Can. J. Math.Vol. **45** (3), 1993 pp. 627–637

POLYNOMIALS WITH A PRESCRIBED ZERO AND THE BERNSTEIN'S INEQUALITY

P. F. OLIVIER AND A. O. WATT

ABSTRACT Let $\mathcal{P}_{n,1}$ be the class of all polynomials p of degree at most n such that $|p(z)| \leq 1$ for $|z| \leq 1$ In view of the example z^n it follows from Bernstein's inequality for polynomials that $\sup_{p\in\mathcal{P}_{n,1}} |p'(z_0)| = n$ at each point z_0 of the unit cirle It was shown by A Giroux and Q I Rahman [2] that if $\mathcal{P}^*_{n,1}$ denotes the subclass of polynomials in $\mathcal{P}_{n,1}$ which vanish at 1, then

$$n - \frac{c_1}{n} < \sup_{p\in\mathcal{P}^*_{n,1}} \max_{|z|=1} \left|p'(z)\right| < n - \frac{c_2}{n}$$

where c_1 and c_2 are constants not depending on n Here we find the exact value of $\sup_{p\in\mathcal{P}^*_{n,1}} |p'(z)|$ at $z = -1$ which has some special significance and also at certain other points of the unit circle

1. **Introduction.** Let $\mathcal{P}_n$ be the class of all polynomials $p(z) := \sum_{\nu=0}^{n} a_\nu z^\nu$ of degree at most n. We shall abbreviate $\max_{|z|=1} |p(z)|$ by $\|p\|$. The subclass of $\mathcal{P}_n$ consisting of polynomials p with $\|p\| \leq 1$ will be denoted by $\mathcal{P}_{n,1}$. Polynomials in $\mathcal{P}_{n,1}$ which vanish at 1 will be said to belong to $\mathcal{P}^*_{n,1}$.

According to Bernstein's inequality for polynomials

$$\sup_{p\in\mathcal{P}_{n,1}} |p'(z)| = n \tag{1}$$

at each point z of the unit circle. Further, the supremum is attained only for $p(z) := e^{i\gamma} z^n$, $\gamma \in \mathbb{R}$.

In this paper we seek to determine how large $|p'(z)|$ can be at a prescribed point z of the unit circle if p is restricted to the subclass $\mathcal{P}^*_{n,1}$ of $\mathcal{P}_{n,1}$. A priori the supremum can be different at different points. We obtain the sharp answer for z belonging to a certain set E_n which will be specified below.

2. **Statement of result.** For $n \in \mathbb{N}$ let T_n denote as usual the n-th Chebyshev polynomial of the first kind. For each integer ν, $1 \leq \nu \leq 2n-1$ let ρ_ν be the only root of the equation

$$T_n(\rho) = \frac{1}{n\sin(\nu\pi/2n)} \sqrt{\frac{1-\rho^2}{1-\rho^2\cos^2(\nu\pi/2n)}} \tag{2}$$

Received by the editors October 17, 1991
AMS subject classification Primary 30C10, 26D05, secondary 41A17
Key words and phrases polynomials, Bernstein's inequality, Chebyshev polynomials

in $\big(\cos(\pi/2n), 1\big)$ if ν is even; otherwise let $\rho_\nu = \cos(\pi/2n)$. Denote by φ_ν the unique root of the equation

$$\cos\frac{\varphi}{2} = \rho_\nu \cos\frac{\nu\pi}{2n} \tag{3}$$

in $(0, 2\pi)$. The set E_n alluded to above consists of the points $z_{n,\nu} = e^{i\varphi_\nu}$, $1 \le \nu \le 2n-1$. It was proved in [1] that if n is odd then for $p \in \mathcal{P}^*_{n,1}$ we have

$$|p'(-1)| \le n\cos^2\frac{\pi}{4n}. \tag{4}$$

In (4) equality holds if and only if $p = e^{i\gamma}P$, $\gamma \in \mathbb{R}$ where P is defined by

$$P(z^2) = z^n\Big\{T_n\Big(\rho\frac{z+z^{-1}}{2}\Big) + \frac{1}{n}\frac{z-z^{-1}}{2}T_n'\Big(\rho\frac{z+z^{-1}}{2}\Big)\Big\}, \quad \rho = \cos\frac{\pi}{2n}. \tag{5}$$

Here we are able to find the polynomials in $\mathcal{P}^*_{n,1}$ which maximize $|p'(z)|$ at any *prescribed* point z belonging to the set E_n.

Let

$$\zeta_\nu(z) := \rho_\nu\frac{z+z^{-1}}{2} - i\frac{1-\rho_\nu^2}{\rho_\nu}\frac{\cos(\varphi_\nu/2)}{\sin(\varphi_\nu/2)}\frac{z-z^{-1}}{2}, \qquad 1 \le \nu \le 2n-1.$$

Since T_n is even or odd according as n is even or odd respectively it is easily seen that for $1 \le \nu \le 2n-1$ the function

$$z \longmapsto z^n e^{-i(n/2-1)\varphi_\nu}\Big\{\Big(\frac{1}{n}\sqrt{1-\rho_\nu^2}T_n'(\rho_\nu)\Big)T_n\big(\zeta_\nu(z)\big) - \Big(\frac{1}{n}\sqrt{1-\rho_\nu^2}T_n(\rho_\nu)\frac{z+z^{-1}}{2} - \frac{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}{n\rho_\nu\sin(\varphi_\nu/2)}\frac{z-z^{-1}}{2} + i\frac{1}{n}\sqrt{1-\rho_\nu^2}T_n(\rho_\nu)\frac{\cos(\varphi_\nu/2)}{\sin(\varphi_\nu/2)}\frac{z-z^{-1}}{2}\Big)T_n'\big(\zeta_\nu(z)\big)\Big\}$$

is an *even* polynomial of degree $2n$ and so can be written as $P_{n,\nu}(z^2)$ where $P_{n,\nu}$ is a polynomial of degree n.

We prove the following

THEOREM. *The polynomial $P_{n,\nu}$ belongs to $\mathcal{P}^*_{n,1}$ for $1 \le \nu \le 2n-1$ and*

$$\sup_{p\in\mathcal{P}^*_{n,1}} |p'(e^{i\varphi_\nu})| = \frac{n}{2}\left|\frac{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}{\sin(\varphi_\nu/2)} + \frac{1}{n}\sqrt{1-\rho_\nu^2}T_n'(\rho_\nu)\right| = |P_{n,\nu}'(e^{i\varphi_\nu})|. \tag{6}$$

The supremum is attained if and only if $p = e^{i\gamma}P_{n,\nu}$ where $\gamma \in \mathbb{R}$.

REMARK 1. From (2) and (3) it follows that

$$T_n(\rho_\nu) = \frac{1}{n\sin(\varphi_\nu/2)}\frac{\rho_\nu\sqrt{1-\rho_\nu^2}}{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}$$

and

$$T_n'(\rho_\nu) = \frac{n}{\sqrt{1-\rho_\nu^2}}\sqrt{1-\frac{\rho_\nu^2(1-\rho_\nu^2)}{n^2\sin^2(\varphi_\nu/2)(\rho_\nu^2-\cos^2(\varphi_\nu/2))}}.$$

REMARK 2. In order to simplify the presentation we introduce for $1 \leq \nu \leq 2n-1$ the functions

$$\xi_{n,\nu}(\theta) := \zeta_\nu(e^{\imath\theta/2}) = \rho_\nu \cos(\theta/2) + \frac{1-\rho_\nu^2}{\rho_\nu}\frac{\cos(\varphi_\nu/2)}{\sin(\varphi_\nu/2)}\sin(\theta/2) \tag{7}$$

$$R_{n,\nu}(\theta) := \frac{\sqrt{1-\rho_\nu^2}}{n}\Big\{T_n'(\rho_\nu)T_n\big(\xi_{n,\nu}(\theta)\big) - T_n(\rho_\nu)\frac{\sin\big((\varphi_\nu-\theta)/2\big)}{\sin(\varphi_\nu/2)}T_n'(\xi_{n,\nu}(\theta))\Big\} \tag{8}$$

$$I_{n,\nu}(\theta) := \frac{1}{n\rho_\nu\sin(\varphi_\nu/2)}\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}\sin(\theta/2)T_n'\big(\xi_{n,\nu}(\theta)\big) \tag{9}$$

$$\omega_{n,\nu}(\theta) := T_n(\rho_\nu)T_n\big(\xi_{n,\nu}(\theta)\big) + \frac{1-\rho_\nu^2}{n^2}T_n'(\rho_\nu)\frac{\sin\big((\varphi_\nu-\theta)/2\big)}{\sin(\varphi_\nu/2)}T_n'\big(\xi_{n,\nu}(\theta)\big), \tag{10}$$

which are defined on $[0, 2\pi)$. It is clear that

$$P_{n,\nu}(e^{\imath\theta}) = e^{-\imath(n/2-1)\varphi_\nu}e^{\imath n\theta/2}\big(R_{n,\nu}(\theta) + iI_{n,\nu}(\theta)\big). \tag{11}$$

3. **Some properties of** $P_{n,\nu}$**.** In Lemmas 1–4 presented below we give certain properties of $P_{n,\nu}$ which are relevant in the present context.

LEMMA 1. *The polynomial $P_{n,\nu}$ belongs to $\mathcal{P}_{n,1}^*$ for* $1 \leq \nu \leq 2n-1$.

PROOF. We have already seen that $P_{n,\nu}$ is a polynomial of degree n. Now using formulas (8), (9), (10) and (11) of Remark 2 we obtain

$$\begin{aligned}
|P_{n,\nu}(e^{\imath\theta})|^2 &\leq |P_{n,\nu}(e^{\imath\theta})|^2 + \omega_{n,\nu}^2(\theta) \\
&= R_{n,\nu}^2(\theta) + I_{n,\nu}^2(\theta) + \omega_{n,\nu}^2(\theta) \\
&= \frac{1-\rho_\nu^2}{n^2}\Big\{T_n'(\rho_\nu)T_n\big(\xi_{n,\nu}(\theta)\big) - T_n(\rho_\nu)\frac{\sin\big((\varphi_\nu-\theta)/2\big)}{\sin(\varphi_\nu/2)}T_n'\big(\xi_{n,\nu}(\theta)\big)\Big\}^2 \\
&\quad + \frac{1}{n^2}\frac{\rho_\nu^2-\cos^2(\varphi_\nu/2)}{\rho_\nu^2\sin^2(\varphi_\nu/2)}\sin^2(\theta/2)T_n'^2\big(\xi_{n,\nu}(\theta)\big) \\
&\quad + \Big\{T_n(\rho_\nu)T_n\big(\xi_{n,\nu}(\theta)\big) + \frac{1-\rho_\nu^2}{n^2}T_n'(\rho_\nu)\frac{\sin\big((\varphi_\nu-\theta)/2\big)}{\sin(\varphi_\nu/2)}T_n'\big(\xi_{n,\nu}(\theta)\big)\Big\}^2 \\
&= T_n^2\big(\xi_{n,\nu}(\theta)\big) + \big(1-\xi_{n,\nu}^2(\theta)\big)\frac{1}{n^2}T_n'^2\big(\xi_{n,\nu}(\theta)\big) \\
&= 1.
\end{aligned}$$

Finally a simple verification gives

$$P_{n,\nu}(1) = 0.$$

REMARK 3. The proof of Lemma 1 shows in particular that

$$|P_{n,\nu}(e^{\imath\theta})|^2 + \omega_{n,\nu}^2(\theta) = 1.$$

In Lemma 2 we describe the points where $|P_{n,\nu}(z)|$ attains its maximum on the unit circle.

LEMMA 2 *Let ν be an ınteger such that $1 \le \nu \le 2n-1$ The maxımum of $|P_{n\,\imath}(z)|$ on the unıt cırcle ıs 1 whıch ıs attaıned at n poınts $z_k \;= e^{\imath\theta_k}$ $1 \le k \le n$ where $0 < \theta_1 < \theta_2 < \quad < \theta_n < 2\pi$ The numbers θ_k depend on n and ν, ıf ν ıs odd they are characterızed by*

$$(12)\quad \begin{cases} \xi_{n\,\nu}(\theta_k) = \cos k\frac{\pi}{n}, & 1 \le k \le \frac{\nu+1}{2} - 1 \text{ (to be dıscounted ıf } \nu = 1) \\ \xi_{n\,\nu}(\theta_{(\nu+1)/2}) = \cos\frac{\nu}{2}\frac{\pi}{n}, & \theta_{(\nu+1)/2} \;= \varphi_\imath \\ \xi_{n\,\nu}(\theta_k) = \cos(k-1)\frac{\pi}{n}, & \frac{\nu+1}{2} + 1 \le k \le n \text{ (to be dıscounted ıf } \nu = 2n-1) \end{cases}$$

whereas ıf ν ıs even they satısfy

$$T_n\Big(\xi_{n\,\nu}(\theta_k)\Big) = (-1)^{k+1}\frac{\sqrt{n^2(\rho_\nu^2-\cos^2(\varphi_\nu/2))\sin^2(\varphi_\nu/2)-\rho_\nu^2(1-\rho_\nu^2)}}{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}\sqrt{n^2\sin^2(\varphi\quad/2)\sin^2\Big((\varphi_\nu-\theta_k)/2\Big)+\sin^2(\theta_k/2)}}\sin\frac{\varphi\qquad\theta_k}{2}$$

$$T_n'\Big(\xi_n\quad(\theta_k)\Big) = (-1)^k\frac{n\rho_\nu\sin(\varphi_\nu/2)}{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}\sqrt{n^2\sin^2(\varphi_\nu/2)\sin^2\Big((\varphi\quad-\theta_k)/2\Big)+\sin^2(\theta_k/2)}}\qquad 1 \le k < n$$

PROOF Let us wrıte $\xi_{n\,\nu}(\theta)$ ın the form $\xi_{n\,\nu}(\theta) = \rho_*\cos\big(\theta_* - \theta/2\big)$ where

$$\rho_* = \sqrt{\rho_\nu^2 + \frac{(1-\rho_\nu^2)^2}{\rho_\nu^2}\frac{\cos^2(\varphi_\imath\;/2)}{\sin^2(\varphi_\nu/2)}}$$

and $\theta_* \in (-\pi/2, \pi/2)$ ıs such that $\cos\theta_* = \frac{\rho_\nu}{\rho_*}$ and $\sin\theta_* = \frac{1}{\rho}\frac{1-\rho_\imath^2}{\rho^2}\frac{\cos(\varphi\;/2)}{\sin(\varphi\;/2)}$ As ıs easıly seen, $\xi_{n\,\nu}(\theta)$ decreases from ρ_ν to $-\rho_\nu$ on the ınterval $[4\theta_*, 2\pi] \subseteq [0, 2\pi]$ ın case $\theta_* \in [0, \pi/2)$ and decreases from ρ_ν to $-\rho_\nu$ on the ınterval $[0, 4\theta_* + 2\pi] \subseteq [0, 2\pi]$ ın case $\theta_* \in (-\pi/2, 0]$ It ıs also clear that $|P_{n\,\nu}(e^{\imath\theta})| = 1$ ıf and only ıf $\omega_{n\,\nu}(\theta) = 0$

Let ν be odd We have $T_n(\rho_\nu) = T_n\big(\cos(\pi/2n)\big) = 0$ Then, from (10) ıt follows that $\omega_{n\,\nu}(\theta)$ ıs zero ıf and only ıf $\sin\big((\varphi_\nu - \theta)/2\big)T_n'\big(\xi_{n\,\nu}(\theta)\big) = 0$, ı e ıf $\theta = \varphi_\imath$ or $\xi_{n\,\nu}(\theta_\mu) = \cos(\mu\pi/n)$ for some ınteger μ, $1 \le \mu \le n-1$ Referrıng to (7) and recallıng that φ_ν satısfies the equatıon (3) we obtaın

$$(13)\qquad \xi_{n\,\nu}(\varphi_\nu) = \frac{1}{\rho_\nu}\cos(\varphi_\nu/2) = \cos\frac{\nu\pi}{2n}$$

In vıew of these consıderatıons the zeros of $\omega_{n\,\imath}(\theta)$ get arranged ın ıncreasıng order ıf we ıncrease the subscrıpt μ of θ_μ by 1 for $\mu \ge (\nu+1)/2$ and set $\theta_{(\nu+1)/2} \;= \varphi_\imath$

Now let ν be even Sınce $e^{\imath n\theta/2}\omega_{n\,\nu}(\theta) = Q(e^{\imath\theta})$ where Q ıs a polynomıal of degree n, $\omega_{n\,\nu}(z)$ has exactly n zeros ın the strıp $0 \le \Re(z) < 2\pi$ We wıll show that ın fact all these zeros are real Suppose first that $\theta_* \in [0, \pi/2)$ and examıne $T_n\big(\xi_{n\,\imath}(\theta)\big)$ for θ belongıng to the ınterval $[0, 2\pi)$ For $1 \le k \le n-1$ let θ_k be the value ın $(0, 2\pi)$ for whıch $\xi_{n\,\imath}(\theta_k) = \cos(k\pi/n)$ Then $T_n\big(\xi_{n\,\nu}(\theta_k)\big) = (-1)^k$ and $T_n'\big(\xi_{n\,\imath}(\theta_k)\big) = 0$ So accordıng to (10), $\omega_{n\,\nu}(\theta_k) = (-1)^k T_n(\rho_\nu)$ Studyıng $\xi_{n\,\imath}(\theta)$ we see that $T_n\big(\xi_{n\,\imath}(\theta)\big)$ ıncreases from $T_n(\rho_\nu)$ to $T_n(\rho_*)$ on the ınterval $[0, 2\theta_*]$ and decreases from $T_n(\rho_*)$ to -1 on $[2\theta_*, \theta_1]$ On the interval $[\theta_1, \theta_{n-1}]$ the graph of $T_n\big(\xi_{n\,\imath}(\theta)\big)$ has $n-2$ branches going up from -1 to $+1$ or going down from $+1$ to -1 Finally, when θ varies from θ_{n-1} to 2π, $T_n\big(\xi_{n\,\nu}(\theta)\big)$ increases or decreases from $(-1)^{n\ 1}$ to $(-1)^n T_n(\rho_\nu)$ according as n is even or odd Further, a simple calculation shows that $\omega_{n\,\nu}(0) = 1$ and $\omega_{n\,\nu}(2\pi) = (-1)^n$ The preceding observations allow us to conclude that $\omega_{n\,\nu}(\theta)$ vanishes at least once in each of the n intervals $(0, \theta_1), (\theta_1, \theta_2), \quad , (\theta_{n\ 1}, 2\pi)$

In case $\theta_* \in (-\pi/2, 0]$ the disposition of the curve $T_n\big(\xi_{n\,\nu}(\theta)\big)$ changes, but arguing in roughly the same way as above we arrive at the desired conclusion about the zeros of $\omega_{n\,\imath}(\theta)$

If as before we denote the zeros of $\omega_{n\,\nu}$ by θ_k, $1 \le k \le n$, then from (10) it follows that

(14)
$$T_n(\rho_\imath)T_n\big(\xi_{n\,\nu}(\theta_k)\big) + \frac{1-\rho_\nu^2}{n^2}T_n'(\rho_\nu)\frac{\sin\big((\varphi_\nu - \theta_k)/2\big)}{\sin(\varphi_\nu/2)}T_n'\big(\xi_{n\,\nu}(\theta_k)\big) = 0, \quad 1 \le k \le n$$

Using the expressions for $T_n(\rho_\nu)$ and $T_n'(\rho_\nu)$ contained in Remark 1 we obtain

$$\rho_\nu\sqrt{1-\xi_{n\,\nu}^2(\theta_k)}\sin\frac{\varphi_\imath}{2}T_n\big(\xi_{n\,\nu}(\theta_k)\big) +$$
$$\sqrt{n^2(\rho_\nu^2 - \cos^2\frac{\varphi_\nu}{2})\sin^2\frac{\varphi_\nu}{2} - \rho_\nu^2(1-\rho_\nu^2)}\sin\frac{\varphi_\nu - \theta_k}{2}\sqrt{1-\xi_{n\,\nu}^2(\theta_k)}\frac{T_n'\big(\xi_{n\,\nu}(\theta_k)\big)}{n} = 0$$

This, in conjunction with

$$n^2T_n^2\big(\xi_{n\,\nu}(\theta_k)\big) + \big(1-\xi_{n\,\nu}^2(\theta_k)\big)T_n'^2\big(\xi_{n\,\nu}(\theta_k)\big) = n^2$$

gives us

$$T_n\Big(\xi_{n\,\nu}(\theta_k)\Big)$$
$$= \pm\frac{\sqrt{n^2\Big(\rho_\nu^2 - \cos^2(\varphi_\nu/2)\Big)\sin^2(\varphi_\nu/2) - \rho_\nu^2(1-\rho_\nu^2)}\sin\Big((\varphi\ \ - \theta_k)/2\Big)}{\sqrt{\rho_\nu^2(1-\xi_{n\,\nu}^2(\theta_k))\sin^2(\varphi\ \ /2) + \Big(n^2\Big(\rho_\nu^2 - \cos^2(\varphi_\nu/2)\Big)\sin^2(\varphi_\nu/2) - \rho_\nu^2(1-\rho_\nu^2)\Big)\sin^2\Big((\varphi_\nu - \theta_k)/2\Big)}}$$

and

$$T_n'\Big(\xi_{n\,\nu}(\theta_k)\Big)$$
$$= \pm\frac{n\rho_\nu\sin(\varphi_\nu/2)}{\sqrt{\rho_\nu^2(1-\xi_n^2\ \ (\theta_k))\sin^2(\varphi_\nu/2) + \Big(n^2(\rho_\nu^2 - \cos^2(\varphi_\nu/2))\sin^2(\varphi_\nu/2) - \rho_\nu^2(1-\rho_\nu^2)\Big)\sin^2\Big((\varphi_\nu - \theta_k)/2\Big)}}$$

Now let $\theta_* \in [0, \pi/2)$ Since ν is even and φ_ν satisfies (13) it follows that

$$T_n\big(\xi_{n\,\nu}(\varphi_\nu)\big) = T_n\Big(\cos\frac{\nu\pi}{2n}\Big) = \cos\frac{\nu\pi}{2} = (-1)^{\nu/2},$$

and so φ_ν is one of the values θ_k, $1 \le k \le n-1$. We observe in addition that $T_n'\big(\xi_{n,\nu}(\theta)\big)$ is alternately negative and positive in the intervals $(2\theta_*, \theta_1), (\theta_1, \theta_2), \ldots, (\theta_{n-1}, 2\pi)$; further $T_n\big(\xi_{n,\nu}(\theta_k)\big)T_n'\big(\xi_{n,\nu}(\theta_k)\big)$ is negative or positive according as $\varphi_\nu > \theta_k$ or $\varphi_\nu < \theta_k$ respectively. These remarks and the identity

$$\rho_\nu^2\big(1-\xi_{n,\nu}^2(\theta_k)\big)\sin^2\frac{\varphi_\nu}{2}+\Big(n^2\Big(\rho_\nu^2-\cos^2\frac{\varphi_\nu}{2}\Big)\sin^2\frac{\varphi_\nu}{2}-\rho_\nu^2(1-\rho_\nu^2)\Big)\sin^2\frac{\varphi_\nu-\theta_k}{2}$$
$$=\Big(\rho_\nu^2-\cos^2\frac{\varphi_\nu}{2}\Big)\Big(n^2\sin^2\frac{\varphi_\nu}{2}\sin^2\frac{\varphi_\nu-\theta_k}{2}+\sin^2\frac{\theta_k}{2}\Big)$$

easily lead us to the result stated in the second part of the Lemma. The above argument remains valid in case $\theta_* \in (-\pi/2, 0]$.

Lemma 3 gives the values of $P_{n,\nu}(z)$ at the points $z_k := e^{\imath\theta_k}$, $1 \le k \le n$.

LEMMA 3. *Let ν be an integer such that $1 \le \nu \le 2n-1$. Then at the points $z_k := e^{\imath\theta_k}$, $1 \le k \le n$ defined in Lemma 2 we have for odd ν*

$$P_{n,\nu}(e^{\imath\theta_k}) = (-1)^k e^{-\imath(n/2-1)\varphi_\nu} e^{\imath n\theta_k/2}, \quad 1 \le k \le \frac{\nu+1}{2}-1$$
$$P_{n,\nu}(e^{\imath\theta_{(\nu+1)/2}}) = e^{\imath\nu\pi/2}e^{\imath\varphi_\nu}, \quad \theta_{(\nu+1)/2} := \varphi_\nu$$
$$P_{n,\nu}(e^{\imath\theta_k}) = (-1)^{k-1} e^{-\imath(n/2-1)\varphi_\nu} e^{\imath n\theta_k/2}, \quad \frac{\nu+1}{2}+1 \le k \le n$$

whereas for even ν

$$P_{n,\nu}(e^{\imath\theta_k}) = (-1)^k e^{-\imath(n/2-1)\varphi_\nu} e^{\imath n\theta_k/2}\frac{-n\sin(\varphi_\nu/2)\sin\big((\varphi_\nu-\theta_k)/2\big)+i\sin(\theta_k/2)}{\sqrt{n^2\sin^2(\varphi_\nu/2)\sin^2\big((\varphi_\nu-\theta_k)/2\big)+\sin^2(\theta_k/2)}},$$
$$1 \le k \le n.$$

PROOF. Let ν be odd, then by definition $\rho_\nu = \cos(\pi/2n)$. Using (11), (8) and (9) we have for $1 \le k \le n$,

$$P_{n,\nu}(e^{\imath\theta_k}) = e^{-\imath(n/2-1)\varphi_\nu} e^{\imath n\theta_k/2}\Bigg(T_n\big(\xi_{n,\nu}(\theta_k)\big)+i\frac{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}{n\rho_\nu\sin(\varphi_\nu/2)}\sin\frac{\theta_k}{2}T_n'\big(\xi_{n,\nu}(\theta_k)\big)\Bigg).$$

Then, by the first part of Lemma 2 we obtain the result for odd ν.

Now let ν be even. According to (8), (9), Remark 1 and the second part of Lemma 2 it follows

$$R_{n,\nu}(\theta_k) = (-1)^{k+1}\frac{n\sin(\varphi_\nu/2)\sin\big((\varphi_\nu-\theta_k)/2\big)}{\sqrt{n^2\sin^2(\varphi_\nu/2)\sin^2\big((\varphi_\nu-\theta_k)/2\big)+\sin^2(\theta_k/2)}}$$
$$I_{n,\nu}(\theta_k) = (-1)^k\frac{\sin(\theta_k/2)}{\sqrt{n^2\sin^2(\varphi_\nu/2)\sin^2\big((\varphi_\nu-\theta_k)/2\big)+\sin^2(\theta_k/2)}}.$$

Hence, by (11) we obtain the result.

In Lemma 4 we calculate $|P_{n,\nu}'(z)|$ at the points of E_n.

LEMMA 4. *Let ν be an integer such that $1 \le \nu \le 2n-1$, φ_ν and $P_{n,\nu}$ defined as in Section 2. Then we have*

$$|P'_{n,\nu}(e^{i\varphi_\nu})| = \frac{n}{2}\left|\frac{\sqrt{\rho_\nu^2 - \cos^2(\varphi_\nu/2)}}{\sin(\varphi_\nu/2)} + \frac{1}{n}\sqrt{1-\rho_\nu^2}T'_n(\rho_\nu)\right|, \quad 1 \le \nu \le 2n-1.$$

PROOF. From (11), (8) and (9) it is easily seen that the derivative of $P_{n,\nu}(e^{i\theta})$ with respect to θ at $\theta = \varphi_\nu$ gives

$$P'_{n,\nu}(e^{i\varphi_\nu}) = A_{n,\nu} + iB_{n,\nu} \quad 1 \le \nu \le 2n-1,$$

where

$$A_{n,\nu} := \frac{\sqrt{1-\rho_\nu^2}}{2}T'_n(\rho_\nu)T_n\big(\xi_{n,\nu}(\varphi_\nu)\big) + \frac{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}{2n\rho_\nu\sin(\varphi_\nu/2)}\cos\frac{\varphi_\nu}{2}T'_n\big(\xi_{n,\nu}(\varphi_\nu)\big)$$
$$+ \frac{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}{n\rho_\nu}\xi'_{n,\nu}(\varphi_\nu)T''_n\big(\xi_{n,\nu}(\varphi_\nu)\big)$$

$$B_{n,\nu} := \left(\frac{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}{2\rho_\nu} - \frac{\sqrt{1-\rho_\nu^2}}{n}\xi'_{n,\nu}(\varphi_\nu)T'_n(\rho_\nu)\right.$$
$$\left. - \frac{\sqrt{1-\rho_\nu^2}}{2n\sin(\varphi_\nu/2)}T_n(\rho_\nu)\right)T'_n\big(\xi_{n,\nu}(\varphi_\nu)\big).$$

From (7) we derive

$$\xi'_{n,\nu}(\varphi_\nu) = -\frac{\rho_\nu^2-\cos^2(\varphi_\nu/2)}{2\rho_\nu\sin(\varphi_\nu/2)}. \tag{15}$$

Let ν be odd, then $\rho_\nu = \cos(\pi/2n)$. With the help of (3), (13), (15) and the differential equation $(1-x^2)T''_n(x) - xT'_n(x) + n^2T_n(x) = 0$ we conclude

$$A_{n,\nu} = 0, \quad B_{n,\nu} = \frac{n}{2}e^{i(\nu-1)\pi/2}\left(\frac{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}{\sin(\varphi_\nu/2)} + 1\right),$$

and then

$$P'_{n,\nu}(e^{i\varphi_\nu}) = \frac{n}{2}e^{i\nu\pi/2}\left(\frac{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}{\sin(\varphi_\nu/2)} + 1\right).$$

Now let ν be even. Using again (3), (13), (15) and the quoted differential equation we obtain

$$B_{n,\nu} = 0, \quad P'_{n,\nu}(e^{i\varphi_\nu}) = A_{n,\nu} = \frac{n}{2}e^{i\nu\pi/2}\left(\frac{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}{\sin(\varphi_\nu/2)} + \frac{\sqrt{1-\rho_\nu^2}}{n}T'_n(\rho_\nu)\right).$$

Thus the Lemma is proved.

4. **Proof of the theorem.** Let $n \in \mathbb{N}$ and ν be an integer such that $1 \leq \nu \leq 2n-1$. Further let $p \in \mathcal{P}^*_{n,1}$. By Lagrange's interpolation formula

$$p(z) = \sum_{k=1}^{n+1} p(z_k)L_k(z) = \sum_{k=1}^{n} p(z_k)L_k(z), \; z_k = e^{\imath\theta_k}, \quad 1 \leq k \leq n, \; z_{k+1} = 1 \tag{16}$$

with

$$L_k(z) = \frac{z-1}{z_k-1} \prod_{\substack{m=1\\ m\neq k}}^{n} \frac{z-z_m}{z_k-z_m}$$

and $z_k = e^{\imath\theta_k}$, $1 \leq k \leq n$ are the n points at which $|P_{n,\nu}(z)|$ attains its maximum on the unit circle. From (16) it follows

$$p'(e^{\imath\varphi_\imath}) = \sum_{k=1}^{n} p(e^{\imath\theta_k})L'_k(e^{\imath\varphi_\imath}), \quad 1 \leq \nu \leq 2n-1. \tag{17}$$

Since the θ_k $1 \leq k \leq n$ are the zeros of $\omega_{n,\nu}(\theta)$ we have

$$L_k(e^{\imath\theta}) = \frac{\sin(\theta/2)e^{\imath n\theta/2}}{\sin(\theta_k/2)e^{\imath n\theta_k/2}} \frac{\omega_{n,\nu}(\theta)}{2\sin\big((\theta-\theta_k)/2\big)\omega'_{n,\nu}(\theta_k)}$$

and then

$$\begin{aligned} L'_k(e^{\imath\theta}) = -\frac{ie^{\imath(n/2-1)\theta}}{2\sin(\theta_k/2)\omega'_{n,\nu}(\theta_k)e^{\imath n\theta_k/2}} &\left\{\frac{1}{2}\cos\frac{\theta}{2}\frac{\omega_{n,\nu}(\theta)}{\sin((\theta-\theta_k)/2)}\right. \\ &\left. + \sin\frac{\theta}{2}\left(\frac{\omega_{n,\nu}(\theta)}{\sin((\theta-\theta_k)/2)}\right)' + i\frac{n}{2}\sin\frac{\theta}{2}\frac{\omega_{n,\nu}(\theta)}{\sin\big((\theta-\theta_k)/2\big)}\right\}. \end{aligned} \tag{18}$$

We first prove that for odd ν

$$L'_k(e^{\imath\varphi_\nu}) = e^{\imath\nu\pi/2}\overline{P_{n,\nu}(e^{\imath\theta_k})}|L'_k(e^{\imath\varphi_\imath})|, \quad 1 \leq k \leq n, \tag{19}$$

whereas for even ν

$$L'_k(e^{\imath\varphi_\imath}) = e^{\imath(\nu/2+1)\pi}\overline{P_{n,\nu}(e^{\imath\theta_k})}|L'_k(e^{\imath\varphi_\nu})|, \quad 1 \leq k \leq n. \tag{20}$$

Observe that $\omega'_{n,\nu}(\theta_k) = (-1)^k|\omega'_{n,\nu}(\theta_k)|$ for $1 \leq k \leq n$. Indeed, since $\omega_{n,\nu}(\theta_1) = 0$ and $\omega_{n,\nu}(0) = 1 > 0$ as seen before, then $\omega'_{n,\nu}(\theta_1) < 0$. The same reasoning shows that $\omega'_{n,\nu}(\theta_2) > 0$, $\omega'_{n,\nu}(\theta_3) < 0, \ldots$ *etc.* So $\omega'_{n,\nu}(\theta)$ has alternating signs at the values θ_1, $\theta_2, \ldots, \theta_n$.

If ν is odd we distinguish three cases.

CASE (i). $1 \leq k \leq (\nu+1)/2 - 1$. According to (10) and (13) we have $\omega_{n,\nu}(\varphi_\nu) = 0$, further $\varphi_\nu \neq \theta_k$. Simple calculations give

$$\frac{\omega_{n,\nu}(\theta)}{\sin((\theta-\theta_k)/2)} = \frac{1-\rho_\nu^2}{n^2}T'_n(\rho_\nu)\frac{\sin((\varphi_\nu-\theta)/2)}{\sin(\varphi_\nu/2)}\frac{T'_n(\xi_{n,\nu}(\theta))}{\sin((\theta-\theta_k)/2)},$$

$$\left(\frac{\omega_{n,\nu}(\theta)}{\sin((\theta-\theta_k)/2)}\right)'_{\theta=\varphi_\nu} = e^{\imath(\nu+1)\pi/2}\frac{\rho_\nu\sqrt{1-\rho_\nu^2}}{2\sin(\varphi_\nu/2)\sin((\varphi_\nu-\theta_k)/2)\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}.$$

Then, from (18) it follows that

$$\begin{aligned}
&L_k'(e^{\imath\varphi_\nu})\\
&= -e^{\imath\pi/2}e^{\imath(n/2-1)\varphi_\nu}e^{-\imath n\theta_k/2}\frac{\rho_\nu\sqrt{1-\rho_\nu^2}(-1)^k e^{\imath(\nu+1)\pi/2}}{4|\omega_{n,\nu}'(\theta_k)|\sin(\theta_k/2)\sin((\varphi_\nu-\theta_k)/2)\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}\\
&= e^{\imath\nu\pi/2}(-1)^k e^{\imath(n/2-1)\varphi_\nu}e^{-\imath n\theta_k/2}|L_k'(e^{\imath\varphi_\nu})|\\
&= e^{\imath\nu\pi/2}\overline{P_{n,\nu}(e^{\imath\theta_k})}|L_k'(e^{\imath\varphi_\nu})| \text{ by Lemma 3.}
\end{aligned}$$

CASE (ii). $k = (\nu+1)/2$. We have $\theta_{(\nu+1)/2} = \varphi_\nu$ and $\frac{\omega_{n,\nu}(\theta)}{\sin((\theta-\varphi_\nu)/2)} = -\frac{\sqrt{1-\rho_\nu^2}}{n\sin(\varphi_\nu/2)}T_n'\big(\xi_{n,\nu}(\theta)\big)$. Then, it is easily seen that

$$\left(\frac{\omega_{n,\nu}(\theta)}{\sin((\theta-\varphi_\nu)/2)}\right)_{\theta=\varphi_\nu} = e^{\imath(\nu+1)\pi/2}\frac{\rho_\nu\sqrt{1-\rho_\nu^2}}{\sin(\varphi_\nu/2)\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}$$

and

$$\begin{aligned}
\left(\frac{\omega_{n,\nu}(\theta)}{\sin((\theta-\varphi_\nu)/2)}\right)'_{\theta=\varphi_\nu} &= (-1)^{(\nu+1)/2}\frac{\rho_\nu\sqrt{1-\rho_\nu^2}}{\sin(\varphi_\nu/2)\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}\frac{\xi_{n,\nu}(\varphi_\nu)\xi_{n,\nu}'(\varphi_\nu)}{1-\xi_{n,\nu}^2(\varphi_\nu)}\\
&= -e^{\imath(\nu+1)\pi/2}\frac{\cos(\varphi_\nu/2)}{2\sin^2(\varphi_\nu/2)}\frac{\rho_\nu\sqrt{1-\rho_\nu^2}}{\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}.
\end{aligned}$$

Using (18) in conjunction with the last two relations we obtain

$$\begin{aligned}
L_k'(e^{\imath\varphi_\nu}) &= e^{-\imath\varphi_\nu}\frac{n\rho_\nu\sqrt{1-\rho_\nu^2}}{4|\omega_{n,\nu}'(\varphi_\nu)|\sin(\varphi_\nu/2)\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}\\
&= e^{-\imath\varphi_\nu}|L_k'(e^{\imath\varphi_\nu})|\\
&= e^{\imath\nu\pi/2}\overline{P_{n,\nu}(e^{\imath\varphi_\nu})}|L_k'(e^{\imath\varphi_\nu})| \text{ by Lemma 3.}
\end{aligned}$$

CASE (iii). $(\nu+1)/2+1 \le k \le n$. As in case (i) we have $\omega_{n,\nu}(\varphi_\nu) = 0$ and $\varphi_\nu \neq \theta_k$. Then

$$\left(\frac{\omega_{n,\nu}(\theta)}{\sin\big((\theta-\theta_k)/2\big)}\right)'_{\theta=\varphi_\nu} = e^{\imath(\nu-1)\pi/2}\frac{\rho_\nu\sqrt{1-\rho_\nu^2}}{2\sin(\varphi_\nu/2)\sin\big((\theta_k-\varphi_\nu)/2\big)\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}$$

and

$$\begin{aligned}
L_k'(e^{\imath\varphi_\nu}) &= e^{\imath\nu\pi/2}(-1)^{k-1}e^{\imath(n/2-1)\varphi_\nu}e^{-\imath n\theta_k/2}\\
&\qquad\frac{\rho_\nu\sqrt{1-\rho_\nu^2}}{4|\omega_{n,\nu}'(\theta_k)|\sin(\theta_k/2)\sin\big((\theta_k-\varphi_\nu)/2\big)\sqrt{\rho_\nu^2-\cos^2(\varphi_\nu/2)}}\\
&= e^{\imath\nu\pi/2}(-1)^{k-1}e^{\imath(n/2-1)\varphi_\nu}e^{-\imath n\theta_k/2}|L_k'(e^{\imath\varphi_\nu})|\\
&= e^{\imath\nu\pi/2}\overline{P_{n,\nu}(e^{\imath\theta_k})}|L_k'(e^{\imath\varphi_\nu})| \text{ by Lemma 3.}
\end{aligned}$$

Now let ν be even. From (10) and (13) it follows that $\omega_{n,\nu}(\varphi_\nu) = e^{i\nu\pi/2}T_n(\rho_\nu)$, $\omega'_{n,\nu}(\varphi_\nu) = 0$ and

$$\left(\frac{\omega_{n,\nu}(\theta)}{\sin((\theta-\theta_k)/2)}\right)'_{\theta=\varphi_\nu} = -\frac{\cos((\varphi_\nu-\theta_k)/2)}{2\sin^2\big((\varphi_\nu-\theta_k)/2\big)}e^{i\nu\pi/2}T_n(\rho_\nu).$$

Then,

$$\begin{aligned} L'_k(e^{i\varphi_\nu}) &= e^{i(\nu/2+1)\pi}(-1)^k e^{i(n/2-1)\varphi_\nu}e^{-in\theta_k/2} \\ &\qquad \frac{-n\sin(\varphi_\nu/2)\sin((\varphi_\nu-\theta_k)/2)-i\sin(\theta_k/2)}{4|\omega'_{n,\nu}(\theta_k)|\sin(\theta_k/2)\sin^2((\varphi_\nu-\theta_k)/2)}T_n(\rho_\nu) \\ &= e^{i(\nu/2+1)\pi}\overline{P_{n,\nu}(e^{i\theta_k})}|L'_k(e^{i\varphi_\nu})| \text{ by Lemma 3.} \end{aligned}$$

Finally, applying Lagrange's interpolation formula to $P_{n,\nu}$ we have

$$P_{n,\nu}(z) = \sum_{k=1}^n P_{n,\nu}(e^{i\theta_k})L_k(z).$$

For odd ν

$$\begin{aligned} P'_{n,\nu}(e^{i\varphi_\nu}) &= \sum_{k=1}^n P_{n,\nu}\big(e^{i\theta_k}\big)L'_k(e^{i\varphi_\nu}) \\ &= \sum_{k=1}^n e^{i\nu\pi/2}\frac{\overline{L'_k(e^{i\varphi_\nu})}}{|L'_k(e^{i\varphi_\nu})|}L'_k\big(e^{i\varphi_\nu}\big) \text{ by (19)} \\ &= e^{i\nu\pi/2}\sum_{k=1}^n |L'_k\big(e^{i\varphi_\nu}\big)|, \end{aligned} \tag{21}$$

whereas for even ν

$$\begin{aligned} P'_{n,\nu}(e^{i\varphi_\nu}) &= \sum_{k=1}^n e^{i(\nu/2+1)\pi}\frac{\overline{L'_k\big(e^{i\varphi_\nu}\big)}}{|L'_k\big(e^{i\varphi_\nu}\big)|}L'_k(e^{i\varphi_\nu}) \text{ by (20)} \\ &= e^{i(\nu/2+1)\pi}\sum_{k=1}^n |L'_k(e^{i\varphi_\nu})|. \end{aligned} \tag{22}$$

Then from (17) it follows that

$$\begin{aligned} |p'(e^{i\varphi_\nu})| &\le \sum_{k=1}^n |p(e^{i\theta_k})|\,|L'_k(e^{i\varphi_\nu})| \\ &\le \sum_{k=1}^n |L'_k(e^{i\varphi_\nu})| \\ &= |P'_{n,\nu}(e^{i\varphi_\nu})| \text{ by (21) or (22),} \end{aligned}$$

which is what we wanted to prove.

It remains to show that equality holds if and only if $p = e^{i\gamma}P_{n,\nu}$ where $\gamma \in \mathbb{R}$. Suppose that

$$|p'(e^{i\varphi_\nu})| = |P'_{n,\nu}(e^{i\varphi_\nu})|,$$

i.e.

$$\left|\sum_{k=1}^{n} p(e^{i\theta_k})L_k'(e^{i\varphi_\nu})\right| = \left|\sum_{k=1}^{n} P_{n,\nu}(e^{i\theta_k})L_k'(e^{i\varphi_\nu})\right|$$

$$\left|\sum_{k=1}^{n} p(e^{i\theta_k})L_k'(e^{i\varphi_\nu})\right| = \sum_{k=1}^{n} |L_k'(e^{i\varphi_\nu})| \text{ by (21) or (22).}$$

This holds if and only if

$$p(e^{i\theta_k}) = \epsilon \frac{\overline{L_k'(e^{i\varphi_\nu})}}{|L_k'(e^{i\varphi_\nu})|}, \quad 1 \le k \le n, \text{ with } |\epsilon| = 1.$$

Then, according to (19) and (20) we obtain

$$p(e^{i\theta_k}) = \epsilon e^{-i\nu\pi/2} P_{n,\nu}(e^{i\theta_k})$$

or

$$p(e^{i\theta_k}) = \epsilon e^{-i(\nu/2+1)\pi} P_{n,\nu}(e^{i\theta_k}), \quad 1 \le k \le n;$$

further, $p(1) = P_{n,\nu}(1) = 0$. Hence, $p = e^{i\gamma}P_{n,\nu}$. This completes the proof of the Theorem.

REMARK 4. If $\nu = n$ and n is odd then $\rho_\nu = \cos(\pi/2n)$, $\phi_\nu = \pi$ and $P_{n,\nu}$ coincide with the polynomial P defined by (5).

ACKNOWLEDGMENTS. We wish to thank Professor Q. I. Rahman for proposing this problem to us. We would also like to express our gratitude for his guidance.

REFERENCES

1. C Frappier, Q I Rahman and St Ruscheweyh, *On polynomials with a prescribed zero*, Constr Approx **2**(1986), 171–177

2. A Giroux and Q I Rahman, *Inequalities for polynomials with a prescribed zero*, Trans Amer Math Soc **193**(1974), 67–98

Dalhousie University
Department of Mathematics
Halifax, Nova Scotia
B3H 3J5
email patrick@cs dal ca

Ecole Polytechnique de Thies
B P 10 Thies
Senegal