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Isotropic Immersions with Low Codimension of Complex Space Forms into Real Space Forms

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Abstract. The main purpose of this paper is to determine isotropic immersions of complex space forms into real space forms with low codimension. This is an improvement of a result of S. Maeda.

1 Introduction.

We recall the notion of isotropic immersions introduced by O'Neill[6]: Let σ be the second fundamental form of a Riemannian submanifold M in a Riemannian manifold $\widetilde{M}$. Then the immersion is said to be *isotropic* at $x \in M$ if $\|\sigma(X,X)\|/\|X\|^2$ is constant for any tangent vectors $X(\neq 0) \in T_xM$. If the immersion is isotropic at every point, then there exists a function λ on M defined by $x \mapsto \|\sigma(X,X)\|/\|X\|^2$ and the immersion is said to be λ-*isotropic* or, simply, *isotropic*. If the function λ is constant on M, we call M a *constant isotropic* submanifold. Note that a totally umbilic immersion is isotropic, but not vice versa. There are many examples of isotropic submanifolds which are not totally umbilic in standard spheres. It is known that there are many isotropic immersions of rank one symmetric spaces into real space forms [10]. In particular, we pay attention to isotropic immersions of complex space forms into real space forms with low codimension.

An n-dimensional real space form $M^n(c;\mathbb{R})$ is a Riemannian manifold of constant sectional curvature c, which is locally congruent to either a standard sphere $S^n(c)$, a Euclidean space $\mathbb{R}^n$ or a real hyperbolic space $H^n(c)$, according as c is positive, zero or negative. A complex n-dimensional complex space form $M^n(c;\mathbb{C})$ is a Kähler manifold of constant holomorphic sectional curvature c, which is locally congruent to either a complex projective space $\mathbb{C}P^n(c)$, a complex Euclidean space $\mathbb{C}^n(=\mathbb{R}^{2n})$ or a complex hyperbolic space $\mathbb{C}H^n(c)$, according as c is positive, zero or negative. A quaternionic n-dimensional quaternionic space form $M^n(c;\mathbb{Q})$ is a quaternionic Kähler manifold of constant quaternionic sectional curvature c, which is locally congruent to either a quaternionic projective space $\mathbb{Q}P^n(c)$, a quaternionic Euclidean space $\mathbb{Q}^n(=\mathbb{R}^{4n})$ or a quaternionic hyperbolic space $\mathbb{Q}H^n(c)$, according as c is positive, zero or negative.

The main purpose of this paper is to prove the following:

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Theorem 1 *Let f be a λ-isotropic immersion of a complex space form $M^n(4c;\mathbb{C})$ ($n \geq 2$) of constant holomorphic sectional curvature $4c$ into a real space form $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$.*

If $p \leq n^2+n-2$, then f is a parallel embedding and locally equivalent to one of the following:

(i) *f is a totally geodesic embedding of a complex Euclidean space $\mathbb{C}^n(=\mathbb{R}^{2n})$ into a Euclidean space $\mathbb{R}^{2n+p}$, where $p \leq n^2+n-2$.*
(ii) *f is a totally umbilic embedding of $\mathbb{C}^n(=\mathbb{R}^{2n})$ into a real hyperbolic space $H^{2n+p}(\tilde{c})$, where $p \leq n^2+n-2$.*
(iii) *f is the first standard minimal embedding of a complex projective space $\mathbb{C}P^n(4c)$ into a standard sphere $S^{2n+p}(\tilde{c})$, where $p = n^2-1$ and $\tilde{c} = 2(n+1)c/n$.*
(iv) *f is a parallel embedding defined by*

$$f = f_2 \circ f_1 \colon \mathbb{C}P^n(4c) \xrightarrow{f_1} S^{n^2+2n-1}(2(n+1)c/n)) \xrightarrow{f_2} \widetilde{M}^{2n+p}(\tilde{c};\mathbb{R}),$$

where f_1 is the first standard minimal embedding, f_2 is a totally umbilic embedding, $n^2 \leq p \leq n^2+n-2$ and $2(n+1)c/n \geq \tilde{c}$.

2 Preliminaries.

Let $f\colon M \to \widetilde{M}$ be an isometric immersion of a Riemannian manifold M into a Riemannian manifold $\widetilde{M}$. We denote by ∇ (resp. $\widetilde{\nabla}$) the covariant differentiation of M (resp. $\widetilde{M}$). Then the second fundamental form σ of f is defined by $\sigma(X,Y) = \widetilde{\nabla}_X Y - \nabla_X Y$, where X and Y are vector fields tangent to M. For a vector field ξ normal to M, we write $\widetilde{\nabla}_X\xi = -A_\xi X + D_X\xi$, where X is a vector field tangent to M and $-A_\xi X$ (resp. $D_X\xi$) denotes the tangential (resp. the normal) component of $\widetilde{\nabla}_X\xi$. We define the covariant differentiation ∇' of the second fundamental form σ with respect to the connection in (tangent bundle) $\oplus$ (normal bundle) as follows: $(\nabla'_X\sigma)(Y,Z) = D_X\big(\sigma(Y,Z)\big) - \sigma(\nabla_X Y,Z) - \sigma(Y,\nabla_X Z)$, where X, Y and Z are vector fields tangent to M. The second fundamental form σ is said to be *parallel* if $\nabla'\sigma = 0$.

We write here the definitions of planar geodesic immersions and circles for readers. An isometric immersion f of M into $\widetilde{M}$ is called a *planar geodesic* immersion if every geodesic in M is mapped locally into a 2-dimensional totally geodesic submanifold of the ambient space $\widetilde{M}$ through f. A smooth curve $\gamma = \gamma(s)$ in M parametrized by its arclength s is called a *circle* if it satisfies the condition that there exists a nonnegative constant κ and a field of unit vectors $Y = Y(s)$ along this curve which satisfy the following differential equations: $\nabla_{\dot\gamma}\dot\gamma = \kappa Y$ and $\nabla_{\dot\gamma} Y = -\kappa\dot\gamma$. We call the constant κ the *curvature* of γ. As we see $\kappa = \|\nabla_{\dot\gamma}\dot\gamma\|$, we treat geodesic as circles of null curvature.

Now, we prepare the following lemmas in order to prove our results.

Lemma 1 ([7]). *Let f be an isometric immersion of an n-dimensional Riemannian manifold M^n into a real space form $M^m(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$. Then the following four conditions* (a)–(d) *are mutually equivalent:*

(a) *f is a planar geodesic immersion.*
(b) *f maps every geodesic in M^n to a circle in $\widetilde{M}^m(\tilde{c};\mathbb{R})$.*
(c) *f is an isotropic immersion with parallel second fundamental form.*
(d) *f is locally equivalent to one of the following:*

(d1) *f is a totally umbilic embedding of $M^n(c;\mathbb{R})$ into $\widetilde{M}^m(\tilde{c};\mathbb{R})$, where $c \geq \tilde{c}$.*

(d2) *f is an embedding defined by*

$$f = f_2 \circ f_1 \colon \mathbb{R}P^n\Big(\frac{nc}{2(n+1)}\Big) \xrightarrow{f_1} S^{n(n+3)/2-1}(c) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c};\mathbb{R}),$$

where f_1 is the first standard minimal embedding, f_2 is a totally umbilic embedding and $c \geq \tilde{c}$.

(d3) *f is an embedding defined by*

$$f = f_2 \circ f_1 \colon \mathbb{C}P^n\Big(\frac{2nc}{(n+1)}\Big) \xrightarrow{f_1} S^{n^2+2n-1}(c) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c};\mathbb{R}),$$

where f_1 is the first standard minimal embedding, f_2 is a totally umbilic embedding and $c \geq \tilde{c}$.

(d4) *f is an embedding defined by*

$$f = f_2 \circ f_1 \colon \mathbb{Q}P^n\Big(\frac{2nc}{(n+1)}\Big) \xrightarrow{f_1} S^{2n^2+3n-1}(c) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c};\mathbb{R}),$$

where f_1 is the first standard minimal embedding, f_2 is a totally umbilic embedding and $c \geq \tilde{c}$.

(d5) *f is an embedding defined by*

$$f = f_2 \circ f_1 \colon \mathbf{Cay}P^2(4c/3) \xrightarrow{f_1} S^{25}(c) \xrightarrow{f_2} \widetilde{M}^m(\tilde{c};\mathbb{R}),$$

where $\mathbf{Cay}P^2(c)$ represents Cayley projective plane of maximal sectional curvature c, f_1 is the first standard minimal embedding, f_2 is a totally umbilic embedding and $c \geq \tilde{c}$.

Remark 1 Note that these immersions (of (d1)–(d5)) are minimal in the case of $c = \tilde{c}$. Here the first standard minimal embedding is a minimal embedding of a compact rank one symmetric space M^n into a sphere by using eigenfunctions corresponding to the first eigenvalue of the Laplacian on M^n [8].

The following lemma shows a necessary and sufficient condition that the second fundamental form is parallel.

Lemma 2 ([1, 2, 9]) *Let M be a complex n-dimensional connected Kähler manifold with complex structure J which is isometrically immersed into a real space form $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$. Then the following two conditions are equivalent:*

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(i) *The second fundamental form σ of M in $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$ is parallel.*
(ii) $\sigma(JX, JY) = \sigma(X, Y)$ *for all* $X, Y \in TM$.

Here, we investigate the second fundamental form at one point.

Using O'Neill's method in [6], we also abstract the second fundamental form at one point to a symmetric bilinear form $(u, v) \mapsto \sigma(u, v)$ on $\mathbb{R}^n$ to $\mathbb{R}^p$. He defined *discriminant* $\triangle$ of σ, which is a real-valued function on planes Π (through origin) in $\mathbb{R}^n$ such that if u and v span Π, then

$$\triangle_{uv} = \triangle(\Pi) = \frac{\langle \sigma(u,u), \sigma(v,v)\rangle - \|\sigma(u,v)\|^2}{\|u\|^2\|v\|^2 - \langle u, v\rangle^2}.$$

For an isometric immersion $f\colon M \to \widetilde{M}$, the Gauss equation asserts that $\triangle(\Pi) = K(\Pi) - \widetilde{K}(df(\Pi))$, where K and $\widetilde{K}$ are the sectional curvatures of M and $\widetilde{M}$, respectively, and Π is any plane tangent to M.

Lemma 3 ([6]) *Let σ be a λ-isotropic bilinear form on $\mathbb{R}^n$ into $\mathbb{R}^p$ ($n \geq 2$). If $\lambda > 0$ and the discriminant $\triangle$ is constant, then $-\{(n+2)/2(n-1)\}\lambda^2 \leq \triangle \leq \lambda^2$. Furthermore, we have*

(1) $\triangle = \lambda^2 \iff \sigma$ *is umbilic* $\iff \dim \operatorname{Span}_{\mathbb{R}}\{\sigma(u, v) : u, v \in \mathbb{R}^n\} = 1$,
(2) $\triangle = -\{(n+2)/2(n-1)\}\lambda^2 \iff \sigma$ *is minimal* $\iff \dim \operatorname{Span}_{\mathbb{R}}\{\sigma(u, v) : u, v \in \mathbb{R}^n\} = n(n+1)/2 - 1$,
(3) $-\{(n+2)/2(n-1)\}\lambda^2 < \triangle < \lambda^2 \iff \dim \operatorname{Span}_{\mathbb{R}}\{\sigma(u, v) : u, v \in \mathbb{R}^n\} = n(n+1)/2$.

Next, we shall prove the following lemma.

Lemma 4 *The value of λ in our Theorem 1 is the following:*

$$\text{(i) } \lambda = 0; \qquad \text{(ii) } \lambda^2 = -\tilde{c}; \qquad \text{(iii), (iv) } \lambda^2 = 4c - \tilde{c}.$$

Proof (i) and (ii) are clear from Lemma 3. We shall consider the cases (iii) and (iv). Let ι be a totally real, totally geodesic embedding of a real projective space $\mathbb{R}P^n(c)$ into a complex projective space $\mathbb{C}P^n(4c)$. We denote by f a λ-isotropic minimal embedding of $\mathbb{C}P^n(4c)$ into a standard sphere $S^{n^2+2n-1}(2(n+1)c/n)$ (with parallel second fundamental form σ) and by J the complex structure on $\mathbb{C}P^n(4c)$. We choose a local field of orthonormal frames $\{e_1, \ldots, e_n\}$ on $\mathbb{R}P^n(c)$. Then $\{e_1, \ldots, e_n, Je_1, \ldots, Je_n\}$ is a local field of orthonormal frames on $\mathbb{C}P^n(4c)$. Since $\mathbb{R}P^n(c)$ is totally geodesic in $\mathbb{C}P^n(4c)$, we can denote by the same letter σ the second fundamental form of $\mathbb{R}P^n(c)$ in the ambient space $S^{n^2+2n-1}(2(n+1)c/n)$ through $f \circ \iota$.

We here remark that $\sigma(e_i, e_i) = \sigma(Je_i, Je_i)$ for $1 \leq i \leq n$ (see Lemma 2). This, together with the fact that $\mathbb{C}P^n(4c)$ is minimal in $S^{n^2+2n-1}(2(n+1)c/n)$, implies that our manifold $\mathbb{R}P^n(c)$ is minimal in $S^{n^2+2n-1}(2(n+1)c/n)$. Therefore, by virtue of our discussion, we know that $f \circ \iota$ is a λ-isotropic minimal embedding of $\mathbb{R}P^n(c)$ into $S^{n^2+2n-1}(2(n+1)c/n)$.

Using (2) in Lemma 3, we can see that

$$\begin{aligned}\lambda^2 &= -\frac{2(n-1)}{n+2}\Big(c - \frac{2(n+1)c}{n}\Big)\\ &= 4c - \frac{2(n+1)c}{n} = 4c - \tilde{c}.\end{aligned}$$

Thus, we can check the case (iii). Let g be a totally umbilic embedding of $S^{n^2+2n-1}(2(n+1)c/n)$ into $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$. Then g is $\sqrt{2(n+1)c/n - \tilde{c}}$-isotropic. Hence the above computation yields that

$$\begin{aligned}\lambda^2 &= \Big(4c - \frac{2(n+1)c}{n}\Big) + \Big(\frac{2(n+1)c}{n} - \tilde{c}\Big)\\ &= 4c - \tilde{c}.\end{aligned}$$

So we can check the case (iv). ∎

3 Proof of Theorem 1.

In this section, we shall prove Theorem 1.

Let J be the complex structure on $M^n(4c;\mathbb{C})$. Then the curvature tensor R of $M^n(4c;\mathbb{C})$ is given by

(3.1) $R(X,Y)Z = c\{\langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + 2\langle X,JY\rangle JZ\}$

for all vector fields X, Y and Z tangent to $M^n(4c;\mathbb{C})$.

By hypothesis, for all vector fields X on $M^n(4c;\mathbb{C})$, we have $\langle\sigma(X,X),\sigma(X,X)\rangle = \lambda^2\langle X,X\rangle\langle X,X\rangle$, which is equivalent to

(3.2) $$\begin{aligned}&\langle\sigma(X,Y),\sigma(Z,W)\rangle + \langle\sigma(X,Z),\sigma(W,Y)\rangle + \langle\sigma(X,W),\sigma(Y,Z)\rangle\\ &\qquad = \lambda^2\{\langle X,Y\rangle\langle Z,W\rangle + \langle X,Z\rangle\langle W,Y\rangle + \langle X,W\rangle\langle Y,Z\rangle\}\end{aligned}$$

for all vector fields X, Y, Z and W tangent to $M^n(4c;\mathbb{C})$, where $\langle\ ,\ \rangle$ denotes the Riemannian metric on $M^n(4c;\mathbb{C})$.

The Gauss equation is written as follows:

(3.3) $$\begin{aligned}&\langle\sigma(X,Y),\sigma(Z,W)\rangle - \langle\sigma(Z,Y),\sigma(X,W)\rangle\\ &\qquad = \langle R(Z,X)Y,W\rangle - \tilde{c}\{\langle X,Y\rangle\langle Z,W\rangle - \langle Z,Y\rangle\langle X,W\rangle\}\end{aligned}$$

for all vector fields X, Y, Z and W tangent to $M^n(4c;\mathbb{C})$. It follows from (3.1), (3.2) and (3.3) that

(3.4) $$\begin{aligned}\langle\sigma(X,Y),\sigma(Z,W)\rangle &= \frac{\lambda^2 + 2(c-\tilde{c})}{3}\langle X,Y\rangle\langle Z,W\rangle\\ &\quad + \frac{\lambda^2 - (c-\tilde{c})}{3}\{\langle X,W\rangle\langle Y,Z\rangle + \langle X,Z\rangle\langle Y,W\rangle\}\\ &\quad + c\{\langle JX,W\rangle\langle JY,Z\rangle + \langle JX,Z\rangle\langle JY,W\rangle\}\end{aligned}$$

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for all vector fields X, Y, Z and W tangent to $M^n(4c;\mathbb{C})$.

First we consider the case that $M^n(4c;\mathbb{C})$ is a totally geodesic submanifold in $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$. Then there occurs only the case (i).

Next we consider the case that $M^n(4c;\mathbb{C})$ is not totally geodesic in $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$. Then there exists some point $x_0 \in M^n(4c;\mathbb{C})$ such that $\lambda(x_0) \neq 0$. Since λ is a continuous function on $M^n(4c;\mathbb{C})$, there exists a neighborhood U of x_0 such that $\lambda > 0$ on U. We shall study on the open subset U from now on. Our discussion is divided into the two cases: (i) $\lambda^2(x_0) \neq c - \tilde{c}$ and (ii) $\lambda^2(x_0) = c - \tilde{c}$.

(i) In the following, we study at an arbitrary fixed point x of U. Note that $\lambda^2(x) \neq c-\tilde{c}$. Now we investigate the first normal space $\operatorname{Span}_{\mathbb{R}}\{\sigma(X,Y) : X, Y \in T_xM^n(4c;\mathbb{C})\}$ by using (3.4). We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1} = Je_1, \dots, e_{2n} = Je_n\}$ for $T_xM^n(4c;\mathbb{C})$. Equation (3.1) shows that $\langle R(e_i, e_j)e_j, e_i\rangle = c$ for $1 \leq i \neq j \leq n$. So, we may apply Lemma 3 to the linear subspace of $T_xM^n(4c;\mathbb{C})$, which is generated by $\{e_1, \dots, e_n\}$. Thus either the case (2) or the case (3) of Lemma 3 must hold at x.

Straightforward computation, by virtue of (3.4), yields the orthogonal relations:

$$\langle\sigma(e_i, Je_j), \sigma(e_k, Je_l)\rangle = \frac{\lambda^2 - (c - \tilde{c})}{3} \cdot \delta_{ik}\delta_{jl} \quad \text{for } 1 \leq i < j \leq n \text{ and } 1 \leq k < l \leq n. \tag{3.5}$$

$$\langle\sigma(e_i, e_j), \sigma(e_k, Je_l)\rangle = 0 \quad \text{for } 1 \leq i \leq j \leq n \text{ and } 1 \leq k < l \leq n. \tag{3.6}$$

Then, in consideration of Lemma 3, (3.5) and (3.6), the codimension p satisfies

$$p \geq n(n+1)/2 - 1 + n(n-1)/2 = n^2 - 1$$

at a fixed point x. We note that x is not an umbilic point, since $\sigma(e_i, Je_j) \neq 0$ for $1 \leq i < j \leq n$. Here we take n vectors $\sigma(e_i, Je_i)$ $(i = 1, \dots, n)$.

Similar computation shows the following orthogonal relations:

$$\langle\sigma(e_i, Je_i), \sigma(e_j, Je_j)\rangle = \frac{\lambda^2 - (4c - \tilde{c})}{3} \cdot \delta_{ij} \quad \text{for } i, j = 1, \dots, n. \tag{3.7}$$

$$\langle\sigma(e_i, e_j), \sigma(e_k, Je_k)\rangle = 0 \quad \text{for } 1 \leq i \leq j \leq n \text{ and } 1 \leq k \leq n. \tag{3.8}$$

$$\langle\sigma(e_i, Je_j), \sigma(e_k, Je_k)\rangle = 0 \quad \text{for } 1 \leq i < j \leq n \text{ and } 1 \leq k \leq n. \tag{3.9}$$

Now suppose that $\lambda^2 \neq 4c - \tilde{c}$. Then, in view of (3.7),(3.8) and (3.9), we find that $p \geq (n^2 - 1) + n$, which contradicts our assumption $p \leq n^2 + n - 2$. And hence we have

$$\lambda^2 = 4c - \tilde{c}. \tag{3.10}$$

Substituting (3.10) into the right-hand side of (3.4), we obtain

$$\begin{aligned}\langle\sigma(X,Y), \sigma(Z,W)\rangle = (2c - \tilde{c})\langle X,Y\rangle\langle Z,W\rangle + c\{\langle X,W\rangle\langle Y,Z\rangle \\ + \langle X,Z\rangle\langle Y,W\rangle + \langle JX,W\rangle\langle JY,Z\rangle + \langle JX,Z\rangle\langle JY,W\rangle\}\end{aligned} \tag{3.11}$$

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for all vector fields X, Y, Z and W tangent to $M^n(4c;\mathbb{C})$.

Equation (3.11) implies the following:

$$\langle\sigma(X,Y),\sigma(X,Y)\rangle = \langle\sigma(JX,JY),\sigma(JX,JY)\rangle = (3c-\tilde{c})\langle X,Y\rangle^2 + c\{\|X\|^2\|Y\|^2 - \langle JX,Y\rangle^2\} \tag{3.12}$$

$$\langle\sigma(X,Y),\sigma(JX,JY)\rangle = (3c-\tilde{c})\langle X,Y\rangle^2 + c\{\|X\|^2\|Y\|^2 - \langle JX,Y\rangle^2\}. \tag{3.13}$$

Thus, in view of (3.12) and (3.13), we can get $\sigma(X,Y) = \sigma(JX,JY)$ for all X, Y. And hence, from Lemma 2, we find that the second fundamental form of our immersion is parallel on U. Therefore, due to Lemma 1, there occurs the case (iii) and (iv).

(ii) Lastly, we consider the case of $\lambda^2(x_0) = c - \tilde{c}$. The above discussion asserts that the continuous function λ on U is $\lambda^2 = 4c-\tilde{c}$ or $\lambda^2 = c-\tilde{c}$. And hence, we have only to consider the case that $\lambda^2 = c - \tilde{c}$ on U. Let ι be a totally real totally geodesic embedding of a real space form $M^n(c;\mathbb{R})$ into $M^n(4c;\mathbb{C})$. It follows (1) of Lemma 3 that our manifold $(M^n(c;\mathbb{R}), f\circ\iota)$ is totally umbilic in $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$.

Here, we take an arbitrary geodesic γ in $M^n(4c;\mathbb{C})$. Since $M^n(4c;\mathbb{C})$ is a Euclidean space or a Riemannian symmetric space of rank one, we may think that γ is a geodesic in $M^n(c;\mathbb{R})$. From (b) and (d1) in Lemma 1, the curve $(f\circ\iota)\circ\gamma$ is a circle in $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$, so that the curve $f\circ\gamma$ is a circle in the ambient space $\widetilde{M}^{2n+p}(\tilde{c};\mathbb{R})$. Hence, from the same Lemma, our immersion f is one of (ii), (iii) and (iv) in Theorem 1. However there occurs only the case (ii). In fact, in either the case (iii) or (iv), we know that $\lambda^2 = 4c - \tilde{c}$ (see Lemma 4). On the other hand, in our case, $\lambda^2 = c - \tilde{c}$. This is a contradiction, because $c > 0$. Therefore, we can get the conclusion. ■

4 Quaternionic Version of Theorem 1

In this section, we investigate isotropic immersions of quaternionic space forms into real space forms with low codimension. Our aim here is to prove the following theorem, which is an improvement of the result in [4].

Theorem 2 *Let f be a λ-isotropic immersion of a quaternionic space form $M^n(4c;\mathbb{Q})$ $(n \geq 2)$ of constant quaternionic sectional curvature $4c$ into a real space form $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$ of constant sectional curvature $\tilde{c}$. If $p \leq 2n^2+2n-2$, then f is a parallel embedding and locally equivalent to one of the following:*

(i) *f is a totally geodesic embedding of a quaternionic Euclidean space $\mathbb{Q}^n(=\mathbb{R}^{4n})$ into a Euclidean space $\mathbb{R}^{4n+p}$, where $p \leq 2n^2+2n-2$.*

(ii) *f is a totally umbilic embedding of $\mathbb{Q}^n(=\mathbb{R}^{4n})$ into a real hyperbolic space $H^{4n+p}(\tilde{c})$, where $p \leq 2n^2+2n-2$.*

(iii) *f is the first standard minimal embedding of a quaternionic projective space $\mathbb{Q}P^n(4c)$ into a standard sphere $S^{4n+p}(\tilde{c})$, where $p = 2n^2-n-1$ and $\tilde{c} = 2(n+1)c/n$.*

(iv) *f is a parallel embedding defined by*

$$f = f_2\circ f_1\colon \mathbb{Q}P^n(4c) \xrightarrow{f_1} S^{2n^2+3n-1}(2(n+1)c/n) \xrightarrow{f_2} \widetilde{M}^{4n+p}(\tilde{c};\mathbb{R}),$$

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where f_1 is the first standard minimal embedding, f_2 is a totally umbilic embedding, $2n^2 - n \leq p \leq 2n^2 + 2n - 2$ and $2(n+1)c/n \geq \tilde{c}$.

Here, we prepare the following similar lemmas in order to prove Theorem 2.

Lemma 5 ([1, 5, 9]) *Let M be a quaternionic n-dimensional connected quaternionic Kähler manifold with canonical local basis $\{I, J, K\}$ which is isometrically immersed into a real space form $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$. Then the following two conditions are equivalent.*

(i) *The second fundamental form σ of M in $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$ is parallel.*
(ii) *$\sigma(IX, IY) = \sigma(JX, JY) = \sigma(KX, KY) = \sigma(X, Y)$ for all $X, Y \in TM$.*

Lemma 6 *The value of λ in our Theorem 2 is the following:*

$$\text{(i) } \lambda = 0; \qquad \text{(ii) } \lambda^2 = -\tilde{c}; \qquad \text{(iii), (iv) } \lambda^2 = 4c - \tilde{c}.$$

Proof Let ι be a totally real totally geodesic embedding of a real projective space $\mathbb{R}P^n(c)$ into a quaternionic projective space $\mathbb{Q}P^n(4c)$. The rest of the proof is similar to that of Lemma 4. ■

Now, we shall prove Theorem 2.

Let $\{I, J, K\}$ be the canonical local basis on $M^n(4c;\mathbb{Q})$. Then the curvature tensor R of $M^n(4c;\mathbb{Q})$ is given by

$$\begin{aligned} R(X,Y)Z = c\big\{ & \langle Y,Z\rangle X - \langle X,Z\rangle Y + \langle IY,Z\rangle IX - \langle IX,Z\rangle IY \\ & + \langle JY,Z\rangle JX - \langle JX,Z\rangle JY + \langle KY,Z\rangle KX - \langle KX,Z\rangle KY \\ & + 2\langle X,IY\rangle IZ + 2\langle X,JY\rangle JZ + 2\langle X,KY\rangle KZ\big\} \end{aligned} \tag{4.1}$$

for all vector fields X, Y and Z tangent to $M^n(4c;\mathbb{Q})$.

It follows from (3.2), (3.3) and (4.1) that

$$\begin{aligned} \langle \sigma(X,Y), \sigma(Z,W)\rangle = {} & \frac{\lambda^2 + 2(c-\tilde{c})}{3}\langle X,Y\rangle\langle Z,W\rangle \\ & + \frac{\lambda^2 - (c-\tilde{c})}{3}\{\langle X,W\rangle\langle Y,Z\rangle + \langle X,Z\rangle\langle Y,W\rangle\} \\ & + c\big\{\langle IX,W\rangle\langle IY,Z\rangle + \langle IX,Z\rangle\langle IY,W\rangle \\ & \quad + \langle JX,W\rangle\langle JY,Z\rangle + \langle JX,Z\rangle\langle JY,W\rangle \\ & \quad + \langle KX,W\rangle\langle KY,Z\rangle + \langle KX,Z\rangle\langle KY,W\rangle\big\} \end{aligned} \tag{4.2}$$

for all vector fields X, Y, Z and W tangent to $M^n(4c;\mathbb{Q})$, where $\langle\ ,\ \rangle$ denotes the Riemannian metric on $M^n(4c;\mathbb{Q})$.

First we consider the case that $M^n(4c;\mathbb{Q})$ is a totally geodesic submanifold in $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$. Then there occurs only the case (i). Next we consider the case that $M^n(4c;\mathbb{Q})$ is not totally geodesic in $\widetilde{M}^{4n+p}(\tilde{c};\mathbb{R})$. Then there exists some point $x_0 \in$

$M^n(4c;\mathbb{Q})$ such that $\lambda(x_0) \neq 0$. Since λ is a continuous function on $M^n(4c;\mathbb{Q})$, there exists a neighborhood U of x_0 such that $\lambda > 0$ on U. We shall study on the open subset U from now on. Our discussion is divided into the two cases: (i) $\lambda^2(x_0) \neq c - \tilde{c}$ and (ii) $\lambda^2(x_0) = c - \tilde{c}$.

(i) In the following, we study at an arbitrary fixed point x of U. Note that $\lambda^2(x) \neq c - \tilde{c}$. Now we investigate the first normal space

$$\mathrm{Span}_{\mathbb{R}}\{\sigma(X,Y) : X, Y \in T_xM^n(4c;\mathbb{Q})\}$$

by using (4.2). We choose an orthonormal basis $\{e_1, \dots, e_n, e_{n+1} = Ie_1, \dots, e_{2n} = Ie_n, e_{2n+1} = Je_1, \dots, e_{3n} = Je_n, e_{3n+1} = Ke_1, \dots, e_{4n} = Ke_n\}$ for $T_xM^n(4c;\mathbb{Q})$. Equation (4.1) shows that $\langle R(e_i, e_j)e_j, e_i\rangle = c$ for $1 \leq i \neq j \leq n$. So, we may apply Lemma 3 to the linear subspace of $T_xM^n(4c;\mathbb{Q})$, which is generated by $\{e_1, \dots, e_n\}$. Thus either the case (2) or the case (3) of Lemma 3 must hold at x.

Straightforward computation, by virtue of (4.2), yields the orthogonal relations:

$$\begin{aligned} \langle \sigma(e_i, e_j), \sigma(e_k, Ie_l)\rangle &= \langle \sigma(e_i, e_j), \sigma(e_k, Je_l)\rangle \\ &= \langle \sigma(e_i, e_j), \sigma(e_k, Ke_l)\rangle = 0 \end{aligned} \tag{4.3}$$

for $1 \leq i \leq j \leq n$ and $1 \leq k < l \leq n$.

$$\begin{aligned} \langle \sigma(e_i, Ie_j), \sigma(e_k, Je_l)\rangle &= \langle \sigma(e_i, Je_j), \sigma(e_k, Ke_l)\rangle \\ &= \langle \sigma(e_i, Ke_j), \sigma(e_k, Ie_l)\rangle = 0 \end{aligned} \tag{4.4}$$

for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$.

$$\begin{aligned} \langle \sigma(e_i, Ie_j), \sigma(e_k, Ie_l)\rangle &= \langle \sigma(e_i, Je_j), \sigma(e_k, Je_l)\rangle \\ &= \langle \sigma(e_i, Ke_j), \sigma(e_k, Ke_l)\rangle = \frac{\lambda^2 - (c - \tilde{c})}{3} \cdot \delta_{ik}\delta_{jl} \end{aligned} \tag{4.5}$$

for $1 \leq i < j \leq n$ and $1 \leq k < l \leq n$.

Then, in consideration of Lemma 3, (4.3),(4.4) and (4.5), the codimension p satisfies

$$p \geq n(n+1)/2 - 1 + 3n(n-1)/2 = 2n^2 - n - 1.$$

at a fixed point x. We note that x is not an umbilic point, since $\sigma(e_i, Je_j) \neq 0$ for $1 \leq i < j \leq n$. Here we take $3n$ vectors $\sigma(e_i, Ie_i), \sigma(e_i, Je_i)$ and $\sigma(e_i, Ke_i)$ $(i = 1, \dots, n)$.

Similar computation shows the following orthogonal relations:

$$\begin{aligned} \langle \sigma(e_i, e_j), \sigma(e_k, Ie_k)\rangle &= \langle \sigma(e_i, e_j), \sigma(e_k, Je_k)\rangle \\ &= \langle \sigma(e_i, e_j), \sigma(e_k, Ke_k)\rangle = 0 \end{aligned} \tag{4.6}$$

for $1 \leq i \leq j \leq n$ and $1 \leq k \leq n$.

$$\begin{aligned} \langle \sigma(e_i, Ie_j), \sigma(e_k, Ie_k)\rangle &= \langle \sigma(e_i, Je_j), \sigma(e_k, Je_k)\rangle \\ &= \langle \sigma(e_i, Ke_j), \sigma(e_k, Ke_k)\rangle = 0 \end{aligned} \tag{4.7}$$

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for $1 \le i < j \le n$ and $1 \le k \le n$.

$$\begin{aligned}(4.8)\qquad \langle \sigma(e_i, Ie_i), \sigma(e_j, Je_j)\rangle &= \langle \sigma(e_i, Je_i), \sigma(e_j, Ke_j)\rangle \\ &= \langle \sigma(e_i, Ke_i), \sigma(e_j, Ie_j)\rangle = 0\end{aligned}$$

for $i, j = 1, \dots, n$.

$$\begin{aligned}(4.9)\qquad \langle \sigma(e_i, Ie_i), \sigma(e_j, Je_k)\rangle &= \langle \sigma(e_i, Ie_i), \sigma(e_j, Ke_k)\rangle = \langle \sigma(e_i, Je_i), \sigma(e_j, Ie_k)\rangle \\ &= \langle \sigma(e_i, Je_i), \sigma(e_j, Ke_k)\rangle = \langle \sigma(e_i, Ke_i), \sigma(e_j, Ie_k)\rangle \\ &= \langle \sigma(e_i, Ke_i), \sigma(e_j, Je_k)\rangle = 0\end{aligned}$$

for $1 \le i \le n$ and $1 \le j < k \le n$.

$$\begin{aligned}(4.10)\qquad \langle \sigma(e_i, Ie_i), \sigma(e_j, Ie_j)\rangle &= \langle \sigma(e_i, Je_i), \sigma(e_j, Je_j)\rangle \\ &= \langle \sigma(e_i, Ke_i), \sigma(e_j, Ke_j)\rangle = \frac{\lambda^2 - (4c - \tilde{c})}{3} \cdot \delta_{ij}\end{aligned}$$

for $i, j = 1, \dots, n$.

Now suppose that $\lambda^2 \neq 4c - \tilde{c}$. Then, in view of (4.6),(4.7),(4.8),(4.9) and (4.10), we find that $p \ge (2n^2 - n - 1) + 3n = 2n^2 + 2n - 1$, which contradicts our assumption $p \le 2n^2 + 2n - 2$. And hence we have

$$(4.11)\qquad \lambda^2 = 4c - \tilde{c}.$$

Substituting (4.11) into the right-hand side of (4.2), we obtain

$$\begin{aligned}(4.12)\qquad \langle \sigma(X, Y), \sigma(Z, W)\rangle = (2c - \tilde{c})&\langle X, Y\rangle\langle Z, W\rangle \\ + c\big\{ &\langle X, W\rangle\langle Y, Z\rangle + \langle X, Z\rangle\langle Y, W\rangle \\ &+ \langle IX, W\rangle\langle IY, Z\rangle + \langle IX, Z\rangle\langle IY, W\rangle \\ &+ \langle JX, W\rangle\langle JY, Z\rangle + \langle JX, Z\rangle\langle JY, W\rangle \\ &+ \langle KX, W\rangle\langle KY, Z\rangle + \langle KX, Z\rangle\langle KY, W\rangle \big\}\end{aligned}$$

for all vector fields X, Y, Z and W tangent to $M^n(4c; \mathbb{Q})$.

Equation (4.12) implies the following:

$$\begin{aligned}(4.13)\qquad \langle \sigma(X, Y), \sigma(X, Y)\rangle &= \langle \sigma(IX, IY), \sigma(IX, IY)\rangle \\ &= \langle \sigma(JX, JY), \sigma(JX, JY)\rangle \\ &= \langle \sigma(KX, KY), \sigma(KX, KY)\rangle \\ &= (3c - \tilde{c})\langle X, Y\rangle^2 \\ &\quad + c\{\|X\|^2\|Y\|^2 - \langle IX, Y\rangle^2 - \langle JX, Y\rangle^2 - \langle KX, Y\rangle^2\}.\end{aligned}$$

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(4.14)
$$\begin{aligned}\langle\sigma(X,Y),\sigma(IX,IY)\rangle &= \langle\sigma(X,Y),\sigma(JX,JY)\rangle\\ &= \langle\sigma(X,Y),\sigma(KX,KY)\rangle\\ &= (3c-\tilde{c})\langle X,Y\rangle^2\\ &\quad + c\{\|X\|^2\|Y\|^2-\langle IX,Y\rangle^2-\langle JX,Y\rangle^2-\langle KX,Y\rangle^2\}.\end{aligned}$$

Thus, in consideration of (4.13) and (4.14), we can get $\sigma(X,Y)=\sigma(IX,IY)=\sigma(JX,JY)=\sigma(KX,KY)$ for all X,Y. And hence, from Lemma 5, we find that the second fundamental form of our immersion is parallel on U. Therefore, due to Lemma 1, there occurs the case (iii) and (iv).

(ii) Lastly, we consider the case of $\lambda^2(x_0)=c-\tilde{c}$. From Lemma 6 and the same discussion as in Theorem 1, we can get the conclusion. ∎

Remark 2 The following problem is still open.

Problem *Let f be a λ-isotropic immersion of a real space form $M^n(c;\mathbb{R})$ into a real space form $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$. If $p\leq n(n+1)/2$, is f locally equivalent to one of the following?*

(i) *f is a totally umbilic embedding of $M^n(c;\mathbb{R})$ into $\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$, where $p\leq n(n+1)/2$.*

(ii) *f is the first standard minimal embedding of a real projective space $\mathbb{R}P^n(c)$ into a standard sphere $S^{n+p}(\tilde{c})$, where $p=n(n+1)/2-1$ and $\tilde{c}=2(n+1)c/n$.*

(iii) *f is a parallel embedding defined by*

$$f=f_2\circ f_1\colon \mathbb{R}P^n(c)\xrightarrow{f_1} S^{n(n+3)/2-1}\Big(\frac{2(n+1)c}{n}\Big)\xrightarrow{f_2}\widetilde{M}^{n+p}(\tilde{c};\mathbb{R})$$

where f_1 is the first standard minimal embedding, f_2 is a totally umbilic embedding, $p=n(n+1)/2$ and $2(n+1)c/n\geq\tilde{c}$.

S. Maeda [3] gave an affirmative partial answer to this problem.

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