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AN EXISTENCE RESULT FOR A VARIATIONAL-LIKE INEQUALITY

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Abstract

In this paper we establish an existence result for a class of generalised variational-like inequalities, when the functions used in their definition are of type ql and satisfy some general continuity assumptions. We use a Brézis–Nirenberg–Stampacchia type result.

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1. Introduction and preliminaries

Variational-like inequality problems were studied for the first time by Parida *et al.* [13] for functions defined on $\mathbb{R}^n$. They generalised the classical variational inequalities by replacing the difference $y - x$ that appears in their formulation by a more general function $\eta(y, x)$. Subsequently, many authors continued the research in infinite-dimensional Banach spaces, for various extensions of the original problem (see, for instance, [3, 4, 6, 14, 15] and the references therein). A related class of problems, called general variational inequalities, was introduced and studied by Noor [12].

Let X be a real Banach space, X^* its dual, and $\langle \cdot, \cdot \rangle$ the usual duality pairing between X^* and X.

Let $K \subset X$ be nonempty closed convex, $T : K \to 2^{X^*}$ a set-valued mapping with nonempty values and $\eta : X \times X \to X$.

Consider the following *generalised variational-like inequality* (of Stampacchia type).

$$\text{Find } x \in K \text{ such that } \sup_{x^* \in T(x)} \langle x^*, \eta(y, x) \rangle \geq 0 \text{ for all } y \in K. \tag{1.1}$$

If $\eta(x, y) = x - y$ the problem (1.1) becomes a classical generalised variational inequality.

In many existing papers on variational-like problems, it is supposed that the function η is affine in the first variable [6, 15] or that the mapping $y \mapsto \langle z^*, \eta(y, x) \rangle$ is convex [3]. In this paper we obtain an existence result in the case where η satisfies a weaker assumption.

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The notion of *operator of type ql* was introduced in a recent paper [11] and generalises two concepts: the monotonicity of a real-valued function of a real variable and the linearity of an operator. In what follows we will use a closely related notion.

For w_1 and w_2 belonging to a linear space, we denote by $[w_1, w_2]$ the closed segment $[w_1, w_2] = \{(1-t)w_1 + tw_2 \mid t \in [0,1]\}$. For a subset A of a linear space, we denote by $\operatorname{co} A$ its convex hull.

Definition 1.1. Let U, V, W be real linear spaces. A function $\eta : U \times V \to W$ is said to be of *type ql relative to the first variable* if, for any $v \in V$ fixed, for any $u_1, u_2 \in U$ and for any $u \in [u_1, u_2]$,

$$\eta(u, v) \in [\eta(u_1, v), \eta(u_2, v)].$$

It is clear that if η is affine in the first variable it is also of type ql relative to the first variable, but the converse is not true.

We will need the following property of type ql operators, proved in [11, Theorem 3.2].

Lemma 1.2. *Let U, V, W be real linear spaces, D a convex subset of U, $v \in V$ fixed and $\eta : D \times V \to W$ a function of type ql relative to the first variable. Then, for every $n \in \mathbb{N}$, every $u_1, \ldots, u_n \in D$ and every $u \in \operatorname{co}\{u_1, \ldots, u_n\}$,*

$$\eta(u, v) \in \operatorname{co}\{\eta(u_1, v), \ldots, \eta(u_n, v)\}.$$

In what follows we will denote by $\to$ the convergence in the norm of the Banach space X, by $\rightharpoonup$ the weak convergence in X and by $\overset{*}{\rightharpoonup}$ the weak* convergence in X^*. If A is a subset of X, we denote by $\operatorname{cl} A$ its closure in the strong topology and by $\text{w-cl}\, A$ its closure in the weak topology.

Definition 1.3. A mapping $T : X \to 2^{X^*}$ is said to be $0 - \eta$ *segmentary closed* if, for every $x, y \in X$ and every net $\{x_i\}_{i \in I} \subset X$ with $x_i \rightharpoonup x$,

$$\sup_{x_i^* \in T(x_i)} \langle x_i^*, \eta((1-t)x + ty, x_i)\rangle \ge 0 \quad \text{for all } t \in [0,1],\ i \in I,$$

implies $\sup_{x^* \in T(x)} \langle x^*, \eta(y, x)\rangle \ge 0$.

A similar concept (for the particular case $\eta(x, y) = x - y$) appears in several articles (see, for instance, [1, 2, 7, 10]). In [10] this concept is called C-pseudomonotonicity.

Let K be a nonempty subset of a Hausdorff topological vector space and Y a nonempty set.

Definition 1.4 [5, 7]. A mapping $\Gamma : Y \to 2^K$ is said to be *transfer-closed valued* if for any $(y, x) \in Y \times K$ with $x \notin \Gamma(y)$ there exists $y' \in Y$ such that $x \notin \operatorname{cl}_K \Gamma(y')$. If $Y = K$ and K_0 is a subset of K then Γ is said to be *transfer-closed valued on* K_0 if the map $y \mapsto \Gamma(y) \cap K_0$, $y \in K_0$, is transfer-closed valued.

Definition 1.5. A mapping $\Gamma : K \to 2^K$ is said to be a KKM mapping if, for any finite subset A of K,

$$\operatorname{co} A \subset \bigcup_{z \in A} \Gamma(z).$$

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To prove our results, we will use the following lemmas.

Lemma 1.6 [5]. *Let K be a nonempty convex subset of a Hausdorff topological vector space E and $\Gamma : K \to 2^K$. Suppose that the following conditions are satisfied:*

(i) *Γ is a KKM mapping;*
(ii) *for every finite subset A of K, Γ is transfer-closed valued on* co A*;*
(iii) *for every $x, y \in K$,*

$$\mathrm{cl}_K\Big(\bigcap_{z\in[x,y]} \Gamma(z)\Big) \cap [x, y] = \Big(\bigcap_{z\in[x,y]} \Gamma(z)\Big) \cap [x, y];$$

(iv) there exist $B \subset K$ nonempty convex compact and $D \subset K$ nonempty compact such that, for each $x \in K \setminus D$, there exists $z \in \mathrm{co}\,(B \cup \{x\})$ such that $x \notin \Gamma(z)$.

Then $\bigcap_{x\in K} \Gamma(x) \neq \emptyset$.

Lemma 1.7 [8]. *Let U and V be topological spaces, $G : U \to 2^V$ a set-valued mapping and $g : V \times U \to \mathbb{R}$. Denote by $h : U \to \mathbb{R}$, $h(u) = \sup_{v\in G(u)} g(u, v)$ the marginal function. Suppose that the following conditions are satisfied:*

(i) *g is upper semi-continuous on $V \times U$;*
(ii) *$G(u_0)$ is compact for some $u_0 \in U$;*
(iii) *G is upper semi-continuous at u_0.*

Then h is upper semi-continuous at u_0.

Lemma 1.8 [8]. *Let U and V be Hausdorff topological spaces, $F : U \to 2^V$ a set-valued mapping. If F has nonempty compact values and is upper semi-continuous then the graph of F is closed.*

2. An existence result

The next theorem extends [5, Corollary 4.3] for a general function η.

Theorem 2.1. *Let K be a nonempty closed convex subset of the Banach space X, $T : X \to 2^{X^*}$ and $\eta : X \times X \to X$. Suppose that the following hypotheses are satisfied:*

(H1) *T is $0 - \eta$ segmentary closed;*
(H2) *for every $x \in K$, $T(x)$ is weakly* compact;*
(H3) *for every finite set $A \subset K$, T is upper semi-continuous on* co A*, with the weak* topology on X^*;*
(H4) *there exist a nonempty convex weakly compact subset $B \subset K$ and a nonempty weakly compact subset $D \subset K$ such that for each $x \in K \setminus D$ there exists $z \in \mathrm{co}\,(B \cup \{x\})$ such that $\sup_{x^*\in T(x)}\langle x^*, \eta(z, x)\rangle < 0$;*
(H5) *η is of type ql in the first variable;*

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(H6) *for every finite set $A \subset K$, for every $(z, x) \in \text{co } A \times \text{co } A$, there exists $z' \in \text{co } A$ such that, for any net $\{x_i\}_{i \in I} \subset \text{co } A$ with $x_i \to x$, we have $\eta(z', x_i) \to \eta(z, x)$;*

(H7) *for every $x \in K$ and $x^* \in T(x)$, $\langle x^*, \eta(x, x)\rangle \geq 0$.*

Then problem (1.1) admits a solution.

Proof. We will use Lemma 1.6, with E being the space X endowed with the weak topology.

Define the set-valued mapping $\Gamma : K \to 2^K$ by

$$\Gamma(z) = \Big\{x \in K \ \Big| \ \sup_{x^* \in T(x)} \langle x^*, \eta(z, x)\rangle \geq 0\Big\}.$$

Clearly, $x \in \bigcap_{z \in K} \Gamma(z)$ if and only if it is a solution of (1.1). In what follows, we will check the hypotheses of Lemma 1.6.

(i) Let $z_1, \dots, z_n \in K$ and suppose that there exists $x \in \text{co}\{z_1, \dots, z_n\}$ such that $x \notin \Gamma(z_i)$, for any $i \in \{1, \dots, n\}$. Since η is of type ql in the first variable it follows from Lemma 1.2 that $\eta(x, x)$ belongs to $\text{co}\{\eta(z_1, x), \dots, \eta(z_n, x)\}$, that is, there exist $\lambda_i \in [0, 1]$ with $\sum_{i=1}^n \lambda_i = 1$ and $\eta(x, x) = \sum_{i=1}^n \lambda_i \eta(z_i, x)$. Using (H7) we have that, for every $x^* \in T(x)$,

$$0 \leq \langle x^*, \eta(x, x)\rangle = \Big\langle x^*, \sum_{i=1}^n \lambda_i \eta(z_i, x)\Big\rangle = \sum_{i=1}^n \lambda_i \langle x^*, \eta(z_i, x)\rangle < 0,$$

which is a contradiction. So Γ is a KKM mapping.

(ii) Let $A \subset K$ be a finite set and $(z, x) \in \text{co } A \times \text{co } A$ with $x \notin \Gamma(z)$, that is,

$$\sup_{x^* \in T(x)} \langle x^*, \eta(z, x)\rangle < 0. \tag{2.1}$$

Let $z' \in \text{co } A$ be the corresponding element from (H6). Suppose by contradiction that $x \in \text{w-cl}_K(\Gamma(z') \cap \text{co } A)$. Then there exists a net $\{x_i\}_{i \in I}$, $x_i \in \Gamma(z') \cap \text{co } A$, such that $x_i \to x$. So we have $x_i \in \text{co } A$ and $\sup_{x^* \in T(x_i)} \langle x^*, \eta(z', x_i)\rangle \geq 0$.

Given $\varepsilon > 0$, from the definition of the supremum we know that for any $i \in I$, there exists $x_i^* \in T(x_i)$ such that

$$\langle x_i^*, \eta(z', x_i)\rangle > -\varepsilon. \tag{2.2}$$

Since, according to (H3), T is upper semi-continuous, it follows that $\overline{\bigcup_{i \in I} T(x_i)}$ is weakly*-compact, so there exists a convergent subnet, $\{x_j^*\}_{j \in J}$ with $x_j^* \overset{*}{\rightharpoonup} x^*$. From (H2) and (H3), using Lemma 1.8, we get that $x^* \in T(x)$.

On the other hand, (H6) implies that $\eta(z', x_j) \to \eta(z, x)$ and next

$$\langle x_j^*, \eta(z', x_j)\rangle \to \langle x^*, \eta(z, x)\rangle.$$

It follows from (2.2) that $\langle x^*, \eta(z, x)\rangle \geq -\varepsilon$ for $x^* \in T(x)$, so

$$\sup_{x^* \in T(x)} \langle x^*, \eta(z, x)\rangle \geq -\varepsilon.$$

Taking $\varepsilon \to 0$, we get a contradiction with (2.1).

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(iii) Let $x, y \in K$. We prove first that

$$\text{if } x \in \text{w-cl}_K\Big(\bigcap_{z\in[x,y]} \Gamma(z)\Big) \text{ then } x \in \bigcap_{z\in[x,y]} \Gamma(z). \tag{2.3}$$

Let $\{x_i\}_{i\in I}$ be a net such that $x_i \rightharpoonup x$ and $x_i \in \Gamma(z)$ for every $z = (1-t)x + ty \in [x, y]$. This means $x_i \in K$ and

$$\sup_{x_i^*\in T(x_i)} \langle x_i^*, \eta((1-t)x + ty, x_i)\rangle \geq 0 \quad \text{for all } t \in [0, 1],\ i \in I.$$

Since T is $0-\eta$ segmentary closed it follows that $\sup_{x^*\in T(x)}\langle x^*, \eta(y, x)\rangle \geq 0$. Also, from the closedness of K, $x \in K$.

For any $z \in [x, y]$, we have from (H5) that there exists $\lambda \in [0, 1]$ such that $\eta(z, x) = \lambda\eta(x, x) + (1-\lambda)\eta(y, x)$. It follows, using also (H7), that

$$\begin{aligned}
\sup_{x^*\in T(x)} \langle x^*, \eta(z, x)\rangle &= \sup_{x^*\in T(x)} \langle x^*, \lambda\eta(x, x) + (1-\lambda)\eta(y, x)\rangle \\
&= \sup_{x^*\in T(x)} \{\lambda\langle x^*, \eta(x, x)\rangle + (1-\lambda)\langle x^*, \eta(y, x)\rangle\} \\
&\geq \lambda \inf_{x^*\in T(x)} \langle x^*, \eta(x, x)\rangle + (1-\lambda) \sup_{x^*\in T(x)} \langle x^*, \eta(y, x)\rangle \geq 0.
\end{aligned}$$

Assumption (2.3) is now proved.

Let $u \in \text{w-cl}_K(\bigcap_{z\in[x,y]} \Gamma(z)) \cap [x, y]$. Then we also have

$$u \in \text{w-cl}_K\Big(\bigcap_{z\in[x,u]} \Gamma(z)\Big) \cap [x, u] \quad \text{and} \quad u \in \text{w-cl}_K\Big(\bigcap_{z\in[u,y]} \Gamma(z)\Big) \cap [u, y].$$

From (2.3) it follows that $u \in (\bigcap_{z\in[x,u]} \Gamma(z)) \cap [x, u]$ and also $u \in (\bigcap_{z\in[u,y]} \Gamma(z)) \cap [u, y]$, that is,

$$u \in \Big(\bigcap_{z\in[x,y]} \Gamma(z)\Big) \cap [x, y].$$

(iv) This follows directly from (H4). □

Remark 2.2. It is obvious that hypothesis (H6) is fulfilled if for any finite subset $A \subset K$, η is continuous in the second variable on co A. However, there exist functions that are not continuous in the second variable and still (H6) is satisfied, for instance $\eta_1 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$\eta_1(u, v) = \begin{cases} u^3 - v^3 & \text{for } u \neq 0 \text{ and } v \neq 0, \\ 1 & \text{for } u = 0 \text{ or } v = 0. \end{cases}$$

If the first variable is fixed, $u = u_0 \neq 0$, then the function $\eta_1(u_0, \cdot)$ is not continuous at the point 0.

To verify (H6), let $A \subset \mathbb{R}$ be finite and $(z, x) \in \text{co } A \times \text{co } A$. If $x \neq 0$ we can take $z' = z$; also if $z = x = 0$, we take $z' = z$. If $z \neq 0$ and $x = 0$, we choose $z' = 0$.

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REMARK 2.3. There exist functions that are of type ql in the first variable but are not affine in the first variable and the mapping $u \mapsto \langle w, \eta(u, v)\rangle$ (with w and v fixed) is not convex. Such a function is, for instance, $\eta_2 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by $\eta_2(u, v) = u^3 - v^3$. It is clear that it is of type ql in the first variable since it is strictly increasing in the first variable.

Another example of a function satisfying the same property is $\eta_3 : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$,

$$\eta_3(u, v) = \begin{cases} u^3 & \text{for } u < 0 \text{ and } v \neq 0, \\ u^2 + v^2 + 1 & \text{for } u > 0 \text{ and } v \neq 0, \\ 1 & \text{for } u = 0 \text{ or } v = 0. \end{cases}$$

Moreover, this function also satisfies (H6), but is not continuous in the second variable.

We investigate in the following the connection between the notion of $0 - \eta$ segmentary closedness and algebraic pseudomonotonicity, extending a result from [9].

DEFINITION 2.4. We say that a mapping $T : X \to 2^{X^*}$ is *generalised algebraic η-pseudomonotone* if for any $x, y \in X$,

$$\sup_{x^* \in T(x)} \langle x^*, \eta(y, x)\rangle \geq 0 \quad \text{implies} \quad \inf_{y^* \in T(y)} \langle x^*, \eta(y, x)\rangle \geq 0.$$

For $\eta(y, x) = y - x$ and T a single-valued function we get the notion of pseudomonotonicity introduced by S. Karamardian in 1976. A definition in the case of set-valued mappings was given in [16]: T is said to be K-pseudomonotone if for every $x, y \in X$ and $x^* \in T(x)$, $y^* \in T(y)$, $\langle x^*, y - x\rangle \geq 0$ implies $\langle y^*, y - x\rangle \geq 0$. If T has nonempty compact values this is equivalent to Definition 2.4.

THEOREM 2.5. *Let $T : X \to 2^{X^*}$ and $\eta : X \times X \to X$. Suppose that:*

(C1) *T is generalised algebraic η-pseudomonotone;*

(C2) *for every $x, y \in X$, the mapping $h : [0, 1] \to \mathbb{R}$ defined by $h(t) = \sup_{y^* \in T((1-t)x+ty)} \langle y^*, \eta(y, x)\rangle$, is upper semi-continuous at* 0*;*

(C3) *η is of type ql in the first variable and is injective;*

(C4) *η is continuous in the second variable, with the weak topology of X;*

(C5) *for every $x \in X$ and $x^* \in T(X)$, $\langle x^*, \eta(x, x)\rangle = 0$.*

Then T is $0 - \eta$ segmentary closed.

PROOF. Let $x, y \in X$ and $\{x_i\}_{i \in I}$ be a net in X, with $x_i \rightharpoonup x$ and

$$\sup_{x_i^* \in T(x_i)} \langle x_i^*, \eta((1 - t)x + ty, x_i)\rangle \geq 0 \quad \text{for all } t \in [0, 1],\ i \in I.$$

From (C1) it follows that, for each $t \in [0, 1]$, $y^* \in T((1 - t)x + ty)$,

$$\langle y^*, \eta((1 - t)x + ty, x_i)\rangle \geq 0.$$

Since η is continuous in the second variable,

$$\langle y^*, \eta((1 - t)x + ty, x)\rangle \geq 0, \tag{2.4}$$

for each $t \in [0, 1]$ and $y^* \in T((1 - t)x + ty)$.

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Let $t \in (0, 1]$. From (C3) it follows that there exists $\lambda \in (0, 1]$ such that

$$\eta((1-t)x + ty, x) = (1-\lambda)\eta(x, x) + \lambda\eta(y, x).$$

This and (2.4) imply that $\langle y^*, (1-\lambda)\eta(x, x) + \lambda\eta(y, x)\rangle \geq 0$. Taking account also of (C5),

$$\sup_{y^* \in T((1-t)x+ty)} \langle y^*, \eta(y, x)\rangle \geq 0 \quad \text{for each } t \in (0, 1]. \tag{2.5}$$

From the upper semi-continuity of h at 0 and (2.5), it follows directly that

$$\sup_{x^* \in T(x)} \langle x^*, \eta(y, x)\rangle = h(0) \geq \limsup_{t \to 0} h(t) \geq 0.$$

This proves that T is $0-\eta$ segmentary closed. □

Remark 2.6. By Lemma 1.7 it can easily be proved that hypothesis (C2) is satisfied if, for every $x \in X$, $T(x)$ is weakly* compact, and for every $x, y \in X$, the mapping $t \to T((1-t)x + ty)$ is upper semi continuous at $t = 0$ with the weak topology of X^*.

Remark 2.7. It is an immediate consequence that if T is generalised algebraic η-pseudomonotone and problem (2.4) admits a solution, then also the following Minty type generalised variational-like inequality admits a solution:

$$\text{find } x \in K \text{ such that } \inf_{y^* \in T(y)} \langle y^*, \eta(y, x)\rangle \geq 0 \text{ for all } y \in K.$$

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