AN EXISTENCE THEOREM FOR OPTIMAL STOCHASTIC PROGRAMMING

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In [4], Hanson has obtained necessary conditions and sufficient conditions for optimality of a program in stochastic systems. However, in many cases, especially in a general treatment, a program satisfying these conditions cannot be determined explicitly, so that the question of existence of an optimal program in such systems is significant. In this paper, we obtain conditions sufficient for existence of an optimal program by applying the direct methods of the calculus of variations [9], [6] and the theory of optimal control [7], [5].

1. Definitions

Consider a probability space $(A, \boldsymbol{S}, \mu)$ [3, p. 191]. We shall assume that

(1) the probability measure μ is regular with respect to some topology on A [3, p. 224].

Let R be a closed set in E_n, and U a closed convex set in E_m. Consider a fixed measurable mapping $r : A \to R$.

We shall call a real function $f(r, u)$ on $R \times U$ "linearly bounded below in u" if

$$f(r(w), u) \geqq p(w) + u \cdot q(w)$$

for some integrable function p and bounded integrable function q on A.

Consider any number of real continuous functions $g_i(r, u)$, $h_j(r, u)$ on $R \times U$, each linearly bounded below and convex in u.

Let Γ be the class of all integrable mappings $u : A \to E_m$ such that

(2) $u(w) \in U$ for almost every $w \in A$;

(3) $(A) \int g_i(r(w), u(w))\, d\mu \leqq 0$ for each i;

(4) $h_j(r(w), u(w)) \leqq 0$ for almost every $w \in A$, for each j.

Let $\phi(r, u)$ be a real continuous function on $R \times U$. We say that a program $u_0 \in \Gamma$ is optimal (with respect to ϕ) in Γ if

$$I(u) = (A) \int \phi(r(w), u(w))\, d\mu$$

has a minimum at u_0; that is, $I(u_0) \leqq I(u)$ for all $u \in \Gamma$. For Γ nonempty, we shall prove that an optimal program exists under suitable extra conditions. Our general approach will be to make Γ closed and compact under weak convergence in $L_1 = L_1(A, S, \mu)$. Lower semicontinuity of $I(u)$ in that convergence is then sufficient for the existence of an optimal program in Γ.

We obtain the equivalent of Hanson's system if $A \subseteqq E_n$, $r(w) = w$, and $\mu = \int \psi d\lambda$ for ψ a probability density with respect to Lebesgue measure λ on E_n.

2. Semicontinuity

We shall make repeated use of the following theorem of lower semicontinuity.

THEOREM 1. *Let $f(r, u)$ be a continuous real function on $R \times U$, linearly bounded below and convex in u. Suppose that*

(5) $(A) \int |u(w)|\, d\mu$ *is bounded on* Γ.

(Conditions sufficient for this will be discussed in Section 5.) Consider a sequence of programs $u_n \in \Gamma$ converging weakly in L_1 to an integrable mapping u such that $u(w) \in U$ almost everywhere. Then, for any set $E \in S$,

$$(E) \int f(r(w), u(w))\, d\mu \leqq \liminf\, (E) \int f(r(w), u_n(w))\, d\mu.$$

Theorem 1 follows immediately from Theorem 4 and Section 6 of [8]. Note that $(E) \int (p + u_n \cdot q) d\mu$ is continuous under weak L_1 convergence, so that the discussion of Section 6 of [8] applies. Our assumption (1) about the measure μ is used in the general semicontinuity theorem of [8].

3. Closure

THEOREM 2. *Suppose that Γ satisfies condition* (5). *Then Γ is closed under weak L_1 convergence.*

PROOF. Consider a sequence of programs $u_n \in \Gamma$, and an integrable mapping $u : A \to E_m$ such that $(A) \int u_n v d\mu \to (A) \int u v d\mu$ for each $v \in L_\infty(A, S, \mu)$.

The closed convex set U is the intersection of a countable number of half spaces $\{u : b + u \cdot c \leqq 0\}$. Let $E = \{w : w \in A,\ b + u(w) \cdot c > 0\}$. By taking v above as the characteristic function of E, we have

$$0 \geqq (E) \int (b+u_n(w) \cdot c) \, d\mu \to (E) \int (b+u(w) \cdot c) \, d\mu \geqq 0.$$

Hence $(E) \int (b+u(w) \cdot c) \, d\mu = 0$, while $b+u(w) \cdot c > 0$ on E. Thus $\mu(E) = 0$, and so $u(w) \in U$ almost everywhere.

Now $(A) \int g_i(r(w), u_n(w)) d\mu \leqq 0$ for each i, and $(A) \int g_i(r(w), u_n(w)) d\mu$ is lower semicontinuous with respect to weak L_1 convergence. Hence

$$(A) \int g_i(r(w), u(w)) \, d\mu \leqq 0,$$

so that u satisfies condition (3).

Let $E_j = \{w : w \in A, h_j(r(w), u(w)) > 0\}$. Now $(E_j) \int h_j(r(w), u_n(w)) d\mu$ is lower semicontinuous with respect to weak L_1 convergence. Hence

$$0 \leqq (E_j) \int h_j(r(w), u(w)) \, d\mu \leqq \liminf \, (E_j) \int h_j(r(w), u_n(w)) \, d\mu \leqq 0.$$

Consequently $\mu(E_j) = 0$, so u satisfies condition (4).

4. The existence theorem

THEOREM 3. *Let $\phi(r, u)$ be a continuous real function on $R \times U$, linearly bounded below and convex in u. Suppose that Γ satisfies condition* (5) *and*

(6) $\int |u(w)| \, d\mu$ *is equi absolutely continuous on Γ.*

(Conditions sufficient for this will be discussed in Section 5.) Then, if Γ is not empty,

$$I(u) = (A) \int \phi(r(w), u(w)) \, d\mu$$

has a minimum on Γ.

PROOF. Since we assume that $\int |u| d\mu$ is equi absolutely continuous and $(A) \int |u| d\mu$ is bounded on Γ, Γ is compact in L_1 with respect to weak convergence [2, p. 294]. By Theorem 2, Γ is closed under weak convergence in L_1. Thus Γ is compact in itself.

By Theorem 1, $I(u)$ is lower semicontinuous with respect to weak convergence in L_1. A lower semicontinuous functional on a compact space has a minimum [1, p. 63]. Hence the result.

5. Equi absolute continuity of $\int |u| d\mu$

Conditions (5) and (6) play a key part in our existence theorem. We now study conditions sufficient for equi absolute continuity of $\int |u| d\mu$ and bounding of $(A) \int |u| d\mu$ on Γ.

For example, if U and the functions h_j are such that some, at most countable, intersection

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$$\bigcap \{u : u \in U, h_j(r(w), u) \leqq 0\}$$

is bounded uniformly on A, then conditions (5) and (6) are satisied immediately.

Alternatively, the following integral condition could be used.

THEOREM 4. *Let $\psi(u)$ be a real function on U, bounded below and such that $\psi(u)/|u| \to \infty$ as $|u| \to \infty$ on U. If $(A)\int \psi(u(w))d\mu$ is bounded on Γ, then $\int |u(w)|d\mu$ is equi absolutely continuous and $(A)\int |u(w)|d\mu$ is bounded on Γ.*

PROOF. Suppose that $(A)\int \psi(u(w))d\mu \leqq c$ on Γ; $\psi(u) \geqq b$; and, for any $\varepsilon > 0$, $(\psi(u)-b)/|u| > 1/\varepsilon$ for $|u| > m(\varepsilon)$, $u \in U$. For any program $u \in \Gamma$ and any set $M \in S$, define $M^+ = M \cap \{w : |u(w)| > m(\varepsilon)\}$, $M^- = M - M^+$. Then

$$\begin{aligned}(M)\int |u(w)|\,d\mu &\leqq \varepsilon(M^+)\int (\psi(u(w))-b)\,d\mu + m(\varepsilon)\mu(M^-)\\ &\leqq \varepsilon(c-b)+m(\varepsilon)\mu(M)\\ &< \varepsilon(c-b+1)\end{aligned}$$

if $\mu(M) < \varepsilon/m(\varepsilon)$. Thus $\int |u(w)|d\mu$ is equi absolutely continuous. Similarly,

$$(A)\int |u(w)|d\mu < c-b+m(1)$$

so that $(A)\int |u(w)|d\mu$ is bounded on Γ.

For example, a "growth condition" $g_i(r(w), u) \geqq \psi(u)$ on some g_i would be sufficient for the bounding of $(A)\int \psi(u(w))d\mu$ on Γ. Alternatively, the bounding of $(A)\int \psi(u(w))d\mu$, sufficiently for our purpose, would follow from a similar growth condition on ϕ.

THEOREM 5. *Suppose that $\phi(r(w), u) \geqq \psi(u)$ where ψ has the properties stated in Theorem 4. Then our existence theorem, Theorem 3, holds without the direct assumption of conditions (5) and (6).*

PROOF. If $I(u) = \infty$ for all $u \in \Gamma$, then the result is trivial. Otherwise, there exists $u_1 \in \Gamma$ with $I(u_1) < \infty$. In considering a minimum for $I(u)$ on Γ, we can restrict consideration to the class

$$\Gamma_1 = \Gamma \cap \{u : I(u) \leqq I(u_1)\}.$$

Now $(A)\int \psi(u(w))d\mu$ is bounded on Γ_1. Theorems 3 and 4 show that $I(u)$ has a minimum on Γ_1, which is also a minimum on Γ.

References

[1] G. Birkhoff, *Lattice theory* (Amer. Math. Soc. Coll. Publ. 25, New York, 1948).
[2] N. Dunford and J. T. Schwartz, *Linear operators, Part I* (Interscience Publishers, New York, 1958).

[3] P. R. Halmos, *Measure theory* (Van Nostrand, New York, 1950).
[4] M. A. Hanson, 'Stochastic non-linear programming', *J. Austral. Math. Soc.* 4 (1964), 347—353.
[5] E. B. Lee and L. Markus, 'Optimal control for nonlinear processes', *Arch. Rational Mech. Anal.* 8 (1961), 36—58.
[6] C. B. Morrey, 'Multiple integral problems in the calculus of variations and related topics', *Ann. Scuola Norm. Sup. Pisa* (3) 14 (1960), 1—61.
[7] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, *The mathematical theory of optimal processes* (English translation, Interscience Publishers, New York, 1962).
[8] A. W. J. Stoddart, 'Semicontinuity of integrals', *Trans. Amer. Math. Soc.* 122 (1966), 120—135.
[9] L. Tonelli, *Fondamenti di calcolo delle variazioni, Vols. I, II.* (Zanichelli, Bologna, 1921, 1923).

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