mathbf{x},\mathbf{y}\rangle$ denote the corresponding bilinear form. Here, we take the convention $Q(\mathbf{x})=\langle \mathbf{x},\mathbf{x}\rangle$. If $\mathbf{x}\in V$ and $\mathbf{y}=(\mathbf{y}_1,\dots,\mathbf{y}_n)\in V^n$, we set $\langle \mathbf{x},\mathbf{y}\rangle=(\langle \mathbf{x},\mathbf{y}_1\rangle,\dots,\langle \mathbf{x},\mathbf{y}_n\rangle)\in \mathrm{Mat}_{1,n}(F)$.
- For $i=1,\dots d$, we set $V_i=V\otimes_{F,\sigma_i}\mathbb{R}$.
- Let

$$\mathbb{H}_n^d=\{\boldsymbol{\tau}=\mathbf{u}+i\mathbf{v}\in \mathrm{Sym}_n(F\otimes_{\mathbb{Q}}\mathbb{R}) \mid \mathbf{v}\gg 0\} \tag{2.2}$$

denote the Hilbert-Siegel upper half-space of genus n attached to F. Via the fixed embeddings $\sigma_1,\dots,\sigma_d$, we may identify $\mathrm{Sym}_n(F\otimes\mathbb{R})\simeq \mathrm{Sym}_n(\mathbb{R})^d$; we let

[1] More broadly, Kudla's program seeks to establish the modularity of generating series of the form

$$\widehat{\Phi}_n(\tau)=\sum_T \widehat{\mathcal{Z}}(T,\mathbf{v})\,q^T,$$

where $\widehat{\mathcal{Z}}(T,\mathbf{v})=(\mathcal{Z}(T),\mathfrak{g}(T,\mathbf{v}))\in \widehat{\mathrm{CH}}^n(\mathcal{X})\otimes S(L^n)^\vee$; here, $\mathcal{Z}(T)$ is a suitable integral model of $Z(T)$, defined on a suitable model $\mathcal{X}/\mathcal{O}_F$ of X. Putting aside what 'suitable' should mean here, we note that there is a natural map $\widehat{\mathrm{CH}}^n(\mathcal{X})\to \widehat{\mathrm{CH}}^n(X)$ given by passing to the generic fibre; applying this map coefficient-wise, the modularity of a generating series of the above form would imply the modularity of (1.11).

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$\sigma_i(\boldsymbol{\tau}) = \sigma_i(\mathbf{u}) + i\sigma_i(\mathbf{v})$ denote the corresponding component, so that, in particular, $\sigma_i(\mathbf{v}) \in \mathrm{Sym}_n(\mathbb{R})_{>0}$ for $i = 1,\ldots,d$. If $\boldsymbol{\tau} \in \mathbb{H}_n^d$ and $T \in \mathrm{Sym}_n(F)$, we write

$$q^T = e(\boldsymbol{\tau}T). \tag{2.3}$$

2.2. Arithmetic Chow groups

An *arithmetic cycle* of codimension n is a pair (Z,g), where Z is a formal $\mathbb{C}$-linear combination of codimension n subvarieties of X, and g is a *Green current* for Z; more precisely, g is a current of degree $(n-1,n-1)$ on $X(\mathbb{C})$ such that Green's equation

$$\mathrm{dd^c} g \; + \; \delta_{Z(\mathbb{C})} = \omega \tag{2.4}$$

holds, where the right-hand side is the current defined by integration[2] against some smooth form ω. We write $\widehat{\mathcal{Z}}^n_{\mathbb{C}}(X)$ for the complex vector space of arithmetic cycles.

Given a codimension $n-1$ subvariety Y and a rational function $f \in E(Y)^\times$ on Y, let

$$\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|^2\,\delta_Y) \tag{2.5}$$

denote the corresponding principal arithmetic cycle. Let $\widehat{\mathrm{Rat}}^n_{\mathbb{C}}(X)$ denote the subspace spanned by (a) the principal arithmetic divisors and (b) classes of the form $(0,\eta)$ with $\eta \in \mathrm{im}(\partial) + \mathrm{im}(\overline{\partial})$ a current of degree $(n-1,n-1)$. Then, by definition, the codimension n arithmetic Chow group is the quotient

$$\widehat{\mathrm{CH}}^n_{\mathbb{C}}(X) = \widehat{\mathcal{Z}}^n_{\mathbb{C}}(X)/\widehat{\mathrm{Rat}}^n_{\mathbb{C}}(X). \tag{2.6}$$

In addition, Gillet and Soulé define an intersection product for these Chow groups, giving $\widehat{\mathrm{CH}}^*_{\mathbb{C}}(X) = \oplus_n \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X)$ the structure of a ring; for more details on all these constructions, see [9, 16].

In their paper [5], Burgos, Kramer and Kühn give an abstract reformulation and generalization of this theory: their main results describe the construction of an arithmetic Chow group

$$\widehat{\mathrm{CH}}^n(X,\mathcal{C}) = \widehat{\mathcal{Z}}^n_{\mathbb{C}}(X,\mathcal{C})/\widehat{\mathrm{Rat}}^n_{\mathbb{C}}(X,\mathcal{C}) \tag{2.7}$$

attached to a 'Gillet complex' $\mathcal{C}$. One of the examples they describe is the group attached to the complex of currents $\mathcal{D}_{\mathrm{cur}}$; we will content ourselves with the superficial description of this group given below, which will suffice for our purposes, and the reader is invited to consult [5, §6.2] for a thorough treatment.

Unwinding the formal definitions in [5], one finds that the space of arithmetic classes $\widehat{\mathcal{Z}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}})$ admits a description as the space of tuples $(Z,[T,g])$, with Z as before, but now T and g are currents of degree (n,n) and $(n-1,n-1)$, respectively, such that[3]

$$\mathrm{dd^c} g + \delta_{Z(\mathbb{C})} = T + \mathrm{dd^c}(\eta) \tag{2.8}$$

[2]Here and throughout this paper, we will abuse notation and write ω both for a smooth form and the current it defines.

[3]The reader is cautioned that in [5], the authors normalize delta currents and currents defined via integration by powers of $2\pi i$, resulting in formulas that look slightly different from those presented here; because we are working with $\mathbb{C}$-coefficients, the formulations are equivalent.

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for some current η with support contained in $Z(\mathbb{C})$; we can view this as a relaxation of the condition that the right-hand side of (2.4) is smooth. There is a natural map

$$\widehat{\mathcal{Z}}^n_{\mathbb{C}}(X) \to \widehat{\mathcal{Z}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}}), \qquad (Z,g) \mapsto (Z,[\omega,g]). \tag{2.9}$$

In this description, it turns out that the space of relations $\widehat{\mathrm{Rat}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}})$ is the image of $\widehat{\mathrm{Rat}}^n_{\mathbb{C}}(X)$ under this map; as a consequence, we obtain an injective map

$$\widehat{\mathrm{CH}}^n_{\mathbb{C}}(X) \to \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}}), \tag{2.10}$$

cf. [5, Theorem 6.35]. Moreover, while $\widehat{\mathrm{CH}}^*_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}})$ is not a ring in general, it is a module over $\widehat{\mathrm{CH}}^*_{\mathbb{C}}(X)$.

A nice consequence of preceding description of $\widehat{\mathcal{Z}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}})$ is that any codimension n cycle Z on X gives rise to a *canonical class* (see [5, Definition 6.37])

$$\widehat{Z}^{\mathrm{can}} \;:=\; (Z,[\delta_Z,0])\,. \tag{2.11}$$

Finally, we record the following intersection formula, which will be useful in the sequel. Let $(Z,g) \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X)$ be an arithmetic divisor, where g is a Green function with logarithmic singularities along the divisor Z. Suppose $\widehat{Y}^{\mathrm{can}} \in \widehat{\mathrm{CH}}^m(X,\mathcal{D}_{\mathrm{cur}})$ is the canonical class attached to a cycle Y that intersects Z properly; then by inspecting the proofs of [5, Theorem 6.23, Proposition 6.32], we find

$$(Z,g)\cdot \widehat{Y}^{\mathrm{can}} \;=\; \big(Z\cdot Y,[\omega \wedge \delta_{Y(\mathbb{C})}, g\wedge \delta_{Y(\mathbb{C})}]\big) \in \widehat{\mathrm{CH}}^{m+1}_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}}). \tag{2.12}$$

Remark 2.1. One consequence of our setup is the vanishing of certain 'archimedean rational' classes in $\widehat{\mathrm{CH}}^n(X)$ and $\widehat{\mathrm{CH}}^n(X,\mathcal{D}_{\mathrm{cur}})$. More precisely, if Y is a codimension $n-1$ subvariety, then

$$(0,\delta_{Y(\mathbb{C})}) = 0 \;\in\; \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X). \tag{2.13}$$

To see this, let $c \in \mathbb{Q}$ be any rational number such that $c \neq 0$ or ± 1, and view c as a rational function on Y; its divisor is trivial, and so

$$0 = \widehat{\mathrm{div}}(c) \;=\; (0,-\log|c|^2\delta_{Y(\mathbb{C})}) \;=\; -\log|c|^2\cdot(0,\delta_{Y(\mathbb{C})}), \tag{2.14}$$

and hence, $(0,\delta_{Y(\mathbb{C})}) = 0$. As a special case, we have $(0,1) = 0 \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X)$.

2.3. Special cycles

Here, we review Kudla's construction of the family $\{Z(T)\}$ of special cycles on X, [13]. First, recall that the symmetric space $\mathbb{D}$ has a concrete realization

$$\mathbb{D} = \{z \in \mathbb{P}^1(V\otimes_{\sigma_1,F}\mathbb{C}) \mid \langle z,z\rangle = 0, \langle z,\overline{z}\rangle < 0\}, \tag{2.15}$$

where $\langle\cdot,\cdot\rangle$ is the $\mathbb{C}$-bilinear extension of the bilinear form on V; the space $\mathbb{D}$ has two connected components, denoted $\mathbb{D}^{\pm}$, which are interchanged by conjugation.

Given a collection of vectors $\mathbf{x} = (\mathbf{x}_1,\ldots,\mathbf{x}_n) \in V^n$, let

$$\mathbb{D}^+_{\mathbf{x}} \;:=\; \{z \in \mathbb{D}^+ \mid z \perp \sigma_1(\mathbf{x}_i) \;\text{ for } i = 1,\ldots,n\}, \tag{2.16}$$

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where, abusing notation, we denote by $\sigma_1 \colon V \to V_1 = V \otimes_{F,\sigma_1} \mathbb{R}$ the map induced by inclusion in the first factor.

Let $\Gamma_{\mathbf{x}}$ denote the pointwise stabilizer of $\mathbf{x}$ in Γ; then the inclusion $\mathbb{D}^+_{\mathbf{x}} \subset \mathbb{D}^+$ induces a map

$$\Gamma_{\mathbf{x}} \backslash \mathbb{D}^+_{\mathbf{x}} \to \Gamma \backslash \mathbb{D}^+ = X, \tag{2.17}$$

which defines a complex algebraic cycle that we denote $Z(\mathbf{x})$. If the span of $\{\mathbf{x}_1, \ldots, \mathbf{x}_r\}$ is not totally positive definite, then $\mathbb{D}^+_{\mathbf{x}} = \emptyset$ and $Z(\mathbf{x}) = 0$; otherwise, the codimension of $Z(\mathbf{x})$ is the dimension of this span.

Now suppose $T \in \mathrm{Sym}_n(F)$ and $\varphi \in S(L^n)$, and set

$$Z(T,\varphi)^{\natural} \ := \sum_{\substack{\mathbf{x} \in \Omega(T) \\ \mathrm{mod}\ \Gamma}} \varphi(\mathbf{x}) \cdot Z(\mathbf{x}), \tag{2.18}$$

where

$$\Omega(T) \ := \ \{\mathbf{x} = (\mathbf{x}_1, \ldots, \mathbf{x}_n) \in V^n \mid \langle \mathbf{x}_i, \mathbf{x}_j \rangle = T_{ij}\}. \tag{2.19}$$

This cycle is rational over E. If $Z(T,\varphi)^{\natural} \neq 0$, then T is necessarily totally positive semidefinite, and in this case, $Z(T,\varphi)^{\natural}$ has codimension equal to the rank of T.

Finally, we define a $S(L^n)^{\vee}$-valued cycle $Z(T)^{\natural}$ by the rule

$$Z(T)^{\natural} \colon \varphi \mapsto Z(T,\varphi)^{\natural} \tag{2.20}$$

for $\varphi \in S(L^n)$.

2.4. The cotautological bundle

Let $\mathcal{E} \to X$ denote the tautological bundle: over the complex points $X(\mathbb{C}) = \Gamma \backslash \mathbb{D}^+$, the fibre $\mathcal{E}_z$ at a point $z \in \mathbb{D}^+$ is simply the line corresponding to z in the model (2.15). There is a natural Hermitian metric $\| \cdot \|^2_{\mathcal{E}}$ on $\mathcal{E}(\mathbb{C})$, defined at a point $z \in \mathbb{D}^+$ by the formula $\|v_z\|^2_{\mathcal{E},z} \ = \ -\langle v_z, v_z \rangle$ for $v_z \in z$.

Consider the arithmetic class

$$\widehat{\omega} = -\widehat{c}_1(\mathcal{E}, \| \cdot \|_{\mathcal{E}}) \ \in \ \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X); \tag{2.21}$$

concretely, $\widehat{\omega} = -(\mathrm{div} s, -\log \|s\|^2_{\mathcal{E}})$, where s is any meromorphic section of $\mathcal{E}$. Finally, for future use, we set

$$\Omega := -c_1(\mathcal{E}, \| \cdot \|_{\mathcal{E}}) \in A^{1,1}(X(\mathbb{C})), \tag{2.22}$$

where $-\Omega = c_1(\mathcal{E}, \| \cdot \|_{\mathcal{E}})$ is the first Chern form attached to $(\mathcal{E}, \| \cdot \|_{\mathcal{E}})$; here, the Chern form is normalized as in [16, §4.2]. Note that $-\Omega$ is a Kähler form; cf. [8, §2.2].

Remark 2.2. Elsewhere in the literature, one often finds a different normalization (i.e., an overall multiplicative constant) for the metric $\| \cdot \|_{\mathcal{E}}$ that is better suited to certain arithmetic applications; for example, see [15, §3.3]. In our setting, however, Remark 2.1 implies that rescaling the metric does not change the Chern class in $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(X)$.

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2.5. Green forms and arithmetic cycles

In this section, we sketch the construction of a family of Green forms for the special cycles, following [8].

We begin by recalling that for any tuple $x = (x_1, \dots x_n) \in V_1^n = (V \otimes_{\sigma_1, F} \mathbb{R})^n$, Kudla and Millson (see [14]) have defined a Schwartz form $\varphi_{\mathrm{KM}}(x)$, which is valued in the space of closed (n,n) forms on $\mathbb{D}^+$, and is of exponential decay in x. Let $T(x) \in \mathrm{Sym}_n(\mathbb{R})$ denote the matrix of inner products (i.e., $T(x)_{ij} = \langle x_i, x_j \rangle$), and consider the normalized form

$$\varphi^o_{\mathrm{KM}}(x) \; := \; \varphi_{\mathrm{KM}}(x)\, e^{2\pi \mathrm{tr} T(x)}. \tag{2.23}$$

In [8, §2.2], another form $\nu^o(x)$, valued in the space of smooth $(n-1, n-1)$ forms on $\mathbb{D}^+$ is defined (this form is denoted by $\nu^o(x)_{[2n-2]}$ there). It satisfies the relation

$$\mathrm{dd^c} \nu^o(\sqrt{u}x) \; = \; -u \frac{\partial}{\partial u} \varphi^o_{\mathrm{KM}}(\sqrt{u}x), \qquad u \in \mathbb{R}_{>0}. \tag{2.24}$$

For a complex parameter $\rho \gg 0$, let

$$\mathfrak{g}^o(x;\rho) \; := \; \int_1^\infty \nu^o(\sqrt{u}x) \frac{du}{u^{1+\rho}}; \tag{2.25}$$

then $\mathfrak{g}^o(x,\rho)$ defines a smooth form for $Re(\rho) \gg 0$. The corresponding current admits a meromorphic continuation to a neighbourhood of $\rho = 0$, and we set

$$\mathfrak{g}^o(x) := \underset{\rho=0}{CT}\, \mathfrak{g}^o(x;\rho). \tag{2.26}$$

Note that, for example,

$$\mathfrak{g}^o(0) \; = \; \nu^o(0) \underset{\rho=0}{CT} \int_1^\infty \frac{du}{u^{1+\rho}} = 0. \tag{2.27}$$

In general, the current $\mathfrak{g}^o(x)$ satisfies the equation

$$\mathrm{dd^c} \mathfrak{g}^o(x) + \delta_{\mathbb{D}^+_{\mathbf{x}}} \wedge \Omega^{n-r(x)} = \varphi^o_{\mathrm{KM}}(x), \tag{2.28}$$

where $r(x) = \dim \mathrm{span}(x) = \dim \mathrm{span}(x_1, \dots, x_n)$; for details regarding all these facts, see [8, §2.6].

Now suppose $T \in \mathrm{Sym}_n(F)$. Following [8, §4], we define an $S(L^n)^\vee$-valued current $\mathfrak{g}^o(T, \mathbf{v})$, depending on a parameter $\mathbf{v} \in \mathrm{Sym}_n(F \otimes_{\mathbb{Q}} \mathbb{R})_{\gg 0}$, as follows: let $v = \sigma_1(\mathbf{v})$ and choose any matrix $a \in \mathrm{GL}_n(\mathbb{R})$ such that $v = aa'$. Then $\mathfrak{g}^o(T, \mathbf{v})$ is defined by the formula

$$\mathfrak{g}^o(T, \mathbf{v})(\varphi) \; := \; \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \; \mathfrak{g}^o\left(\sigma_1(\mathbf{x}) a\right), \tag{2.29}$$

where $\sigma_1(\mathbf{x}) \in V_1^n$; by [8, Proposition 2.12], this is independent of the choice of $a \in \mathrm{GL}_n(\mathbb{R})$. Note that $\mathfrak{g}^o(T, \mathbf{v})$ is a Γ-invariant current on $\mathbb{D}^+$ and hence descends to $X(\mathbb{C}) = \Gamma \backslash \mathbb{D}^+$.

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Next, consider the $S(L^n)^\vee$-valued differential form $\omega(T,\mathbf{v})$, defined by the formula

$$\omega(T,\mathbf{v})(\varphi) \;:=\; \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}(\sigma_1(\mathbf{x})a), \qquad \sigma_1(\mathbf{v}) = aa', \tag{2.30}$$

and which is a q-coefficient of the *Kudla-Millson theta series*

$$\Theta_{\mathrm{KM}}(\boldsymbol{\tau}) \;=\; \sum_{T\in\mathrm{Sym}_n(F)} \omega(T,\mathbf{v})\,q^T, \tag{2.31}$$

where $\tau \in \mathbb{H}_n^d$, and $\mathbf{v} = \mathrm{Im}(\boldsymbol{\tau})$. We then have the equation of currents

$$\mathrm{dd^c}\mathfrak{g}^o(T,\mathbf{v}) \;+\; \delta_{Z(T)(\mathbb{C})}\wedge\Omega^{n-\mathrm{rank}T} \;=\; \omega(T,\mathbf{v}) \tag{2.32}$$

on X; see [8, Proposition 4.4].

In particular, if T is non-degenerate, then $\mathrm{rank}(T) = n$ and $\mathfrak{g}^o(T,\mathbf{v})$ is a Green current for the cycle $Z(T)^\natural$. In this case, we obtain an arithmetic special cycle

$$\widehat{Z}(T,\mathbf{v}) := (Z(T)^\natural, \mathfrak{g}^o(T,\mathbf{v})) \;\in\; \widehat{\mathrm{CH}}{}^n_{\mathbb{C}}(X)\otimes_{\mathbb{C}} S(L^n)^\vee. \tag{2.33}$$

Now suppose $T\in\mathrm{Sym}_n(F)$ is arbitrary. Let $r = \mathrm{rank}(T)$, and fix $\varphi\in S(L^n)$. We may choose a pair (Z_0,g_0) representing the class $\widehat{\omega}^{n-r}\in\widehat{\mathrm{CH}}{}^n_{\mathbb{C}}(X)$, such that Z_0 intersects $Z(T,\varphi)^\natural$ properly and g_0 has logarithmic type [16, §II.2]. We then define

$$\widehat{Z}(T,\mathbf{v},\varphi) := \big(Z(T,\varphi)^\natural\cdot Z_0,\; \mathfrak{g}^o(T,\mathbf{v},\varphi) + g_0\wedge\delta_{Z(T,\varphi)^\natural(\mathbb{C})}\big) \;\in\; \widehat{\mathrm{CH}}{}^n_{\mathbb{C}}(X). \tag{2.34}$$

The reader may consult [8, §5.4] for more detail on this construction, including the fact that it is independent of the choice of (Z_0,g_0).

Finally, we define a class $\widehat{Z}(T,\mathbf{v})\in\widehat{\mathrm{CH}}{}^n_{\mathbb{C}}(X)\otimes S(L^n)^\vee$ by the rule

$$\widehat{Z}(T,\mathbf{v})(\varphi) = \widehat{Z}(T,\mathbf{v},\varphi). \tag{2.35}$$

Remark 2.3. In [8], the Green current $\mathfrak{g}^o(T,\mathbf{v})$ is augmented by an additional term, depending on $\log(\det\mathbf{v})$, when T is degenerate see [8, Definition 4.5]. This term was essential in establishing the archimedean arithmetic Siegel-Weil formula in the degenerate case; however, in the setting of the present paper, Remark 2.1 implies that this additional term vanishes upon passing to $\widehat{\mathrm{CH}}{}^n_{\mathbb{C}}(X)$ and can be omitted from the discussion without consequence. In particular, according to our definitions, we have

$$\widehat{Z}(0_n,\mathbf{v})(\varphi) = \varphi(0)\cdot\widehat{\omega}^n. \tag{2.36}$$

2.6. Hilbert-Jacobi modular forms

In this section, we briefly review the basic definitions of vector-valued (Hilbert) Jacobi modular forms, mainly to fix notions. For convenience, we work in 'classical' coordinates and only with parallel scalar weight. Throughout, we fix an integer $n\geq 1$.

We begin by briefly recalling the theory of metaplectic groups and the Weil representation; a convenient summary for the facts mentioned here, in a form useful to us, is

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[10, §2]. Let Sp_n denote the symplectic group, viewed as an algebraic group: given any ring R, we have

$$\mathrm{Sp}_n(R) \ := \ \left\{ g \in M_{2r}(R) \mid g' \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} g = \begin{pmatrix} & -1 \\ 1 & \end{pmatrix} \right\}.$$

For a place $v \leq \infty$, let $\widetilde{\mathrm{Sp}}_n(F_v)$ denote the metaplectic group, a two-fold cover of $\mathrm{Sp}_n(F_v)$; as a set, $\widetilde{\mathrm{Sp}}_n(F_v) = \mathrm{Sp}_n(F_v) \times \{\pm 1\}$. When $F_v = \mathbb{R}$, the group $\widetilde{\mathrm{Sp}}_n(\mathbb{R})$ is isomorphic to the group of pairs (g,ϕ), where $g = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \mathrm{Sp}_n(\mathbb{R})$ and $\phi\colon \mathbb{H}_n \to \mathbb{C}$ is a function such that $\phi(\tau)^2 = \det(C\tau + D)$. In this model, multiplication is given by

$$(g,\phi(\tau)) \cdot (g',\phi'(\tau)) = (gg',\phi(g'\tau)\phi'(\tau)). \tag{2.37}$$

At a non-dyadic finite place, there exists a canonical embedding $\mathrm{Sp}_n(\mathcal{O}_v) \to \widetilde{\mathrm{Sp}}_n(F_v)$, which splits the double covering $\tilde{K}$ over K; here, $\tilde{K}$ is the inverse image of K under the projection map $\widetilde{\mathrm{Sp}}_n(F_v) \to \mathrm{Sp}_n(F_v)$. Consider the restricted product $\prod'_{v\leq\infty} \widetilde{\mathrm{Sp}}_n(F_v)$ with respect to these embeddings. Then the global double cover $\widetilde{\mathrm{Sp}}_{n,\mathbb{A}}$ of $\mathrm{Sp}_n(\mathbb{A})$ is the quotient $\prod'_{v\leq\infty} \widetilde{\mathrm{Sp}}_n(F_v)/I$ of this restricted direct product by the subgroup

$$I := \left\{ (1,\epsilon_v)_{v\leq\infty} \mid \prod_v \epsilon_v = 1, \epsilon_v = 1 \text{ for almost all } v \right\}. \tag{2.38}$$

Moreover, there is a splitting

$$\iota_F\colon \ \mathrm{Sp}_n(F) \hookrightarrow \widetilde{\mathrm{Sp}}_{n,\mathbb{A}}, \qquad \gamma \mapsto \prod_v (\gamma,1)_v \cdot I. \tag{2.39}$$

Let $\widetilde{\Gamma}'$ denote the full inverse image of $\mathrm{Sp}_n(\mathcal{O}_F)$ under the covering map $\prod_{v|\infty} \widetilde{\mathrm{Sp}}_n(F_v) \to \mathrm{Sp}_n(F \otimes_{\mathbb{Q}} \mathbb{R})$. We obtain an action ρ of $\widetilde{\Gamma}'$ on the space $S(V(\mathbb{A}_f)^n)$ as follows. Let ω denote the[4] Weil representation of $\widetilde{\mathrm{Sp}}_{n,\mathbb{A}}$ on $S(V(\mathbb{A})^n)$. Given $\widetilde{\gamma} \in \widetilde{\Gamma}'$, choose $\widetilde{\gamma}_f \in \prod'_{v<\infty} \widetilde{\mathrm{Sp}}_n(F_v)$ such that $\widetilde{\gamma}\widetilde{\gamma}_f \in \mathrm{im}(\iota_F)$ and set

$$\rho(\widetilde{\gamma}) \ := \ \omega(\widetilde{\gamma}_f). \tag{2.40}$$

Recall that we had fixed a lattice $L \subset V$. The subspace $S(L^n) \subset S(V(\mathbb{A}_f)^n)$, as defined in (1.3), is stable under the action of $\widetilde{\Gamma}'$; when we wish to emphasize this lattice, we denote the corresponding action by ρ_L.

For a half-integer $\kappa \in \frac{1}{2}\mathbb{Z}$, we define a (parallel, scalar) weight κ slash operator, for the group $\widetilde{\Gamma}'$ acting on the space of functions $f\colon \mathbb{H}_n^d \to S(L^n)^\vee$, by the formula

$$f|_\kappa[\widetilde{\gamma}](\boldsymbol{\tau}) \ = \ \prod_{v|\infty} \phi_v(\sigma_v(\boldsymbol{\tau}))^{-2\kappa} \rho_L^\vee(\widetilde{\gamma}^{-1}) \cdot f(g\boldsymbol{\tau}), \qquad \widetilde{\gamma} = (g_v,\phi_v(\tau))_{v|\infty} \tag{2.41}$$

where $g = (g_v)_v$.

[4]Here, we normalize the Weil representation as in [11, Section II.4]. This representation also depends on the choice of an additive character $\psi_F\colon \mathbb{A}_F/F \to \mathbb{C}$; we choose the standard one, and suppress this choice from the notation.

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If $n>1$, consider the Jacobi group $G^J = G^J_{n,n-1} \subset \mathrm{Sp}_n$: for any ring R, its R-points are given by

$$G^J(R) := \left\{ g = \left(\begin{array}{cc|cc} a & 0 & b & a\mu - b\lambda \\ \lambda^t & 1_{n-1} & \mu^t & 0 \\ \hline c & 0 & d & c\mu - d\lambda \\ 0 & 0 & 0 & 1_{n-1} \end{array} \right) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(R),\ \mu,\tau \in M_{1,n-1}(R) \right\}. \tag{2.42}$$

Define $\widetilde{\Gamma}^J \subset \widetilde{\Gamma}'$ to be the inverse image of $G^J(\mathcal{O}_F)$ in $\widetilde{G}'_{\mathbb{R}}$.

Definition 2.4. Suppose $f\colon \mathbb{H}_n^d \to S(L^n)^\vee$ is a smooth function. Given $T_2 \in \mathrm{Sym}_{n-1}(F)$, we say that $f(\boldsymbol{\tau})$ transforms like a Jacobi modular form of genus n, weight κ and index T_2 if the following conditions hold.

(a) For all $\mathbf{u}_2 \in \mathrm{Sym}_{n-1}(F_{\mathbb{R}})$,

$$f\left(\boldsymbol{\tau} + \left(\begin{smallmatrix} 0 & \\ & \mathbf{u}_2 \end{smallmatrix}\right)\right) \;=\; e(T_2\mathbf{u}_2) f(\boldsymbol{\tau}). \tag{2.43}$$

(b) For all $\widetilde{\gamma} \in \widetilde{\Gamma}^J$,

$$f|_\kappa[\widetilde{\gamma}](\boldsymbol{\tau}) \;=\; f(\boldsymbol{\tau}). \tag{2.44}$$

Let $A_{\kappa,T_2}(\rho_L^\vee)$ denote the space of $S(L^n)^\vee$-valued smooth functions that transform like a Jacobi modular form of weight κ and index T_2.

Remark 2.5.

1. If desired, one can impose further analytic properties of f (holomorphic, real analytic, etc.).
2. If $n=1$, then we simply say that a function $f\colon \mathbb{H}_1^d \to S(L)^\vee$ transforms like a (Hilbert) modular form of weight κ if it satisfies $f|_\kappa[\tilde{\gamma}](\boldsymbol{\tau}) = f(\boldsymbol{\tau})$ as usual.
3. An $S(L^n)^\vee$-valued Jacobi modular form f, in the above sense, has a Fourier expansion of the form

$$f(\boldsymbol{\tau}) \;:=\; \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} c_f(T,\mathbf{v})\, q^T, \tag{2.45}$$

   where the coefficients $c_f(T,\mathbf{v})$ are smooth functions $c_f(T,\mathbf{v})\colon \mathrm{Sym}_n(F\otimes_{\mathbb{Q}}\mathbb{R})_{\gg 0} \to S(L^n)^\vee$.
4. For each $i = 1,\ldots,d$, let $(\phi_v(\tau))_v$ be the collection given by $\phi_{v_j}(\tau) = 1$ if $j \neq i$, and $\phi_{v_i}(\tau) = -1$. Let $\tilde{\epsilon}(i) = (\mathrm{Id},(\phi_v))$. Then, using the formulas in [11], we have that

$$f|_\kappa[\tilde{\epsilon}(i)](\boldsymbol{\tau}) = (-1)^{2\kappa + \dim(V)} f(\boldsymbol{\tau}).$$

   In particular, if $2\kappa \not\equiv \dim V \pmod 2$, then $A_{\kappa,T_2}(\rho_L^\vee) = \{0\}$.

We now clarify what it should mean for generating series with coefficients in arithmetic Chow groups, such as those appearing in Theorem 1.1, to be modular.

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First, let $D^{n-1}(X)$ denote the space of currents on $X(\mathbb{C})$ of complex bidegree $(n-1, n-1)$, and note that there is a map

$$a\colon D^{n-1}(X) \to \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}}), \qquad a(g) = (0,[\mathrm{dd}^{\mathrm{c}}g,g]). \tag{2.46}$$

Definition 2.6. Define the space $A_{\kappa,T_2}(\rho^\vee;D^{n-1}(X))$ of 'Jacobi forms valued in $S(L^n)^\vee \otimes_{\mathbb{C}} D^{n-1}(X)$' as the space of functions

$$\xi\colon \mathbb{H}^d_n \to D^{n-1}(X) \otimes_{\mathbb{C}} S(L^n)^\vee \tag{2.47}$$

such that the following two conditions hold.

(a) For every smooth form α on X, the function $\xi(\boldsymbol{\tau})(\alpha)$ is an element of $A_{\kappa,T_2}(\rho_L^\vee)$, and in particular, is smooth in the variable $\boldsymbol{\tau}$.

(b) Fix an integer $k \geq 0$ and let $\|\cdot\|_k$ be an algebra seminorm, on the space of smooth differential forms on X, such that given a sequence $\{\alpha_i\}$, we have $\|\alpha_i\|_k \to 0$ if and only if α_i, together with all partial derivatives of order $\leq k$, tends uniformly to zero. We then require that for every compact subset $C \subset \mathbb{H}^d_n$, there exists a constant $c_{k,C}$ such that

$$|\xi(\boldsymbol{\tau})(\alpha)| \leq c_{k,C}\|\alpha\|_k \tag{2.48}$$

for all $\boldsymbol{\tau} \in C$ and all smooth forms α.

The second condition ensures that any such function admits a Fourier expansion as in (2.45) whose coefficients are continuous in the sense of distributions (i.e., they are again $S(L^n)^\vee$-valued currents).

Definition 2.7. Given a collection of classes

$$\{\widehat{Y}(T,\mathbf{v}) \mid T \in \mathrm{Sym}_n(F), \mathbf{v} \in \mathrm{Sym}_n(F \otimes_{\mathbb{Q}} \mathbb{R})_{\gg 0}\} \tag{2.49}$$

with $\widehat{Y}(T,\mathbf{v}) \in \widehat{\mathrm{CH}}^n(X,\mathcal{D}_{\mathrm{cur}}) \otimes_{\mathbb{C}} S(L^n)^\vee$, consider the formal generating series

$$\widehat{\Phi}_{T_2}(\boldsymbol{\tau}) := \sum_{T=\left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{Y}(T,\mathbf{v})\, q^T. \tag{2.50}$$

Roughly speaking, we say that $\widehat{\Phi}_{T_2}(\boldsymbol{\tau})$ is modular (of weight κ and index T_2) if there is an element

$$\widehat{\phi}(\boldsymbol{\tau}) \in A_{\kappa,T_2}(\rho_L^\vee) \otimes_{\mathbb{C}} \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}}) \;+\; a\left(A_{\kappa,T_2}(\rho_L^\vee;D^{n-1}(X))\right) \tag{2.51}$$

whose Fourier expansion coincides with $\widehat{\Phi}_{T_2}(\boldsymbol{\tau})$. More precisely, we define the modularity of $\widehat{\Phi}_{T_2}(\boldsymbol{\tau})$ to mean that there are finitely many classes

$$\widehat{Z}_1,\ldots,\widehat{Z}_r \in \widehat{\mathrm{CH}}^n(X,\mathcal{D}_{\mathrm{cur}}) \tag{2.52}$$

and Jacobi forms

$$f_1,\ldots f_r \in A_{\kappa,T_2}(\rho^\vee), \qquad g \in A_{\kappa,T_2}(\rho^\vee;D^{n-1}(X)) \tag{2.53}$$

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such that

$$\widehat{Y}(T,\mathbf{v}) = \sum_i c_{f_i}(T,\mathbf{v})\,\widehat{Z}_i + a\left(c_g(T,\mathbf{v})\right) \;\in\; \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}}) \otimes_{\mathbb{C}} S(L^n)^\vee \tag{2.54}$$

for all $T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)$.

Remark 2.8.

1. If $\widehat{Z}_1,\ldots,\widehat{Z}_r \in \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X)$ and $g(\boldsymbol{\tau})$ takes values in the space of (currents represented by) smooth differential forms on X, then we say that $\widehat{\Phi}_{T_2}(\boldsymbol{\tau})$ is valued in $\widehat{\mathrm{CH}}^n_{\mathbb{C}}(X) \otimes S(L^n)^\vee$; indeed, in this case, the right-hand side of (2.54) lands in this latter group.
2. As before, one may also impose additional analytic conditions on the forms f_i, g appearing above if desired.
3. Elsewhere in the literature (e.g., [2, 3, 19]), one finds a notion of modularity that amounts to omitting the second term in (2.51); this notion is well-adapted to the case that the generating series of interest are holomorphic (i.e., the coefficients are independent of the imaginary part of $\boldsymbol{\tau}$).

   In contrast, the generating series that figure in our main theorem depend on these parameters in an essential way. Indeed, the Green forms $\mathfrak{g}^o(T,\mathbf{v})$ vary smoothly in $\mathbf{v}$; however, to the best of the author's knowledge, there is no natural topology on $\widehat{\mathrm{CH}}^n(X)$, or $\widehat{\mathrm{CH}}^n(X,\mathcal{D}_{\mathrm{cur}})$, for which the corresponding family $\widehat{Z}(T,\mathbf{v})$ varies smoothly in v. As we will see in the course of the proof of the main theorem, the additional term in (2.51) will allow us enough flexibility to reflect the non-holomorphic nature of the generating series. Similar considerations appear in [6] in the codimension one case.

3. The genus one case

In this section, we prove the main theorem in the case $n = 1$; later on, this will be a key step in the proof for general n. The proof of this theorem amounts to a comparison with a generating series of special divisors equipped with a different family of Green functions, defined by Bruinier. A similar comparison appears in [6] for unitary groups over imaginary quadratic fields; in the case at hand, however, the compactness of X allows us to apply spectral theory and simplify the argument considerably.

Suppose $t \gg 0$. In [3], Bruinier constructs an $S(L)^\vee$-valued Green function $\Phi(t)$ for the divisor $Z(t) = Z(t)^\natural$. To be a bit more precise about this, recall the Kudla-Millson theta function $\Theta_{\mathrm{KM}}(\boldsymbol{\tau})$ from (2.31). As a function of $\boldsymbol{\tau}$, the theta function Θ_{KM} is non-holomorphic and transforms as a Hilbert modular form of parallel weight $\kappa = p/2+1$. It is moreover of moderate growth, [3, Prop. 3.4] and hence can be paired, via the Petersson pairing, with cusp forms. Let $\Lambda_{\mathrm{KM}}(\boldsymbol{\tau}) \in S_\kappa(\rho_L)$ denote the cuspidal projection, defined by the property

$$\langle \Theta_{\mathrm{KM}}, g\rangle^{\mathrm{Pet}} = \langle \Lambda_{\mathrm{KM}}, g\rangle^{\mathrm{Pet}} \tag{3.1}$$

for all cusp forms $g \in S_\kappa(\rho)$.

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Writing the Fourier expansion

$$\Lambda_{\mathrm{KM}}(\boldsymbol{\tau}) = \sum_t c_\Lambda(t)\ q^t, \qquad \Lambda_{\mathrm{KM},t} \in A^{1,1}(X) \otimes_{\mathbb{C}} S(L)^\vee, \tag{3.2}$$

it follows from [3, Corollary 5.16, Theorem 6.4] that $\Phi(t)$ satisfies the equation

$$\mathrm{dd^c}[\Phi(t)] \ + \ \delta_{Z(t)} \ = \ [c_\Lambda(t) + B(t) \cdot \Omega] \tag{3.3}$$

of currents on X, where

$$B(t) \ := \ -\frac{\deg(Z(t))}{\mathrm{vol}(X, (-\Omega)^p)} \ \in \ S(L)^\vee. \tag{3.4}$$

Recall here that $(-\Omega)^p$ induces a volume form on X.

For future use, we define $\Phi(t) = 0 = B(t)$ if t is not totally positive.

Finally, we define classes $\widehat{Z}_{\mathrm{Br}}(t) \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X) \otimes_{\mathbb{C}} S(L)^\vee$ as follows:

$$\widehat{Z}_{\mathrm{Br}}(t) \ = \ \begin{cases} (Z(t), \Phi(t)), & \text{if } t \gg 0 \\ \widehat{\omega} \otimes \mathrm{ev}_0, & \text{if } t = 0. \\ 0, & \text{otherwise,} \end{cases} \tag{3.5}$$

where $\mathrm{ev}_0 \in S(L)^\vee$ is the functional $\varphi \mapsto \varphi(0)$.

We then form the generating series

$$\widehat{\phi}_{\mathrm{Br}}(\tau) = \sum_t \widehat{Z}_{\mathrm{Br}}(t)\, q^t. \tag{3.6}$$

Theorem 3.1 (Bruinier). *The generating series $\widehat{\phi}_{\mathrm{Br}}(\tau)$ is a (holomorphic) Hilbert modular form of parallel weight $\kappa = p/2 + 1$. More precisely, there are finitely many classes $\widehat{Z}_1, \dots \widehat{Z}_r \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X)$ and holomorphic Hilbert modular forms $f_1, \dots, f_r$ such that $\widehat{Z}_{\mathrm{Br}}(t) = \sum c_{f_i}(t)\, \widehat{Z}_i$ for all $t \in F$.*

Proof. The proof follows the same argument as [3, Theorem 7.1], whose main steps we recall here. Bruinier defines a space $M^!_k(\rho_L)$ of weakly holomorphic forms [3, §4] of a certain 'dual' weight k; each $f \in M^!_k(\rho_L)$ is defined by a finite collection of vectors $c_f(m) \in S(L)^\vee$ indexed by $m \in F$. Applying Bruinier's criterion for the modularity of a generating series (cf. [3, (7.1)]), we need to show that

$$\sum_m c_f(m) \widehat{Z}_{\mathrm{Br}}(m) = 0 \in \widehat{\mathrm{CH}}^1(X) \tag{3.7}$$

for all $f \in M^!_k(\rho_L)$. Given such a form f, let $c_0 = c_f(0)(0)$, and assume $c_0 \in \mathbb{Z}$. By [3, Theorem 6.8], after replacing f by a sufficiently large integer multiple, there exists an analytic meromorphic section Ψ^{an} of $(\omega^{an})^{-c_0}$ such that

$$\mathrm{div}\, \Psi^{an} = \sum_{m \neq 0} c_f(m) \cdot Z(m)^{an} \tag{3.8}$$

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and

$$-\log\|\Psi^{an}\|^2 = \sum_{m\neq 0} c_f(m)\cdot\Phi(m). \tag{3.9}$$

Recall that X is projective; by GAGA and the fact that the $Z(m)$'s are defined over E, there is an E-rational section ψ of ω^{-c_0} and a constant $C\in\mathbb{C}$ such that

$$\operatorname{div}(\psi) = \sum_{m\neq 0} c_f(m)Z(m), \qquad -\log\|\psi^{an}\|^2 = -\log\|\Psi^{an}\|^2 + C. \tag{3.10}$$

Thus,

$$-c_0\cdot\widehat{\omega} \;=\; \widehat{\operatorname{div}}(\psi) = \sum_{m\neq 0} c_f(m)\cdot\widehat{Z}_{\mathrm{Br}}(m) \;+\; (0,C) \;\in\; \widehat{\mathrm{CH}}^1(X). \tag{3.11}$$

However, as in Remark 2.1, the class $(0,C)$ vanishes, and thus, we find

$$\sum_m c_f(m)\widehat{Z}_{\mathrm{Br}}(m) \;=\; c_0\cdot\widehat{\omega} + \sum_{m\neq 0} c_f(m)\widehat{Z}_{\mathrm{Br}}(m) \;=\; 0 \tag{3.12}$$

as required. □

Now we consider the difference

$$\widehat{\phi}_1(\boldsymbol{\tau}) - \widehat{\phi}_{\mathrm{Br}}(\boldsymbol{\tau}) = \sum_t (0, \mathfrak{g}^o(t,\mathbf{v}) - \Phi(t))\, q^t, \tag{3.13}$$

whose terms are classes represented by purely archimedean cycles. Comparing the Green equations (2.28) and (3.3), we have that for $t\neq 0$ and any smooth form η,

$$\mathrm{dd^c}[\mathfrak{g}^o(t,v) - \Phi(t)](\eta) = \int_X (\mathfrak{g}^o(t,\mathbf{v}) - \Phi(t))\,\mathrm{dd^c}\eta = \int_X (\omega(t,\mathbf{v}) - c_\Lambda(t) - B(t)\Omega)\wedge\eta, \tag{3.14}$$

where $\omega(t,\mathbf{v})$ is the t'th q-coefficient of $\Theta_{\mathrm{KM}}(\boldsymbol{\tau})$; in particular, [9, Theorem 1.2.2 (i)] implies that the difference $\mathfrak{g}^o(t,\mathbf{v}) - \Phi(t)$ is smooth on $X(\mathbb{C})$.

Theorem 3.2. *There exists a smooth $S(L)^\vee$-valued function $s(\boldsymbol{\tau},z)$ on $\mathbb{H}_1^d\times X(\mathbb{C})$ such that the following holds.*

(i) *For each fixed $z\in X(\mathbb{C})$, the function $s(\boldsymbol{\tau},z)$ transforms like a Hilbert modular form in $\boldsymbol{\tau}$.*

(ii) *Let*

$$s(\boldsymbol{\tau},z) = \sum_t c_s(t,\mathbf{v},z)\; q^t \tag{3.15}$$

denote its q-expansion in $\boldsymbol{\tau}$; then for each t, we have

$$(0,\mathfrak{g}^o(t,\mathbf{v}) - \Phi(t)) \;=\; (0, c_s(t,\mathbf{v},z)) \;\in\; \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X)\otimes_{\mathbb{C}} S(L)^\vee. \tag{3.16}$$

Combining this theorem with Theorem 3.1, we obtain the following:

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Corollary 3.3. *The generating series $\widehat{\phi}_1(\boldsymbol{\tau})$ is modular, valued in $\widehat{\mathrm{CH}}^1_{\mathbb{C}}(X) \otimes S(L)^\vee$, in the sense of Remark 2.8(i).*

Proof of Theorem 3.2. Recall that the (1,1) form $-\Omega$ is a Kähler form on X. Let $-\Delta_X$ denote the corresponding Laplacian; the eigenvalues of $-\Delta_X$ are non-negative and discrete in $\mathbb{R}_{\geq 0}$, and each eigenspace is finite-dimensional.

Write $\Delta_X = 2(\partial\partial^* + \partial^*\partial)$ and let $L\colon \eta \mapsto -\eta \wedge (-\Omega)$ denote the Lefschetz operator. From the Kähler identities $[L,\partial] = [L,\overline{\partial}] = [L,\Delta_S] = 0$ and $[L,\partial^*] = i\overline{\partial}$, an easy induction argument shows that

$$\partial^* \circ L^k \; = \; L^k \circ \partial^* \; - \; ik\overline{\partial} \circ L^{k-1} \tag{3.17}$$

for $k \geq 1$.

Thus, for a smooth function ϕ on X, we have

$$\begin{aligned}
\Delta_X(\phi) \cdot (-\Omega)^p = \Delta_X \circ L^p(\phi) &= 2\partial\partial^* \circ L^p(\phi) &(3.18)\\
&= \; 2\; \partial \circ \left(L^p \circ \partial^* - ip\overline{\partial} \circ L^{p-1}\right)(\phi) &(3.19)\\
&= \; -2ip\; \partial\overline{\partial}\left(\phi \wedge (-\Omega)^{p-1}\right) &(3.20)\\
&= \; -4\pi p\; \mathrm{dd^c}\left(\phi \wedge (-\Omega)^{p-1}\right); &(3.21)
\end{aligned}$$

note here that $p = \dim_{\mathbb{C}}(X)$.

Consider the Hodge pairing

$$\langle f, g \rangle_{L^2} = \int_X f\,\overline{g}\,(-\Omega)^p = (-1)^p \int_X f\,\overline{g}\,\Omega^p. \tag{3.22}$$

If $\lambda > 0$ and ϕ_λ is a Laplace eigenfunction, we have that for any $t \neq 0$,

$$\begin{aligned}
\langle \mathfrak{g}^o(t,\mathbf{v}) - \Phi(t), \phi_\lambda \rangle_{L^2} &= \lambda^{-1} \langle \mathfrak{g}^o(t,\mathbf{v}) - \Phi(t), -\Delta_X \phi_\lambda \rangle_{L^2} &(3.23)\\
&= (-1)^p \lambda^{-1} \int_X (\mathfrak{g}^o(t,\mathbf{v}) - \Phi(t)) \cdot (\overline{-\Delta_X \phi_\lambda}) \cdot \Omega^p &(3.24)\\
&= (-1)^{p+1} \frac{4\pi p}{\lambda} \int_X (\mathfrak{g}^o(t,\mathbf{v}) - \Phi(t)) \cdot \mathrm{dd^c}\left(\overline{\phi}_\lambda\, \Omega^{p-1}\right) &(3.25)\\
&= (-1)^{p+1} \frac{4\pi p}{\lambda} \int_X (\omega(t,\mathbf{v}) - c_\Lambda(t) - B(t)\Omega) \wedge \overline{\phi}_\lambda \Omega^{p-1}. &(3.26)
\end{aligned}$$

Note that $\int_X \overline{\phi}_\lambda \Omega^p = \langle 1, \phi_\lambda \rangle_{L^2} = 0$, as $\lambda > 0$ and so ϕ_λ is orthogonal to constants; thus, the term involving $B(t)\Omega$ vanishes, and so

$$\langle \mathfrak{g}^o(t,\mathbf{v}) - \Phi(t), \phi_\lambda \rangle_{L^2} = (-1)^{p+1} \frac{4\pi p}{\lambda} \int_X (\omega(t,\mathbf{v}) - c_\Lambda(t)) \wedge \overline{\phi}_\lambda \Omega^{p-1} \tag{3.27}$$

for all $t \neq 0$. This equality also holds for $t = 0$, as both sides of this equation vanish. Indeed, for the left-hand side we have $\mathfrak{g}^o(0,\mathbf{v}) = 0$ (cf. (2.27)), and $\Phi(0) = 0$ by definition;

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on the right-hand side, $c_\Lambda(0) = 0$ as $\Lambda_{\mathrm{KM}}(\boldsymbol{\tau})$ is cuspidal, and the constant term of the Kudla-Millson theta function is given by

$$\omega(0,\mathbf{v}) = \Omega \otimes \mathrm{ev}_0. \tag{3.28}$$

Now define

$$h(\boldsymbol{\tau},z) = (L^*)^{p-1} \circ *(\Theta_{\mathrm{KM}}(\boldsymbol{\tau}) - \Lambda_{\mathrm{KM}}(\boldsymbol{\tau})), \tag{3.29}$$

where $*$ is the Hodge star operator, and L^* is the adjoint of the Lefschetz map L. Then $h(\boldsymbol{\tau},z)$ is smooth, and transforms like a modular form in $\boldsymbol{\tau}$, since both $\Theta_{\mathrm{KM}}(\boldsymbol{\tau})$ and $\Lambda_{\mathrm{KM}}(\boldsymbol{\tau})$ do; writing its Fourier expansion

$$h(\boldsymbol{\tau},z) \;=\; \sum_t c_h(t,\mathbf{v},z)\, q^t, \tag{3.30}$$

we have

$$\langle c_h(t,\mathbf{v},z),\phi\rangle_{L^2} = (-1)^{p-1} \int_X (\omega(t,\mathbf{v}) - c_\Lambda(t)) \wedge \overline{\phi}\,\Omega^{p-1} \tag{3.31}$$

for any smooth function ϕ.

Note that for any integer N and L^2 normalized eigenfunction ϕ_λ with $\lambda \neq 0$,

$$|\langle h,\phi_\lambda\rangle_{L^2}| = \lambda^{-N} |\langle -\Delta_X^N(h),\phi_\lambda\rangle| \le \lambda^{-N} \| -\Delta_X^N(h)\|_{L^2}^2. \tag{3.32}$$

Choose an orthonormal basis $\{\phi_\lambda\}$ of $L^2(X)$ consisting of eigenfunctions, and consider the sum

$$s(\tau,z) = 4\pi p \sum_{\lambda>0} \lambda^{-1} \langle h,\phi_\lambda\rangle_{L^2}\, \phi_\lambda(z). \tag{3.33}$$

By Weyl's law, there are positive constants C_1 and C_2 such that

$$\#\{\lambda \mid \lambda < x\} \sim x^{C_1} \tag{3.34}$$

and $\|\phi_\lambda\|_{L^\infty} = O(\lambda^{C_2})$. These facts imply that there exists an integer $N \gg 0$ such that

$$C_N := \sum_{\lambda>0} \lambda^{-1-N} \|\phi_\lambda\|_{L^\infty} < \infty.$$

For such fixed N, we then have the estimate

$$\sum_{\lambda>0} \lambda^{-1} |\langle h,\phi_\lambda\rangle_{L^2}\, \phi_\lambda(z)| \le \| -\Delta_X^N(h)\|_{L^2}^2 \sum_{\lambda>0} \lambda^{-1-N} \|\phi_\lambda\|_{L^\infty} = \| -\Delta_X^N(h)\|_{L^2}^2\, C_N < \infty.$$

In particular, the sum (3.33) is absolutely convergent, locally uniformly in $\boldsymbol{\tau}$ and z, and hence defines a smooth function in $(\boldsymbol{\tau},z)$.

Writing its Fourier expanison as

$$s(\boldsymbol{\tau},z) = \sum_t c_s(t,\mathbf{v},z) q^t, \tag{3.35}$$

we have

$$\langle c_s(t,\mathbf{v},z),\phi_\lambda\rangle_{L^2} = \langle \mathfrak{g}^o(t,\mathbf{v}) - \Phi(t),\phi_\lambda\rangle_{L^2} \tag{3.36}$$

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for any eigenfunction ϕ_λ with $\lambda \neq 0$. Thus, $c_s(t,\mathbf{v},z)$ and $g^o(t,\mathbf{v}) - \Phi(t)$ differ by a function that is constant in z; as $(0,1) = 0 \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X)$, we have

$$(0,\mathfrak{g}^o(t,\mathbf{v}) - \Phi(t)) = (0,c_s(t,\mathbf{v},z)) \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X) \otimes_{\mathbb{C}} S(L)^\vee, \tag{3.37}$$

which concludes the proof of the theorem. □

4. Decomposing Green currents

We now suppose $n > 1$ and fix $T_2 \in \mathrm{Sym}_{n-1}(F)$.

The aim of this section is to establish a decomposition $\widehat{Z}(T,\mathbf{v}) = \widehat{A}(T,\mathbf{v}) + \widehat{B}(T,\mathbf{v})$, where $T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)$. Our first step is to decompose Green forms in a useful way; the result can be seen as an extension of the star product formula [8, Theorem 4.10] to the degenerate case.

Let $x = (x_1,\ldots,x_n) \in (V_1)^n = (V \otimes_{F,\sigma_1} \mathbb{R})^n$ and set $y = (x_2,\ldots,x_n) \in V_1^{n-1}$. By [8, Proposition 2.6. (a)], we may decompose

$$\mathfrak{g}^o(x,\rho) = \int_1^\infty \nu^o(\sqrt{t}x_1) \wedge \varphi^o_{\mathrm{KM}}(\sqrt{t}y)\, \frac{dt}{t^{1+\rho}} + \int_1^\infty \varphi^o_{\mathrm{KM}}(\sqrt{t}x_1) \wedge \nu^o(\sqrt{t}y)\, \frac{dt}{t^{1+\rho}} \tag{4.1}$$

for $Re(\rho) \gg 0$.

By the transgression formula (2.24), we may rewrite the second term in (4.1) as

$$\begin{aligned} &\int_1^\infty \varphi^o_{\mathrm{KM}}(\sqrt{t}x_1) \wedge \nu^o(\sqrt{t}y)\, \frac{dt}{t^{1+\rho}} \\ &\quad = \int_1^\infty \left(\int_1^t \frac{\partial}{\partial u} \varphi^o_{\mathrm{KM}}(\sqrt{u}x_1)\, du \right) \wedge \nu^o(\sqrt{t}y)\, \frac{dt}{t^{1+\rho}} + \varphi^o_{\mathrm{KM}}(x_1) \wedge \int_1^\infty \nu^o(\sqrt{t}y) \frac{dt}{t^{1+\rho}} \\ &\quad = \int_1^\infty \left(\int_1^t -\mathrm{dd^c}\nu^o(\sqrt{u}x_1)\, \frac{du}{u} \right) \wedge \nu^o(\sqrt{t}y)\, \frac{dt}{t^{1+\rho}} + \varphi^o_{\mathrm{KM}}(x_1) \wedge \mathfrak{g}^o(y,\rho). \end{aligned} \tag{4.2}$$

For $t > 1$, define smooth forms

$$\alpha_t(x_1,y) := \int_1^t \overline{\partial} \nu^o(\sqrt{u}x_1)\, \frac{du}{u} \wedge \nu^o(\sqrt{t}y) \tag{4.3}$$

and

$$\beta_t(x_1,y) := \int_1^t \nu^o(\sqrt{u}x_1)\, \frac{du}{u} \wedge \partial \nu^o(\sqrt{t}y) \tag{4.4}$$

so that

$$\begin{aligned} (4.2) = \frac{i}{2\pi} \int_1^\infty \partial \alpha_t(x_1,y) + \overline{\partial} \beta_t(x_1,y)\, \frac{dt}{t^{1+\rho}} \;-\; \int_1^\infty \left[\int_1^t \nu^o(\sqrt{u}x_1)\, \frac{du}{u} \right] \wedge \mathrm{dd^c} \nu^o(\sqrt{t}y)\, \frac{dt}{t^{1+\rho}} \\ + \varphi^o_{\mathrm{KM}}(x_1) \wedge \mathfrak{g}^o(y,\rho). \end{aligned} \tag{4.5}$$

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Finally, we consider the second integral above; as $Re(\rho)$ is large, we may interchange the order of integration and obtain

$$\begin{aligned}
&\int_1^{\infty}\left(\int_1^{t}\nu^o(\sqrt{u}x_1)\,\frac{du}{u}\right)\wedge \mathrm{dd^c}\nu^o(\sqrt{t}y)\,\frac{dt}{t^{1+\rho}}\\
&\quad=\int_1^{\infty}\nu^o(\sqrt{u}x_1)\wedge\left(\int_u^{\infty}\mathrm{dd^c}\nu^o(\sqrt{t}y)\frac{dt}{t^{1+\rho}}\right)\frac{du}{u}\\
&\quad=\int_1^{\infty}\nu^o(\sqrt{u}x_1)\wedge\left(\int_u^{\infty}-\frac{\partial}{\partial t}\varphi^o_{\mathrm{KM}}(\sqrt{t}y)\frac{dt}{t^{\rho}}\right)\frac{du}{u}\\
&\quad=\int_1^{\infty}\nu^o(\sqrt{u}x_1)\wedge\varphi^o_{\mathrm{KM}}(\sqrt{u}y)\,\frac{du}{u^{1+\rho}}-\rho\int_1^{\infty}\nu^o(\sqrt{u}x_1)\wedge\left(\int_u^{\infty}\varphi^o_{\mathrm{KM}}(\sqrt{t}y)\frac{dt}{t^{1+\rho}}\right)\frac{du}{u}.
\end{aligned}\tag{4.6}$$

Note that the first term in the last line above coincides with the first term in (4.1). Combining these computations, it follows that

$$\begin{aligned}
\mathfrak{g}^o(x,\rho)=\varphi^o_{\mathrm{KM}}(x_1)\wedge\mathfrak{g}^o(y,\rho)+\frac{i}{2\pi}\int_1^{\infty}\partial\alpha_t(x_1,y)+\overline{\partial}\beta_t(x_1,y)\,\frac{dt}{t^{1+\rho}}\\
+\rho\int_1^{\infty}\nu^o(\sqrt{u}x_1)\wedge\left(\int_u^{\infty}\varphi^o_{\mathrm{KM}}(\sqrt{t}y)\frac{dt}{t^{1+\rho}}\right)\frac{du}{u}.
\end{aligned}\tag{4.7}$$

This identity holds for arbitrary $x=(x_1,y)\in V_1^n$ and $Re(\rho)\gg 0$, and is an identity of smooth differential forms on $\mathbb{D}$.

To continue, we view the above line as an identity of currents, and consider meromorphic continuation.[5] Note that (as currents)

$$\begin{aligned}
&\rho\int_1^{\infty}\nu^o(\sqrt{u}x_1)\wedge\left(\int_u^{\infty}\varphi^o_{\mathrm{KM}}(\sqrt{t}y)\frac{dt}{t^{1+\rho}}\right)\frac{du}{u}\\
&\quad=\rho\int_1^{\infty}\nu^o(\sqrt{u}x_1)\wedge\int_u^{\infty}\left(\varphi^o_{\mathrm{KM}}(\sqrt{t}y)-\delta_{\mathbb{D}^+_y}\wedge\Omega^{n-1-r(y)}\right)\frac{dt}{t^{1+\rho}}\frac{du}{u}\\
&\qquad+\int_1^{\infty}\nu^o(\sqrt{u}x_1)\wedge\delta_{\mathbb{D}^+_y}\wedge\Omega^{n-1-r(y)}\;\frac{du}{u^{1+\rho}}
\end{aligned}\tag{4.8}$$

where $r(y)=\dim\operatorname{span}(y)$. The first term vanishes at $\rho=0$; indeed, the double integral in the first term is holomorphic at $\rho=0$, as can easily seen by by Bismut's asymptotic[6] [1, Theorem 3.2]

[5] More precisely, we mean that for every smooth form α, the function $[\mathfrak{g}^o(x,\rho)](\alpha)=\int_X\mathfrak{g}^o(x,\rho)\wedge\alpha$ admits a meromorphic continuation in ρ, such that the Laurent coefficients are continuous in α in the sense of currents.

[6] This asymptotic is meant in the sense of distributions. To be more precise, we say that a family of currents T_t, parametrized by $t>0$, is $O(t^{-\frac{1}{2}})$ if the following holds. Let $k>0$ be any integer, and let $\|\cdot\|$ denote a norm on the space of smooth differential forms on X such that for a sequence η_n, we have $\|\eta_n\|\to 0$ if and only if η_n, together with all partial derivatives up to order k, converge uniformly to 0 on X. Then there exists a constant $C=C_{\|\cdot\|}$ such that

$$|T_t(\eta)|\le Ct^{-\frac{1}{2}}\,\|\eta\|$$

for all smooth forms η.

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$$\varphi^o_{\mathrm{KM}}(\sqrt{t}y) - \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)} \;=\; O(t^{-1/2}) \tag{4.9}$$

as $t \to \infty$.

Next, let

$$\alpha(x_1, y; \rho) \;:=\; \int_1^\infty \alpha_t(x_1, y) \frac{dt}{t^{1+\rho}}, \qquad \beta(x_1, y; \rho) \;:=\; \int_1^\infty \beta_t(x, y) \frac{dt}{t^{1+\rho}}. \tag{4.10}$$

A straightforward modification of the proof of [8, Proposition 2.12.(iii)] can be used to show that $\alpha(x_1, y; \rho)$ and $\beta(x_1, y; \rho)$ have meromorphic extensions, as currents, to a neighbourhood of $\rho = 0$. We denote the constant terms in the Laurent expansion at $\rho = 0$ by $\alpha(x_1, y)$ and $\beta(x_1, y)$ respectively. Thus, as currents on $\mathbb{D}$, we have

$$\begin{aligned} \mathfrak{g}^o(x_1, y) &= \varphi^o_{\mathrm{KM}}(x_1) \wedge \mathfrak{g}^o(y) + d\alpha(x_1, y) + d^c\beta(x_1, y) \\ &\quad + CT_{\rho=0} \int_1^\infty \nu^o(\sqrt{u}x_1) \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)} \; \frac{du}{u^{1+\rho}} \end{aligned} \tag{4.11}$$

for all $x_1 \in V_1$ and $y \in (V_1)^{n-1}$.

As a final observation, note that if $x_1 \in \operatorname{span}(y)$, then $\nu^o(\sqrt{u}x_1) \wedge \delta_{\mathbb{D}_y^+} = \delta_{\mathbb{D}_y^+}$; see [8, Lemma 2.4]. Thus,

$$\begin{aligned} \gamma(x_1, y) &:= CT_{\rho=0} \int_1^\infty \nu^o(\sqrt{u}x_1) \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)} \; \frac{du}{u^{1+\rho}} \\ &= \begin{cases} \mathfrak{g}^o(x_1) \wedge \delta_{\mathbb{D}_y^+} \wedge \Omega^{n-1-r(y)}, & \text{if } x_1 \notin \operatorname{span}(y) \\ 0, & \text{if } x_1 \in \operatorname{span}(y). \end{cases} \end{aligned} \tag{4.12}$$

In the case that the components of $x = (x_1, y) = (x_1, \ldots, x_n)$ are linearly independent, we recover the star product formula from [8, Theorem 2.16].

Now we discuss a decomposition of the global Green current $\mathfrak{g}^o(T, \mathbf{v})$, for $\mathbf{v} \in \operatorname{Sym}_n(F_{\mathbb{R}})_{\gg 0}$. Write

$$v := \sigma_1(\mathbf{v}) = \begin{pmatrix} v_1 & v_{12} \\ v_{12}' & v_2 \end{pmatrix} \tag{4.13}$$

with $v_1 \in \mathbb{R}_{>0}$ and $v_{12} \in M_{1,n-1}(\mathbb{R})$; recall that $\sigma_1 \colon F \to \mathbb{R}$ is the distinguished real embedding. Set

$$v_2^* := v_2 - v_{12}' v_{12}/v_1 \in \operatorname{Sym}_{n-1}(\mathbb{R})_{>0}, \tag{4.14}$$

and fix a matrix $a_2^* \in \mathrm{GL}_{n-1}(\mathbb{R})$ such that $v_2^* = a_2^* \cdot (a_2^*)'$.

Proposition 4.1. *Let $T \in \operatorname{Sym}_n(F)$ and $\mathbf{v} \in \operatorname{Sym}_n(F_{\mathbb{R}})_{\gg 0}$ as above, and define $S(L)^\vee$-valued currents $\mathfrak{a}(T, \mathbf{v})$ and $\mathfrak{b}(T, \mathbf{v})$ on X by the formulas*

$$\mathfrak{a}(T, \mathbf{v})(\varphi) := \sum_{\mathbf{x} \in \Omega(T)} \varphi(\mathbf{x}) \gamma(\sqrt{v_1} x_1, y), \tag{4.15}$$

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where we have written $\sigma_1(\mathbf{x}) = (x_1, y) \in V_1 \oplus (V_1)^{n-1}$, *and*

$$\mathfrak{b}(T,\mathbf{v})(\varphi) = \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}\right) \wedge \mathfrak{g}^o(ya_2^*). \tag{4.16}$$

Then

$$\mathfrak{g}^o(T,\mathbf{v})(\varphi) \equiv \mathfrak{a}(T,\mathbf{v})(\varphi) + \mathfrak{b}(T,\mathbf{v})(\varphi) \quad (\mathrm{mod}\ \mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}). \tag{4.17}$$

Proof. First, the fact that the sums defining $\mathfrak{a}(T,\mathbf{v})$ and $\mathfrak{b}(T,\mathbf{v})$ converge to currents on X follows from the same argument as [8, Proposition 4.3].

Now recall that

$$\mathfrak{g}^o(T,\mathbf{v})(\varphi) = \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\mathfrak{g}^o(xa), \tag{4.18}$$

where $x = \sigma_1(\mathbf{x})$, and $a \in \mathrm{GL}_n(\mathbb{R})$ is any matrix satisfying $v = aa'$. Note that

$$v = \begin{pmatrix} v_1 & v_{12} \\ v_{12}' & v_2 \end{pmatrix} = \theta \begin{pmatrix} v_1 & \\ & v_2^* \end{pmatrix} \theta', \qquad \text{where } \theta = \begin{pmatrix} 1 & \\ v_{12}'/v_1 & 1_{n-1} \end{pmatrix}. \tag{4.19}$$

Thus, we may take

$$a = \theta \cdot \begin{pmatrix} \sqrt{v_1} & \\ & a_2^* \end{pmatrix}, \tag{4.20}$$

and so, applying (4.11) , we find

$$\begin{aligned} \mathfrak{g}^o(T,\mathbf{v})(\varphi) &= \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\mathfrak{g}^o\left((x_1,y)\theta\begin{pmatrix} \sqrt{v_1} & \\ & a_2^* \end{pmatrix}\right) \qquad x = (x_1,y) \\ &= \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\mathfrak{g}^o\left(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}, ya_2^*\right) \\ &= \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\Big(\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}\right) \wedge \mathfrak{g}^o(ya_2^*) + \partial\alpha(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}, ya_2^*) \\ &\qquad + \overline{\partial}\beta(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}, ya_2^*) + \gamma(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}, ya_2^*)\Big). \end{aligned} \tag{4.21}$$

Again, an argument as in [8, Proposition 4.3] shows that the sums

$$\eta_1 := \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\alpha(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}, ya_2^*) \tag{4.22}$$

and

$$\eta_2 := \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\beta(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}, ya_2^*) \tag{4.23}$$

converge to Γ-invariant currents on $\mathbb{D}$, and hence define currents on X. Moreover, it follows easily from the definitions that

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$$\gamma(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}, ya_2^*) = \gamma(\sqrt{v_1}x_1, y). \tag{4.24}$$

Thus, we find

$$\mathfrak{g}^o(T,\mathbf{v})(\varphi) = \mathfrak{a}(T,\mathbf{v})(\varphi) + \mathfrak{b}(T,\mathbf{v})(\varphi) + \partial\eta_1 + \overline{\partial}\eta_2, \tag{4.25}$$

as required. □

Next, we define an $S(L^n)^\vee$-valued current $\psi(T,\mathbf{v})$ as follows. For $\mathbf{x}\in\Omega(T)$, write $\sigma_1(\mathbf{x}) = x = (x_1,y)\in V_1\oplus V_1^{n-1}$ as above; then

$$\psi(T,\mathbf{v})(\varphi) := \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}(\sqrt{v_1}x_1)\wedge\delta_{\mathbb{D}^+_y}\wedge\Omega^{n-1-r(y)} \tag{4.26}$$

defines a Γ-equivariant current on $\mathbb{D}^+$, and hence descends to a current (also denoted $\psi(T,\mathbf{v})$) on $X(\mathbb{C})$.

Lemma 4.2.

(i) *Let $\omega(T,\mathbf{v})$ be the Tth coefficient of the Kudla-Millson theta function, as in (2.30); then*

$$\mathrm{dd^c}\mathfrak{b}(T,\mathbf{v}) = \omega(T,\mathbf{v}) - \psi(T,\mathbf{v}). \tag{4.27}$$

(ii) *We have*

$$\mathrm{dd^c}\,\mathfrak{a}(T,\mathbf{v}) + \delta_{Z(T)^\natural(\mathbb{C})}\wedge\Omega^{n-r(T)} = \psi(T,\mathbf{v}), \tag{4.28}$$

where $r(T) = \mathrm{rank}(T)$.

Proof. With $v = \sigma_1(\mathbf{v})$ and taking $a = \theta\cdot\begin{pmatrix}\sqrt{v_1} & \\ & a_2^*\end{pmatrix}$ as (4.20), we have

$$\begin{aligned}\omega(T,\mathbf{v})(\varphi) &= \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}(xa)\\ &= \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y v_{12}'}{\sqrt{v_1}},\ ya_2^*\right)\\ &= \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y v_{12}'}{\sqrt{v_1}}\right)\wedge\varphi^o_{\mathrm{KM}}(ya_2^*)\end{aligned} \tag{4.29}$$

for $\varphi\in S(L^n)$, where the last line follows from [14, Theorem 5.2(i)]. Therefore,

$$\mathrm{dd^c}\mathfrak{b}(T,\mathbf{v})(\varphi) = \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y v_{12}'}{\sqrt{v_1}}\right)\wedge\mathrm{dd^c}\mathfrak{g}^o(ya_2^*) \tag{4.30}$$

$$= \sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y v_{12}'}{\sqrt{v_1}}\right)\wedge\left\{-\delta_{\mathbb{D}^+_y}\wedge\Omega^{n-1-r(y)} + \varphi^o_{\mathrm{KM}}(ya_2^*)\right\} \tag{4.31}$$

$$= -\sum_{\mathbf{x}\in\Omega(T)} \varphi(\mathbf{x})\,\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y v_{12}'}{\sqrt{v_1}}\right)\wedge\delta_{\mathbb{D}^+_y}\wedge\Omega^{n-1-r(y)} + \omega(T,\mathbf{v})(\varphi). \tag{4.32}$$

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For $v \in V_1$, the restriction $\varphi^o_{\mathrm{KM}}(v) \wedge \delta_{\mathbb{D}^+_y}$ depends only on the orthogonal projection of v onto $\mathrm{span}(y)^{\perp}$; see, for example, [8, Lemma 2.4]. In particular,

$$\varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y\,v'_{12}}{\sqrt{v_1}}\right) \wedge \delta_{\mathbb{D}^+_y} = \varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1\right) \wedge \delta_{\mathbb{D}^+_y}. \tag{4.33}$$

The first part of the lemma follows upon applying the definition of $\gamma(T,\mathbf{v})$ in (4.26).

The second part then follows from the first, together with Proposition 4.1 and (2.32). □

We finally arrived at the promised decomposition of $\widehat{Z}(T,\mathbf{v})$. Recall that in defining the cycle $\widehat{Z}(T,\mathbf{v})$ in Section 2.5, we fixed a representative (Z_0,g_0) for $\widehat{\omega}^{n-r(T)}$ such that Z_0 intersects $Z(T)$ properly. By the previous proposition,

$$\mathrm{dd}^{\mathrm{c}}\left(\mathfrak{a}(T,\mathbf{v}) + g_0 \wedge \delta_{Z(T)^{\natural}(\mathbb{C})}\right) + \delta_{Z(T)^{\natural}\cap Z_0(\mathbb{C})} \;=\; \psi(T,\mathbf{v}); \tag{4.34}$$

we then obtain classes in $\widehat{\mathrm{CH}}{}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}}) \otimes_{\mathbb{C}} S(L^n)$ by setting

$$\widehat{A}(T,\mathbf{v}) \;:=\; \left(Z(T)^{\natural}\cdot Z_0, [\psi(T,\mathbf{v}), \mathfrak{a}(T,\mathbf{v}) + g_0 \wedge \delta_{Z(T)^{\natural}(\mathbb{C})}]\right) \tag{4.35}$$

and

$$\widehat{B}(T,\mathbf{v}) \;:=\; (0, [\omega(T,\mathbf{v}) - \psi(T,\mathbf{v}), \mathfrak{b}(T,\mathbf{v})]), \tag{4.36}$$

so that

$$\widehat{Z}(T,\mathbf{v}) = \widehat{A}(T,\mathbf{v}) + \widehat{B}(T,\mathbf{v}) \;\in\; \widehat{\mathrm{CH}}{}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}}) \otimes_{\mathbb{C}} S(L)^{\vee}. \tag{4.37}$$

Remark 4.3. Suppose $T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)$ as above; if T_2 is not totally positive semidefinite, then $\mathbb{D}^+_y = \emptyset$ for any $\mathbf{y} \in \Omega(T_2)$, and hence, $\widehat{A}(T,\mathbf{v}) = 0$.

5. Modularity I

In this section, we establish the modularity of the generating series

$$\widehat{\phi}_B(\boldsymbol{\tau}) = \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{B}(T,\mathbf{v})\, q^T. \tag{5.1}$$

Note that

$$\widehat{B}(T,\mathbf{v}) = (0, [\mathrm{dd}^{\mathrm{c}}\mathfrak{b}(T,\mathbf{v}), \mathfrak{b}(T,\mathbf{v})]) = a(\mathfrak{b}(T,v)); \tag{5.2}$$

thus, in light of Definition 2.7, it suffices to establish the following theorem.

Theorem 5.1. *Fix $T_2 \in \mathrm{Sym}_{n-1}(F)$, and consider the generating series*

$$\xi(\boldsymbol{\tau}) = \sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \mathfrak{b}(T,\mathbf{v})\, q^T, \tag{5.3}$$

Then $\xi(\boldsymbol{\tau})$ is an element of $A_{\kappa,T_2}(\rho_L^{\vee}; D^*(X))$ with $\kappa = p/2+1$; see Definition 2.6.

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Proof. We begin by showing the convergence of the series (5.3). By definition,

$$\sum_{T=\begin{pmatrix} T_1 & T_{12} \\ T_{12}' & T_2 \end{pmatrix}} \mathfrak{b}(T,\mathbf{v})(\varphi)\, q^T = \sum_T \sum_{(\mathbf{x}_1,\mathbf{y})\in\Omega(T)} \varphi(\mathbf{x}_1,\mathbf{y})\, \varphi^o_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}\right) \wedge \mathfrak{g}^o(ya_2^*)\; q^T, \tag{5.4}$$

where $x_1 = \sigma_1(\mathbf{x}_1)$ and $y = \sigma_1(\mathbf{y})$, and here we are working with Γ-equivariant currents on $\mathbb{D}^+$.

For $v \in V_1$, consider the normalized Kudla-Millson form

$$\varphi_{\mathrm{KM}}(v) \;:=\; e^{-2\pi\langle v,v\rangle}\, \varphi^o_{\mathrm{KM}}(v), \tag{5.5}$$

which is a Schwartz form on V_1, valued in closed forms on $\mathbb{D}^+$. More precisely, fix an integer k, a relatively compact open subset $U \subset \mathbb{D}^+$, and an algebra seminorm $\|\cdot\|_{k,\overline{U}}$ measuring uniform convergence of all derivatives of order $\le k$ on the space of smooth forms supported on $\overline{U}$. Then there exists a totally positive definite quadratic form Q_U on V such that

$$\|\varphi_{\mathrm{KM}}(v)\|_{k,\overline{U}} \ll e^{-Q_U(v)}, \tag{5.6}$$

where the implied constant depends on k and $\overline{U}$, and we abuse notation and write Q_U for the induced positive definite quadratic form on V_1. Similarly, for $y \in V_1^{n-1}$, write

$$\mathfrak{g}(y) = e^{-2\pi\sum\langle y_i,y_i\rangle}\mathfrak{g}^o(y). \tag{5.7}$$

If $\mathbb{D}^+_y \cap \overline{U} = \emptyset$, then $\mathfrak{g}(y)$ is smooth on U, and the form Q_U may be chosen so that

$$\|\mathfrak{g}(y)\|_{k,\overline{U}} \;\ll\; e^{-\sum_{i=1}^{n-1} Q_U(y_i)}, \qquad y = (y_1,\dots,y_{n-1}) \in V_1^{n-1}; \tag{5.8}$$

see [8, §2.1.5].

Finally, for the remaining real embeddings $\sigma_2,\dots\sigma_d$, let $\varphi_{\infty_i} \in S(V_i^n)$ denote the standard Gaussian on the positive definite space $V_i = V \otimes_{F,\sigma_i} \mathbb{R}$, defined by $\varphi_{\infty_i}(x_1,\dots,x_n) = e^{-2\pi\sum\langle x_i,x_i\rangle}$. Then a brief calculation gives

$$\xi(\boldsymbol{\tau})(\varphi) = \sum_T \sum_{(\mathbf{x}_1,\mathbf{y})\in\Omega(T)} \varphi(\mathbf{x}_1,\mathbf{y})\varphi_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}\right) \wedge \mathfrak{g}(ya_2^*)\cdot \prod_{i=2}^d \varphi_{\infty_i}(\sigma_i(\mathbf{x}_1,\mathbf{y})a_i)\; e(T\mathbf{u}), \tag{5.9}$$

where we have chosen matrices $a_i \in GL_n(\mathbb{R})$ for $i = 2,\dots,d$, such that $\sigma_i(\mathbf{v}) = a_i \cdot a_i'$.

Let

$$S_1 := \{\mathbf{y} \in (L')^{n-1} \mid \langle \mathbf{y},\mathbf{y}\rangle = T_2 \text{ and } \mathbb{D}^+_y \cap \overline{U} \neq \emptyset\} \tag{5.10}$$

and

$$S_2 := \{\mathbf{y} \in (L')^{n-1} \mid \langle \mathbf{y},\mathbf{y}\rangle = T_2 \text{ and } \mathbb{D}^+_y \cap \overline{U} = \emptyset\}. \tag{5.11}$$

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We claim that S_1 is finite. To see this, first note that for a point $z \in \mathbb{D}^+ \subset \mathbb{P}(V_{1,\mathbb{C}})$, there is an associated positive definite form $Q_z$on V_1, determined by the formulas

$$Q_z(v) = \begin{cases} -Q(v), & \text{if } v \in \mathrm{Re}(z) + \mathrm{Im}(z) \\ Q(v), & \text{if } v \in (\mathrm{Re}(z) + \mathrm{Im}(z))^\perp \end{cases}$$

for $v \in V_1$. Moreover, this quadratic form varies smoothly in z, and we have the equivalence

$$z \in \mathbb{D}_y^+ \iff Q_z(y_i) = Q(y_i) \text{ for all } i = 1,\dots,n-1$$

for $y = (y_1,\dots,y_{n-1}) \in V_1^{n-1}$.

Without loss of generality, we may suppose that Q_U is chosen so that $Q_U(v) < Q_z(v)$ for all $v \in V_1$ and $z \in \overline{U}$ on V_1, and that $Q_U(v) < \frac{1}{2}\langle v,v\rangle$ on V_i for $i > 1$. Then

$$S_1 \subset \left\{ \mathbf{y} = (\mathbf{y}_1,\dots,\mathbf{y}_{n-1}) \in (L')^{n-1} \mid Q_U(\mathbf{y}_i) \ll tr(T_2) \text{ for } i = 1,\dots,n-1 \right\}.$$

The latter set is finite, as Q_U is totally positive definite and L' is a lattice.

Using the estimates (5.6) and (5.8), and standard arguments for convergence of theta series, it follows that the sum

$$\sum_{T = \left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \sum_{\substack{(\mathbf{x}_1,\mathbf{y}) \in \Omega(T) \\ \mathbf{y} \in S_2}} \varphi(\mathbf{x}_1,\mathbf{y}) \varphi_{\mathrm{KM}}\left(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}} \right) \wedge \mathfrak{g}(y a_2^*) \prod_{i=2}^{d} \varphi_{\infty_i}(\sigma_i(\mathbf{x}_1,\mathbf{y}) a_i)\ e(T\mathbf{u}) \tag{5.12}$$

converges absolutely to a smooth form on $\mathbb{H}_n^d \times U$. The (finitely many) remaining terms, corresponding to $\mathbf{y} \in S_1$, can be written as

$$\sum_{\mathbf{y} \in S_1} f_{\mathbf{y}}(\boldsymbol{\tau})(\varphi) \wedge \mathfrak{g}(y a_2^*), \tag{5.13}$$

where, for any $\mathbf{y} \in V^{n-1}$ and $\varphi \in S(L^n)$, we set

$$f_{\mathbf{y}}(\boldsymbol{\tau})(\varphi) \;=\; \sum_{\mathbf{x}_1 \in V} \varphi(\mathbf{x}_1,\mathbf{y}) \varphi_{\mathrm{KM}}\left(\sqrt{v_1} x_1 + \frac{y \cdot v_{12}'}{\sqrt{v_1}} \right) \prod_{i=2}^{d} \varphi_{\infty_i}(\sigma_i(\mathbf{x}_1,\mathbf{y}) a_i)\ e(T(\mathbf{x}_1,\mathbf{y})\mathbf{u}), \tag{5.14}$$

where $T(\mathbf{x}_1,\mathbf{y}) = \left(\begin{smallmatrix} \langle \mathbf{x}_1,\mathbf{x}_1\rangle & \langle \mathbf{x}_1,\mathbf{y}\rangle \\ \langle \mathbf{x}_1,\mathbf{y}\rangle' & \langle \mathbf{y},\mathbf{y}\rangle \end{smallmatrix}\right)$. Again, the estimate (5.6) shows that the series defining $f_{\mathbf{y}}(\boldsymbol{\tau})$ converges absolutely to a smooth form on $\mathbb{H}_d^n \times \mathbb{D}^+$. Moreover, for a fixed $y \in V_1^{n-1}$ and any compactly supported test form α on $\mathbb{D}^+$, the value of the current $\mathfrak{g}^o(y a_2^*)[\alpha]$ varies smoothly in the entries of a_2^* (this fact follows easily from the discussion in [8, §2.1.4]).

Taken together, the above considerations imply that the series $\xi(\boldsymbol{\tau})(\varphi)$ converges absolutely to a Γ-invariant current on $\mathbb{D}^+$, and therefore descends to a current on X that satisfies part (b) of Definition 2.6 as $\boldsymbol{\tau}$ varies. In addition, this discussion shows that given any test form α, the value of the current $\xi(\boldsymbol{\tau})[\alpha]$ is smooth in $\boldsymbol{\tau}$.

It remains to show that $\xi(\boldsymbol{\tau})$ transforms like a Jacobi modular form (i.e. is invariant under the slash operators (2.41)). Recall that the form φ_{KM} is of weight $p/2 + 1$; more

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precisely, let $\widetilde{U}(1) \subset \widetilde{\mathrm{Sp}}_1(\mathbb{R})$ denote the inverse image of $U(1)$, which admits a genuine character χ whose square is the identity on $U(1)$. Then $\omega(\widetilde{k})\varphi_{\mathrm{KM}} = (\chi(\widetilde{k}))^{p+2}\varphi_{\mathrm{KM}}$ for all $\widetilde{k} \in \widetilde{U}(1)$, where ω is the Weil representation attached to V_1; cf. [14, Theorem 5.2].

To show that $\xi(\boldsymbol{\tau})$ transforms like a Jacobi form, note that (by Vaserstein's theorem [17]), every element of $\widetilde{\Gamma}^J$ can be written as a product of the following elements.

(i) For each $i = 1, \dots, d$, let

$$\widetilde{\epsilon}(i) = (\widetilde{\epsilon}(i))_v \in \prod_{v|\infty} \widetilde{\mathrm{Sp}}_n(F_v) \tag{5.15}$$

be the element whose v'th component is $(\mathrm{Id}, 1)$ if $v \neq \sigma_i$, and $(\mathrm{Id}, -1)$ if $v = \sigma_i$.

(ii) For $\mu, \lambda \in \mathrm{M}_{1,n-1}(\mathcal{O}_F)$, let

$$\gamma_{\lambda,\mu} = \left(\begin{array}{cc|cc} 1 & 0 & 0 & \mu \\ \lambda' & 1_{n-1} & \mu' & 0 \\ \hline 0 & 0 & 1 & -\lambda \\ 0 & 0 & 0 & 1_{n-1} \end{array}\right) \in G^J(\mathcal{O}_F). \tag{5.16}$$

Let $\iota_F(\gamma_{\lambda,\mu}) \in \widetilde{\mathrm{Sp}}_{n,\mathbb{A}}$ denote its image under the splitting (2.39); we choose $\widetilde{\gamma}_{\lambda,\mu} \in \widetilde{\Gamma}^J$ to be the archimedean part of a representative $\iota_F(\gamma_{\lambda,\mu}) = \widetilde{\gamma}_{\lambda,\mu} \cdot \widetilde{\gamma}_f$.

(iii) For $r \in \mathcal{O}_F$, let

$$\gamma_r = \left(\begin{array}{cc|cc} 1 & 0 & r & 0 \\ 0 & 1_{n-1} & 0 & 0 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1_{n-1} \end{array}\right) \in G^J(\mathcal{O}_F), \tag{5.17}$$

and choose an element $\widetilde{\gamma}_r$ as the archimedean part of a representative of $\iota_F(\gamma_r)$, as before.

(iv) Finally, let

$$S = \left(\begin{array}{cc|cc} 0 & 0 & -1 & 0 \\ 0 & 1_{n-1} & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{n-1} \end{array}\right) \tag{5.18}$$

and take $\widetilde{S} \in \widetilde{\Gamma}^J$ to be the archimedean part of a representative of $\iota_F(S)$.

Now, rearranging the absolutely convergent sum (5.12), we may write

$$\xi(\boldsymbol{\tau}) = \sum_{\mathbf{y} \in \Omega(T_2)} f_{\mathbf{y}}(\boldsymbol{\tau}) \wedge \mathfrak{g}(ya_2^*). \tag{5.19}$$

Using the aforementioned generators, a direct computation shows that $\mathbf{v}_2^* = \mathbf{v}_2 - \mathbf{v}_{12}'\mathbf{v}_{12}/\mathbf{v}_1$, viewed as a function on $\mathbb{H}_n^d$, is invariant under the action of $\widetilde{\Gamma}^J$; it therefore suffices to show that for a fixed $\mathbf{y}$, the $S(L^n)^\vee$-valued function $f_{\mathbf{y}}(\boldsymbol{\tau})$ transforms like a Jacobi form.

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It is a straightforward verification to check that $f_{\mathbf{y}}(\boldsymbol{\tau})$ is invariant under the action of $\widetilde{\epsilon}(i)$, $\widetilde{\gamma}_{\lambda,\mu}$, and $\widetilde{\gamma}_r$. For example, the invariance with respect to $\widetilde{\epsilon}(i)$ follows from Remark 2.5.

The element $\widetilde{\gamma}_{\lambda,\mu}^{-1}$ acts on $S(L^n)$ by the formula

$$\rho(\widetilde{\gamma}_{\lambda,\mu}^{-1})(\varphi)(\mathbf{x}_1,\mathbf{y}) \ = \ e\big(2\langle \mathbf{x}_1,\mathbf{y}\rangle\mu' - \langle \mathbf{y}\lambda',\mathbf{y}'\rangle\mu'\big)\,\varphi(\mathbf{x}_1 - \mathbf{y}\lambda',\mathbf{y}), \tag{5.20}$$

and $\gamma_{\lambda,\mu}$ acts on $\mathbb{H}_n^d$ by the formula

$$\gamma_{\lambda,\mu}\cdot\boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{\tau}_1 & \boldsymbol{\tau}_{12}+\boldsymbol{\tau}_1\lambda+\mu \\ \boldsymbol{\tau}_{12}'+\boldsymbol{\tau}_1\lambda'+\mu' & \boldsymbol{\tau}_2+(\lambda'\cdot\boldsymbol{\tau}_{12}+\boldsymbol{\tau}_{12}'\cdot\lambda)+\mu'\cdot\lambda \end{pmatrix}, \tag{5.21}$$

where $\boldsymbol{\tau} = \left(\begin{smallmatrix} \boldsymbol{\tau}_1 & \boldsymbol{\tau}_{12} \\ \boldsymbol{\tau}_{12}' & \boldsymbol{\tau}_2 \end{smallmatrix}\right)$. Moreover, writing $\widetilde{\gamma}_{\lambda,\mu} = (\gamma_{\lambda,\mu},(\phi_v))_v$ as in Section 2.6, we have $\prod \phi_v(\tau) = 1$. For $\mathbf{x}_1 \in V$ and $\mathbf{y} \in V^{n-1}$, a direct computation gives

$$\operatorname{tr}\Big(T(\mathbf{x}_1,\mathbf{y})\cdot\operatorname{Re}(\gamma_{\lambda,\mu}\cdot\boldsymbol{\tau})\Big) = \operatorname{tr}\Big(T(\mathbf{x}_1+\mathbf{y}\lambda',\mathbf{y})\mathbf{u}\Big) + 2\langle\mathbf{x}_1,\mathbf{y}\rangle\mu' + \langle\mathbf{y}\lambda',\mathbf{y}\rangle\mu'; \tag{5.22}$$

therefore, applying the above identity and the change of variables $\mathbf{x}_1 \mapsto \mathbf{x}_1 - \mathbf{y}\cdot\lambda'$, we find

$$\begin{aligned} f_{\mathbf{y}}(\gamma_{\lambda,\mu}\cdot\boldsymbol{\tau})(\varphi) &= \sum_{\mathbf{x}_1\in V}\varphi(\mathbf{x}_1,\mathbf{y})\varphi_{\mathrm{KM}}\left(\sqrt{v_1}(x_1+y\cdot\lambda') + \frac{y\cdot v_{12}'}{\sqrt{v_1}}\right) \\ &\qquad\times\left\{\prod_{i=2}^d \varphi_{\infty_i}\left(\sigma_i(\mathbf{x}_1,\mathbf{y})\left(\begin{smallmatrix} 1 & \\ \lambda' & 1\end{smallmatrix}\right)a_i\right)\right\} e\Big(T(\mathbf{x}_1,\mathbf{y})\operatorname{Re}(\gamma_{\lambda,\mu}\boldsymbol{\tau})\Big) \qquad (5.23)\\ &= \sum_{\mathbf{x}_1\in V}\left\{\varphi(\mathbf{x}_1-\mathbf{y}\boldsymbol{\lambda}')\,e(2\langle\mathbf{x}_1,\mathbf{y}\rangle\mu' - \langle\mathbf{y}\lambda',\mathbf{y}\rangle\mu')\right\}\varphi_{\mathrm{KM}}\left(\sqrt{v_1}x_1 + \frac{y\cdot v_{12}'}{\sqrt{v_1}}\right) \\ &\qquad\times\prod_{i=2}^d \varphi_{\infty_i}(\sigma_i(\mathbf{x}_1,\mathbf{y})a_i)\ e(T(\mathbf{x}_1,\mathbf{y})\mathbf{u}) \qquad (5.24)\\ &= f_{\mathbf{y}}(\boldsymbol{\tau})\,(\rho(\widetilde{\gamma}_{\lambda,\mu})\varphi) \qquad (5.25) \end{aligned}$$

as required.

Similarly, for $r \in \mathcal{O}_F$, the element $\tilde{\gamma}_r^{-1} = \tilde{\gamma}_{-r}$ acts on $S(L^n)$ by the formula

$$\rho(\tilde{\gamma}_{-r})(\varphi)(\mathbf{x}_1,\mathbf{y}) = e\left(T(\mathbf{x}_1,\mathbf{y})\left(\begin{smallmatrix} -r & \\ & 0\end{smallmatrix}\right)\right)\varphi(\mathbf{x}_1,\mathbf{y}).$$

However, γ_r acts on $\mathbb{H}_n^d$ by

$$\gamma_r\cdot\tau = \tau + \left(\begin{smallmatrix} r & \\ & 0_{n-1}\end{smallmatrix}\right).$$

The invariance of $f_{\mathbf{y}}(\tau)$ under the action of $\tilde{\gamma}_r$ follows immediately.

As for $\widetilde{S}$, recall that $\iota_F(S)$ acts on $S(V(\mathbb{A})^n)$ by the partial Fourier transform in the first variable; the desired invariance follows from Poisson summation on $\mathbf{x}_1$ and straightforward identities for the behaviour of the Fourier transform under translations and dilations. □

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6. Modularity II

In this section, we prove the modularity of the generating series $\widehat{\phi}_A(\boldsymbol{\tau})$. By Remark 4.3, we only need to consider totally positive semidefinite matrices T_2, and we assume that this is the case throughout this section.

We begin by fixing an element $\mathbf{y} = (\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}) \in \Omega(T_2)$, and setting $y = \sigma_1(\mathbf{y})$. Let

$$U_{\mathbf{y}} = \operatorname{span}(\mathbf{y}_1, \ldots, \mathbf{y}_{n-1}) \subset V, \tag{6.1}$$

so that $U_{\mathbf{y}}$ is totally positive definite. Let

$$\Lambda_{\mathbf{y}} := U_{\mathbf{y}} \cap L, \qquad \text{and} \qquad \Lambda_{\mathbf{y}}^{\perp} := U_{\mathbf{y}}^{\perp} \cap L \tag{6.2}$$

and set

$$\Lambda := \Lambda_{\mathbf{y}} \oplus \Lambda_{\mathbf{y}}^{\perp} \subset L, \tag{6.3}$$

so that

$$\Lambda \subset L \subset L' \subset \Lambda'. \tag{6.4}$$

In light of the definition (1.3), we have a natural inclusion $S(L^n) \to S(\Lambda^n)$, and the composition

$$S(L^n) \to S(\Lambda^n) \xrightarrow{\sim} S(\Lambda_{\mathbf{y}}^n) \otimes_{\mathbb{C}} S((\Lambda_{\mathbf{y}}^{\perp})^n) \tag{6.5}$$

is equivariant for the action of $\widetilde{\Gamma}^J$, via ρ_L on the left-hand side, and via $\rho_{\Lambda_{\mathbf{y}}} \otimes \rho_{\Lambda_{\mathbf{y}}^{\perp}}$ on the right; this latter fact can be deduced from explicit formulas for the Weil representation; cf. [11, Proposition II.4.3].

Note that $U_{\mathbf{y}}^{\perp}$ is a quadratic space of signature $((p',2),(p'+2,0),\ldots(p'+2,0))$ with $p' = p - \operatorname{rank}(T_2)$, so the constructions in Section 2 apply equally well in this case. In particular, let $X_{\mathbf{y}}(\mathbb{C}) = \Gamma_{\mathbf{y}} \backslash \mathbb{D}_y^+$. Then for $m \in F$ and $\mathbf{v}_1 \in (F \otimes_{\mathbb{R}} \mathbb{R})_{\gg 0}$, we have a special divisor

$$\widehat{Z}_{U_{\mathbf{y}}^{\perp}}(m, \mathbf{v}_1) = \left(Z_{U_{\mathbf{y}}^{\perp}}(m), \mathfrak{g}^o_{U_{\mathbf{y}}^{\perp}}(m, \mathbf{v}_1) \right) \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X_{\mathbf{y}}) \otimes S(\Lambda_{\mathbf{y}}^{\perp})^{\vee}, \tag{6.6}$$

where we introduce the subscript $U_{\mathbf{y}}^{\perp}$ in the notation to emphasize the underlying quadratic space being considered.

Let

$$\pi_{\mathbf{y}} \colon X_{\mathbf{y}} \to X \tag{6.7}$$

denote the natural map, which defines the cycle $Z(\mathbf{y})$ of codimension $\operatorname{rank}(T_2)$. We define a class

$$\widehat{Z}_{\mathbf{y}}(m, \mathbf{v}_1) \in \widehat{\mathrm{CH}}^{\operatorname{rk}(T_2)+1}(X, \mathcal{D}_{\mathrm{cur}}) \otimes_{\mathbb{C}} S((\Lambda_{\mathbf{y}}^{\perp})^n) \tag{6.8}$$

as follows: if $\varphi \in S((\Lambda_{\mathbf{y}}^{\perp})^n)$ is of the form $\varphi_1 \otimes \varphi_2$ with $\varphi_1 \in S(\Lambda_{\mathbf{y}}^{\perp})$ and $\varphi_2 \in S((\Lambda_{\mathbf{y}}^{\perp})^{n-1})$, we set

$$\widehat{Z}_{\mathbf{y}}(m, \mathbf{v}_1)(\varphi_1 \otimes \varphi_2) := \varphi_2(0) \cdot \pi_{\mathbf{y},*}\left(\widehat{Z}_{U_{\mathbf{y}}^{\perp}}(m, \mathbf{v}_1, \varphi_1) \right) \in \widehat{\mathrm{CH}}^{\operatorname{rk}(T_2)+1}(X, \mathcal{D}_{\mathrm{cur}}), \tag{6.9}$$

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and extend this definition to arbitrary φ by linearity. Here, the pushforward is given explicitly as

$$\pi_{\mathbf{y},*}\left(\widehat{Z}_{U_{\mathbf{y}}^{\perp}}(m,\mathbf{v}_1,\varphi_1)\right) = \left(\pi_{\mathbf{y},*}Z_{U_{\mathbf{y}}^{\perp}}(m)(\varphi_1),\ \left[\omega_{U_{\mathbf{y}}^{\perp}}(m,\mathbf{v}_1,\varphi_1)\wedge\delta_{Z(\mathbf{y})},\mathfrak{g}^o(m,\mathbf{v}_1,\varphi_1)\delta_{Z(\mathbf{y})}\right]\right).$$

Observe that this pushforward is an element of $\widehat{\mathrm{CH}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}})$; the existence of pushforward maps along arbitrary proper morphisms, which are not available in general for the Gillet-Soulé Chow groups, are an essential feature of the extended version, [5, §6.2].

Finally, for $\boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{\tau}_1 & \boldsymbol{\tau}_{12} \\ \boldsymbol{\tau}'_{12} & \boldsymbol{\tau}_2 \end{pmatrix} \in \mathbb{H}^n_d$, we define the generating series

$$\widehat{\phi}_{\mathbf{y}}(\boldsymbol{\tau}_1) \ := \ \sum_{m\in F} \widehat{Z}_{\mathbf{y}}(m,\mathbf{v}_1)\, q_1^m, \tag{6.10}$$

where $\boldsymbol{\tau}_1 \in \mathbb{H}^d_1$ with $\mathbf{v}_1 = \mathrm{Im}(\boldsymbol{\tau}_1)$, and $q_1^m = e(m\boldsymbol{\tau}_1)$.

There is also a classical theta function attached to the totally positive definite space $U_{\mathbf{y}}$, defined as follows: let $\varphi \in S(\Lambda^n_{\mathbf{y}})$ and suppose $\varphi = \varphi_1 \otimes \varphi_2$ with $\varphi_1 \in S(\Lambda_{\mathbf{y}})$ and $\varphi_2 \in S(\Lambda^{n-1}_{\mathbf{y}})$. Then we set

$$\theta_{\mathbf{y}}(\boldsymbol{\tau})(\varphi_1\otimes\varphi_2) \ := \ \varphi_2(\mathbf{y}) \sum_{\lambda\in U_{\mathbf{y}}} \varphi_1(\lambda)\ e\left(\langle\lambda,\lambda\rangle\boldsymbol{\tau}_1 + 2\langle\lambda,\mathbf{y}\rangle\boldsymbol{\tau}'_{12}\right) e(T_2\cdot\boldsymbol{\tau}_2), \tag{6.11}$$

and again, extend to all $\varphi \in S(\Lambda^n_{\mathbf{y}})$ by linearity. It is well known that $\theta_{\mathbf{y}}(\boldsymbol{\tau})$ is a holomorphic Jacobi modular form of weight $\dim U_{\mathbf{y}}/2 = \mathrm{rk}(T_2)/2$ and index T_2; see, for example, [7, §II.7].

The Fourier expansion of $\theta_{\mathbf{y}}(\boldsymbol{\tau})(\varphi)$ can be written, for $\varphi = \varphi_1\otimes\varphi_2$ as above, as

$$\theta_{\mathbf{y}}(\boldsymbol{\tau})(\varphi_1\otimes\varphi_2) \ = \ \varphi_2(\mathbf{y}) \sum_{T=\left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} r_{\mathbf{y}}(T,\varphi_1)\ q^T, \tag{6.12}$$

where $r_{\mathbf{y}}(T) \in S(\Lambda_{\mathbf{y}})^\vee$ is given by the formula

$$r_{\mathbf{y}}\left(\begin{pmatrix} T_1 & T_{12} \\ T'_{12} & T_2 \end{pmatrix},\varphi_1\right) = \sum_{\substack{\lambda\in U_{\mathbf{y}} \\ \langle\lambda,\lambda\rangle = T_1 \\ \langle\lambda,\mathbf{y}\rangle = T_{12}}} \varphi_1(\lambda). \tag{6.13}$$

Finally, note that given T as above, we must have either $\mathrm{rank}(T) = \mathrm{rank}(T_2)+1$, or $\mathrm{rank}(T) = \mathrm{rank}(T_2)$.

Lemma 6.1. *Suppose* $\mathrm{rank}(T) = \mathrm{rank}(T_2)+1$. *Then for any* $\mathbf{y} \in \Omega(T_2)$, *we have* $r_{\mathbf{y}}(T) = 0$.

Proof. Suppose $r_{\mathbf{y}}(T) \neq 0$; by definition, there exists $\lambda \in U_{\mathbf{y}} = \mathrm{span}(\mathbf{y})$ such that $(\lambda,\mathbf{y}) \in \Omega(T)$. Since V is anisotropic, we have $\mathrm{rank}(T) = \dim\mathrm{span}(\lambda,\mathbf{y})$. However, $\lambda \in U_{\mathbf{y}}$, so $\dim\mathrm{span}(\lambda,\mathbf{y}) = \dim\mathrm{span}(\mathbf{y}) = \mathrm{rank}(T_2)$, which contradicts the assumption on $\mathrm{rank}(T)$. □

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Proposition 6.2. *As formal generating series, we have*

$$\widehat{\phi}_A(\boldsymbol{\tau}) = \sum_{T=\left(\begin{smallmatrix} * & * \\ * & T_2 \end{smallmatrix}\right)} \widehat{A}(T,\mathbf{v})\, q^T = \sum_{\substack{\mathbf{y}\in\Omega(T_2)\\ \bmod \Gamma}} \widehat{\phi}_{\mathbf{y}}(\boldsymbol{\tau}) \cdot \widehat{\omega}^{n-r(T_2)-1} \otimes \theta_{\mathbf{y}}(\boldsymbol{\tau}), \tag{6.14}$$

where

$$\widehat{\phi}_{\mathbf{y}}(\boldsymbol{\tau}) \cdot \widehat{\omega}^{n-r(T_2)-1} := \sum_{m\in F} \widehat{Z}_{\mathbf{y}}(m,\mathbf{v}_1) \cdot \widehat{\omega}^{n-r(T_2)-1}\, q_1^m; \tag{6.15}$$

here, we view the right-hand side of (6.14) as valued in $S(L^n)^\vee$ by dualizing (6.5).

Proof. By linearity, it suffices to evaluate both sides of the desired relation at a Schwartz function $\varphi \in S(L^n)$ of the form $\varphi = \varphi_1 \otimes \varphi_2$ for $\varphi_1 \in S(L)$ and $\varphi_2 \in S(L^{n-1})$.

Then we may write

$$\begin{aligned} Z(T)(\varphi_1 \otimes \varphi_2) &= \sum_{\substack{\mathbf{x}\in\Omega(T)\\ \bmod \Gamma}} (\varphi_1 \otimes \varphi_2)(\mathbf{x})\, Z(\mathbf{x}) && (6.16)\\ &= \sum_{\substack{\mathbf{y}\in\Omega(T_2)\\ \bmod \Gamma}} \varphi_2(\mathbf{y}) \sum_{\substack{\mathbf{x}_1\in\Omega(T_1)\\ \langle \mathbf{x}_1,\mathbf{y}\rangle = T_{12}\\ \bmod \Gamma_{\mathbf{y}}}} \varphi_1(\mathbf{x}_1) Z(\mathbf{x}_1,\mathbf{y}). && (6.17) \end{aligned}$$

We may further assume that

$$\varphi_1 = \varphi_1' \otimes \varphi_1'' \in S(U_{\mathbf{y}}) \otimes S(U_{\mathbf{y}}^\perp) \qquad \text{and} \qquad \varphi_2 = \varphi_2' \otimes \varphi_2'' \in S(\Lambda_{\mathbf{y}}^{n-1}) \otimes S((\Lambda_{\mathbf{y}}^\perp)^{n-1}); \tag{6.18}$$

in this case, $\varphi_2(\mathbf{y}) = \varphi_2'(\mathbf{y})\varphi_2''(0)$.

For a vector $\mathbf{x}_1 \in V$ as above, write its orthogonal decomposition as

$$\mathbf{x}_1 = \mathbf{x}_1' + \mathbf{x}_1'' \in U_{\mathbf{y}} \oplus U_{\mathbf{y}}^\perp, \tag{6.19}$$

and note that $\mathbb{D}^+_{(x_1,y)} = \mathbb{D}^+_{(x_1'',y)}$, where $x_1 = \sigma_1(\mathbf{x}_1)$, etc., and $\Gamma_{(\mathbf{x}_1,\mathbf{y})} = \Gamma_{(\mathbf{x}_1'',\mathbf{y})}$.

Thus, decomposing the sum on $\mathbf{x}_1$ as above and writing $T = \left(\begin{smallmatrix} T_1 & T_{12} \\ T_{12}' & T_2 \end{smallmatrix}\right)$, we have

$$\begin{aligned} &Z(T)(\varphi_1 \otimes \varphi_2)\\ &= \sum_{\substack{\mathbf{y}\in\Omega(T_2)\\ \bmod \Gamma}} \varphi_2(\mathbf{y}) \sum_{m\in F} \left(\sum_{\substack{\mathbf{x}_1''\in U_{\mathbf{y}}^\perp\\ \langle \mathbf{x}_1'',\mathbf{x}_1''\rangle = m\\ \bmod \Gamma_{\mathbf{y}}}} \varphi_1''(\mathbf{x}_1'') Z(\mathbf{x}_1'',\mathbf{y}) \right) \cdot \left(\sum_{\substack{\mathbf{x}_1'\in U_{\mathbf{y}}\\ \langle \mathbf{x}_1',\mathbf{x}_1'\rangle = T_1 - m\\ \langle \mathbf{x}_1',\mathbf{y}\rangle = T_{12}}} \varphi_1'(\mathbf{x}_1') \right)\\ &= \sum_{\substack{\mathbf{y}\in\Omega(T_2)\\ \bmod \Gamma}} \varphi_2(\mathbf{y}) \sum_{m\in F} \left(\sum_{\substack{\mathbf{x}_1''\in U_{\mathbf{y}}^\perp\\ \langle \mathbf{x}_1'',\mathbf{x}_1''\rangle = m\\ \bmod \Gamma_{\mathbf{y}}}} \varphi_1''(\mathbf{x}_1'') Z(\mathbf{x}_1'',\mathbf{y}) \right) \cdot r_{\mathbf{y}}\left(\left(\begin{smallmatrix} T_1 - m & T_{12} \\ T_{12'} & T_2 \end{smallmatrix}\right), \varphi_1' \right), \qquad (6.20) \end{aligned}$$

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which we may rewrite as

$$Z(T)(\varphi_1\otimes\varphi_2)=\sum_{\substack{\mathbf{y}\in\Omega(T_2)\\ \bmod \Gamma}}\varphi_2''(0)\varphi_2'(\mathbf{y})\sum_m \pi_{\mathbf{y},*}\left(Z_{U_{\mathbf{y}}^\perp}(m)(\varphi_1'')\right)\cdot r\left(\left(\begin{smallmatrix} T_1-m & T_{12}\\ T_{12'} & T_2\end{smallmatrix}\right),\varphi_1'\right) \quad (6.21)$$

$$=\sum_{\substack{\mathbf{y}\in\Omega(T_2)\\ \bmod \Gamma}}\sum_m Z_{\mathbf{y}}(m)(\varphi_1''\otimes\varphi_2'')\cdot\left\{\varphi_2'(\mathbf{y})\, r\left(\left(\begin{smallmatrix} T_1-m & T_{12}\\ T_{12'} & T_2\end{smallmatrix}\right),\varphi_1'\right)\right\}, \quad (6.22)$$

where in the second line, $Z_{\mathbf{y}}(m)$ denotes the $S((\Lambda_{\mathbf{y}}^\perp)^n)^\vee$-valued cycle

$$Z_{\mathbf{y}}(m)\colon \varphi''\mapsto \varphi_2''(0)\,\pi_{\mathbf{y},*}Z_{U_{\mathbf{y}}^\perp}(m,\varphi_1''). \quad (6.23)$$

Now suppose that $\mathrm{rk}(T)=\mathrm{rk}(T_2)+1$. Then, by Lemma 6.1, the term $m=0$ does not contribute to (6.21), and so all the terms $Z_{U_{\mathbf{y}}^\perp}(m)$ that do contribute are divisors. To incorporate Green currents in the discussion, recall that, at the level of arithmetic Chow groups, the pushforward is given by the formula

$$\widehat{Z}_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi'')=\varphi_2''(0)\ \pi_{\mathbf{y},*}\widehat{Z}_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'') \quad (6.24)$$

$$=\left(\pi_{\mathbf{y},*}Z_{U_{\mathbf{y}}^\perp}(m,\varphi_1''),\ \left[\omega_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'')\wedge\delta_{Z(\mathbf{y})},\mathfrak{g}^o_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'')\wedge\delta_{Z(\mathbf{y})}\right]\right), \quad (6.25)$$

where, as before, we use the subscript $U_{\mathbf{y}}^\perp$ to denote objects defined with respect to that space.

This may be rewritten as

$$\widehat{Z}_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi'')=\widehat{Z}_{\mathbf{y}}(m)^{\mathrm{can}}(\varphi'')$$
$$+\varphi_2''(0)\left(0,\left[\omega_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'')\wedge\delta_{Z(\mathbf{y})}-\delta_{Z_{\mathbf{y}}(m)},\mathfrak{g}^o_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'')\wedge\delta_{Z(\mathbf{y})}\right]\right), \quad (6.26)$$

where $\widehat{Z}_{\mathbf{y}}(m)^{\mathrm{can}}=(Z_{\mathbf{y}}(m),[\delta_{Z_{\mathbf{y}}(m)},0])$ is the canonical class associated to $Z_{\mathbf{y}}(m)$. Thus,

$$\widehat{Z}_{\mathbf{y}}(m,\mathbf{v}_1)\cdot\widehat{\omega}^{n-\mathrm{rk}(T)}=\widehat{Z}_{\mathbf{y}}(m)^{\mathrm{can}}\cdot\widehat{\omega}^{n-\mathrm{rk}(T)}+(0,[\beta_{\mathbf{y}}(m,\mathbf{v}_1),\alpha_{\mathbf{y}}(m,\mathbf{v}_1)]), \quad (6.27)$$

where $\alpha_{\mathbf{y}}(m,\mathbf{v}_1)$ and $\beta_{\mathbf{y}}(m,\mathbf{v}_1)$ are $S((\Lambda_{\mathbf{y}}^\perp)^n)^\vee$-valued currents defined by

$$\alpha_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi'')\ =\ \varphi_2''(0)\ \mathfrak{g}^o_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'')\wedge\delta_{Z(\mathbf{y})}\wedge\Omega^{n-\mathrm{rk}(T)} \quad (6.28)$$

and

$$\beta_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi'')\ =\ \varphi_2''(0)\ \omega_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1,\varphi_1'')\wedge\delta_{Z(\mathbf{y})}\wedge\Omega^{n-\mathrm{rk}(T)}-\delta_{Z_{\mathbf{y}}(m)(\varphi'')}\wedge\Omega^{n-\mathrm{rk}(T)} \quad (6.29)$$

where $\varphi''=\varphi_1''\otimes\varphi_2''$ as before.

Turning to the class $\widehat{A}(T,\mathbf{v})$, it can be readily verified that

$$\widehat{A}(T,\mathbf{v})=\widehat{Z(T)}^{\mathrm{can}}\cdot\widehat{\omega}^{n-\mathrm{rk}(T)}\ +\ \left(0,[\psi(T,\mathbf{v})-\delta_{Z(T)}\wedge\Omega^{n-\mathrm{rk}(T)},\mathfrak{a}(T,\mathbf{v})]\right), \quad (6.30)$$

where the currents $\mathfrak{a}(T,\mathbf{v})$ and $\psi(T,\mathbf{v})$ are defined in (4.15) and (4.26), respectively. Now, by the same argument as in (6.21), and under the assumption $\mathrm{rank}(T) = \mathrm{rank}(T_2)+1$, we have (as a Γ-invariant current on $\mathbb{D}$)

$$\mathfrak{a}(T,\mathbf{v})(\varphi_1\otimes\varphi_2) = \sum_{\mathbf{y}\in\Omega(T_2)} \varphi_2(\mathbf{y}) \sum_{\substack{\mathbf{x}_1\in\Omega(T_1)\\ \langle\mathbf{x}_1,\mathbf{y}\rangle = T_{12}}} \varphi_1(\mathbf{x}_1)\mathfrak{g}^o(\sqrt{v_1}x_1)\wedge\delta_{\mathbb{D}_y^+}\wedge\Omega^{n-r(T)} \tag{6.31}$$

$$= \sum_{\mathbf{y}\in\Omega(T_2)} \varphi_2(\mathbf{y})\cdot\sum_{m\in F}\left(\sum_{\substack{\mathbf{x}_1''\in U_{\mathbf{y}}^{\perp}\\ \langle\mathbf{x}_1'',\mathbf{x}_1''\rangle = m}} \varphi_1''(\mathbf{x}_1'')\,\mathfrak{g}^o(\sqrt{v_1}x_1'')\wedge\delta_{\mathbb{D}_y^+}\wedge\Omega^{n-r(T)}\right) \tag{6.32}$$

$$\times r\left(\begin{pmatrix} T_1-m & T_{12} \\ T_{12'} & T_2\end{pmatrix},\varphi_1'\right), \tag{6.33}$$

where we use the fact that $\mathfrak{g}^o(\sqrt{v_1}x_1)\wedge\delta_{\mathbb{D}_y^+}$ only depends on the orthogonal projection x_1'' of x_1 onto $U_y^{\perp} = \sigma_1(U_{\mathbf{y}}^{\perp})$. Thus, as $S(L^n)^{\vee}$-valued currents on X, we obtain the identity

$$\mathfrak{a}(T,\mathbf{v})(\varphi_1\otimes\varphi_2) = \sum_{\mathbf{y} \bmod \Gamma}\sum_{m\in F}\alpha_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi_1''\otimes\varphi_2'')\cdot\left\{\varphi_2'(\mathbf{y})\,r_{\mathbf{y}}\left(\begin{pmatrix} T_1-m & T_{12} \\ T_{12'} & T_2\end{pmatrix},\varphi_1'\right)\right\} \tag{6.34}$$

with $\varphi_i = \varphi_i'\otimes\varphi_i''$ as above.

A similar argument gives

$$\begin{aligned}&\psi(T,\mathbf{v})(\varphi) - \delta_{Z(T)(\varphi)}\wedge\Omega^{n-\mathrm{rk}(T)} \\ &\quad= \sum_{\mathbf{y} \bmod \Gamma}\sum_{m\in F}\beta_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi_1''\otimes\varphi_2'')\cdot\left\{\varphi_2'(\mathbf{y})\,r_{\mathbf{y}}\left(\begin{pmatrix} T_1-m & T_{12} \\ T_{12'} & T_2\end{pmatrix},\varphi_1'\right)\right\},\end{aligned} \tag{6.35}$$

and so in total, we have

$$\begin{aligned}&\widehat{A}(T,\mathbf{v})(\varphi_1\otimes\varphi_2) \\ &\quad= \sum_{\mathbf{y} \bmod \Gamma}\sum_{m}\widehat{Z}_{\mathbf{y}}(m,\mathbf{v}_1)(\varphi_1''\otimes\varphi_2'')\cdot\widehat{\omega}^{n-\mathrm{rk}(T_2)-1}\cdot\left\{\varphi_2'(\mathbf{y})\,r_{\mathbf{y}}\left(\begin{pmatrix} T_1-m & T_{12} \\ T_{12'} & T_2\end{pmatrix}\right)(\varphi_1')\right\}\end{aligned} \tag{6.36}$$

whenever $\mathrm{rank}(T) = \mathrm{rank}(T_2)+1$.

Now suppose $\mathrm{rank}(T) = \mathrm{rank}(T_2)$. Then for any tuple $(\mathbf{x}_1,\mathbf{y})\in\Omega(T)$, we must have $\mathbf{x}_1\in U_{\mathbf{y}}$, and in particular, the only terms contributing to the right-hand side of (6.36) are those with $m=0$. However, we have

$$\mathfrak{a}(T,\mathbf{v}) = 0, \qquad \psi(T,\mathbf{v}) = \delta_{Z(T)}\wedge\Omega^{n-\mathrm{rk}(T)}, \tag{6.37}$$

and hence,

$$\widehat{A}(T,\mathbf{v}) = \widehat{Z(T)}^{\mathrm{can}}\cdot\widehat{\omega}^{n-\mathrm{rk}(T)}; \tag{6.38}$$

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with these observations, it follows easily from unwinding definitions that (6.36) continues to hold in this case.

Finally, the statement in the proposition follows by observing that the T'th q coefficient on the right-hand side of (6.14) is precisely the right-hand side of (6.36). □

Corollary 6.3. *The series $\widehat{\phi}_A(\boldsymbol{\tau})$ is a Jacobi modular form of weight $\kappa := (p+2)/2$ and index T_2, in the sense of Definition 2.7.*

Proof. Fix $\mathbf{y} \in \Omega(T_2)$. By Corollary 3.3, applied to the space $U_{\mathbf{y}}^{\perp}$, there exist finitely many $\widehat{z}_{\mathbf{y},1},\dots,\widehat{z}_{\mathbf{y},r} \in \widehat{\mathrm{CH}}^1_{\mathbb{C}}(X_{\mathbf{y}})$, finitely many (elliptic) forms $f_{\mathbf{y},1},\dots,f_{\mathbf{y},r} \in A_\kappa(\rho^\vee_{\Lambda_{\mathbf{y}}^\perp})$ and an element $g_{\mathbf{y}} \in A_\kappa(\rho^\vee_{\Lambda_{\mathbf{y}}^\perp}; D^*(X))$ such that the identity

$$\sum_{m\in F} \widehat{Z}_{U_{\mathbf{y}}^\perp}(m,\mathbf{v}_1)q^m = \sum_{i=1}^r f_{\mathbf{y},i}(\boldsymbol{\tau}_1)\widehat{z}_{\mathbf{y},i} \;+\; a(g_{\mathbf{y}}(\boldsymbol{\tau}_1)) \tag{6.39}$$

holds at the level of q-coefficients; here, $\boldsymbol{\tau}_1 \in \mathbb{H}_1^d$ and $\mathbf{v}_1 = \mathrm{Im}(\boldsymbol{\tau}_1)$. Moreover, from the proof of Corollary 3.3, we see that $g_{\mathbf{y}}(\tau)$ is smooth on X.

Therefore, applying Proposition 6.2 and unwinding definitions, we obtain the identity

$$\widehat{\phi}_A(\boldsymbol{\tau}) \;=\; \sum_{\substack{\mathbf{y}\in\Omega(T_2)\\ \bmod \Gamma}} \sum_{i=1}^r (F_{\mathbf{y},i}(\boldsymbol{\tau})\otimes\theta_{\mathbf{y}}(\boldsymbol{\tau}))\,\widehat{Z}_{\mathbf{y},i} \;+\; a\Big((G_{\mathbf{y}}(\boldsymbol{\tau})\otimes\theta_{\mathbf{y}}(\boldsymbol{\tau}))\wedge\delta_{Z(\mathbf{y})}\wedge\Omega^{n-\mathrm{rank}(T_2)-1}\Big) \tag{6.40}$$

of formal generating series, where

$$\widehat{Z}_{\mathbf{y},i} \;:=\; \pi_{\mathbf{y},*}(\widehat{z}_{\mathbf{y},i})\cdot\widehat{\omega}^{n-\mathrm{rk}(T_2)-1} \;\in\; \widehat{\mathrm{CH}}^n_{\mathbb{C}}(X,\mathcal{D}_{\mathrm{cur}}), \tag{6.41}$$

and we promote the elliptic forms $f_{\mathbf{y},i}$ and $g_{\mathbf{y}}$ to $S((\Lambda_{\mathbf{y}}^\perp)^n)^\vee$-valued functions by setting

$$F_{\mathbf{y},i}(\boldsymbol{\tau})(\varphi) \;:=\; \varphi_2(0)\cdot f_{\mathbf{y},i}(\boldsymbol{\tau}_1)(\varphi_1), \qquad G_{\mathbf{y}}(\boldsymbol{\tau}) = \varphi_2(0)\cdot g_{\mathbf{y}}(\boldsymbol{\tau}_1)(\varphi_1) \tag{6.42}$$

for $\varphi = \varphi_1\otimes\varphi_2 \in S(\Lambda_{\mathbf{y}}^\perp)\otimes S((\Lambda_{\mathbf{y}}^\perp)^{n-1})$ and $\boldsymbol{\tau} = \begin{pmatrix} \boldsymbol{\tau}_1 & \boldsymbol{\tau}_{12} \\ \boldsymbol{\tau}_{12}' & \boldsymbol{\tau}_2 \end{pmatrix}$.

It remains to show that $F_{\mathbf{y},i}(\boldsymbol{\tau})\otimes\theta_{\mathbf{y}}(\boldsymbol{\tau})$ and $G_{\mathbf{y}}(\tau)\otimes\theta_{\mathbf{y}}(\boldsymbol{\tau})$ are invariant under the slash operators (2.41) for elements of $\widetilde{\Gamma}^J$; this can be verified directly using the generators (5.15)–(5.18), the modularity in genus one of $f_{\mathbf{y},i}$ and $g_{\mathbf{y}}$, and explicit formulas for the Weil representation (as in, for example, [11]). □

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