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ON STRONGLY RIGHT BOUNDED FINITE RINGS II

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An associative ring R is called a BT-ring if R is strongly right bounded, but not right duo, and not strongly left bounded. We show that the order of the smallest BT-rings (without unity) is 16, while we prove earlier that the order of the smallest unitary BT-rings is 32.

From [3], an associative ring R is *right (left) duo* if every right (left) ideal is an ideal, and R is *strongly right (left) bounded* if every nonzero right (left) ideal contains a nonzero ideal. The interesting result [3, Proposition 6] states that a ring is right duo if and only if every factor ring is strongly right bounded. For convenience, we call R a *BT-ring* in case R is strongly right bounded, but not right duo, and not strongly left bounded.

Let p be a prime number and $\mathbb{Z}_p$ the field with p elements. In view of Birkenmeier [1, Example 9] we note that the unitary ring $\begin{bmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ 0 & \mathbb{Z} \end{bmatrix}$ is a BT-ring. Birkenmeier and Tucci [3, Example 8] constructed a finite unitary BT-ring with 32 elements, and recently the author [5] proved that 32 is just the order of the smallest unitary BT-rings. In this paper, we shall consider finite BT-rings (not necessarily possessing a unity) and show that the smallest such rings have 16 elements. Related questions are also discussed.

If R is a ring and $r \in R$, we let $\langle r\rangle_R$ (${}_R\langle r\rangle$, ${}_R\langle r\rangle_R$) denote the right (left, two-sided) ideal of R generated by r. For a finite ring R, let $|R|$ denote the order of R and $\operatorname{char}(R)$ denote the characteristic of R.

Using an idea of [3, Example 8], we first give a BT-ring with 16 elements.

Example 1: Let $\langle x^3\rangle$ be the ideal of the polynomial ring $\mathbb{Z}_2[x]$ generated by x^3. Let $S = \mathbb{Z}_2[x]/\langle x^3\rangle$ and $R = \begin{bmatrix} 0 & S\overline{x} \\ 0 & S\overline{x} \end{bmatrix}$. Let $a = \begin{bmatrix} 0 & \overline{x} \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 & 0 \\ 0 & \overline{x} \end{bmatrix}$. The right ideal $\langle b\rangle_R$ is not an ideal since $ab \notin \langle b\rangle_R$. So R is not right duo. The minimal right ideals $\langle ab\rangle_R$, $\langle b^2\rangle_R$ and $\langle ab+b^2\rangle_R$ are ideals, so R is strongly right bounded. Since

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$(a+ab)b = ab \notin {}_R\langle a+ab\rangle$, the minimal left ideal ${}_R\langle a+ab\rangle$ is not an ideal. So R is not strongly left bounded. It follows that R is a BT-ring with 16 elements.

We need the following propositions to prove that the ring R in the above example is one of the smallest BT-rings.

Proposition 2. [4] *A finite ring R is commutative if the order $|R|$ of R has square free factorisation.*

Proposition 3. *Let R be a ring with p^2 elements. If R is strongly right bounded then it is right duo.*

Proof: Each non-zero proper right ideal I of R has p elements. Hence I is a minimal right ideal that is two-sided. ∎

Recall that there are exactly two (up to isomorphism) noncommutative rings with order p^2; one is $S_1 = \begin{bmatrix} \mathbb{Z}_p & \mathbb{Z}_p \\ 0 & 0 \end{bmatrix}$ and the other one is $S_2 = \begin{bmatrix} \mathbb{Z}_p & 0 \\ \mathbb{Z}_p & 0 \end{bmatrix}$, where S_1 is right duo but not strongly left bounded and S_2 is left duo but not strongly right bounded. In [5] we proved that a unitary ring with p^4 elements is strongly right bounded if and only if it is strongly left bounded. For rings with p^3 elements we have:

Proposition 4. *Let R be a ring with p^3 elements. If R is strongly right bounded but not right duo, then R is strongly left bounded, that is, any ring with order p^3 is not a BT-ring.*

Proof: Assume we have a BT-ring R with p^3 elements.

Since R is strongly right bounded but not right duo, there is a principal right ideal B which is not an ideal and B contains a non-zero ideal I. Then $|B| = p^2$ and $|I| = p$. Now R is not strongly left bounded, so there is a minimal left ideal L that is not an ideal. We see that R/I is not strongly right bounded, so we may assume $R/I = \begin{bmatrix} \mathbb{Z}_p & 0 \\ \mathbb{Z}_p & 0 \end{bmatrix}$, where $(L+I)/I = \begin{bmatrix} 0 & 0 \\ \mathbb{Z}_p & 0 \end{bmatrix}$ and $B/I = \begin{bmatrix} \mathbb{Z}_p & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$p = |(L+I)/I| = |L/(L\cap I)| = |L|,$$

where $L \cap I = 0$ since L is a minimal left ideal but not an ideal. Therefore $L = {}_R\langle c\rangle = \{jc \mid 0 \leqslant j < p\}$ where $c + I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. And let $B = \langle b\rangle_R$ where $b + I = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. We have that $R = L + B$. Since R is not commutative, $\operatorname{char}(R) = p$ or p^2.

(1) If $\operatorname{char}(R) = p^2$ then $I = {}_R\langle pb\rangle_R$. We have $cb = c + npb$, where $npb \neq 0$ since $L = {}_R\langle c\rangle$ is not a right ideal. Let $b^2 = b + mpb$, where $0 \leqslant m < p$. Then $cb^2 = cb + npb^2$, that is, $c(b+mpb) = cb + np(b+mpb)$. Since $cp = 0$ and $p^2b = 0$ we see that $0 = npb$, a contradiction.

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(2) If $\operatorname{char}(R) = p$, R is a $\mathbb{Z}_p$-algebra. The *Dorroh extension* of R via $\mathbb{Z}_p$, denoted by $(R;\, \mathbb{Z}_p)$, is the standard one with the operations on $R \times \mathbb{Z}_p$ of

(i) $(r_1, a_1) + (r_2, a_2) = (r_1 + r_2, a_1 + a_2)$,

(ii) $(r_1, a_1)(r_2, a_2) = (r_1 r_2 + r_1 a_2 + a_1 r_2, a_2 a_2)$,

yielding $(R;\, \mathbb{Z}_p)$ is a unitary ring with p^4 elements. Since R is not strongly left bounded, neither is $(R; \mathbb{Z}_p)$. By [**5**, Theorem], $(R;\, \mathbb{Z}_p)$ is not strongly right bounded, either. Now R is strongly right bounded, so it follows from [**2**, Corollary 1.5] that there exists $(r, a) \in (R;\, \mathbb{Z}_p)$ with $a \neq 0$ such that $rt + at = 0$ for all $t \in R$. Then $et = t$ where $e = -a^{-1}r$, that is, e is a left identity of R. If $e \in B$, $c = ec \in B$, a contradiction since $L \cap B = 0$. So $e = jc + b'$ for some $b' \in B$ and $0 < j < p$, and then $b = eb = (jc + b')b = jcb + b'b$. Then $jcb = b - b'b \in B$ and so $cb \in B$. It follows that B is a left ideal of R, a contradiction. ∎

Now we are ready to prove our main result.

Theorem 5. *The order of the smallest BT-rings is* 16.

Proof: The BT-ring in Example 1 has 16 elements. Let R be a finite ring and $|R| < 16$. We shall prove that R is not a BT-ring. Since $|R| < 16$, we have the following two cases to be considered:

Case (1): $|R| = p^n$ for some prime number p; then we must have $n \in \{1, 2, 3\}$ since $|R| < 16$. Then by Propositions 2, 3, and 4, R cannot be a BT-ring.

Case (2): $|R| = q^m p^n$ where q and p are two distinct primes. It follows that either $m = 1$ or $n = 1$, since $|R| < 16$. Without loss of generality, we assume that $|R| = qp^n$ where $n \in \{1, 2\}$. Since any finite ring R is a direct sum of rings of prime power order, we let $R = R_1 \oplus R_2$ where $|R_1| = q$ and $|R_2| = p^n$. From Case (1) we see that R_2 is not a BT-ring. Now R_1 is commutative by Proposition 2; hence $R = R_1 \oplus R_2$ is not a BT-ring. ∎

If R is a finite unitary BT-ring, then, according to [5], $|R|$ must have a factor of the form p^5. One may ask, if R is a finite BT-ring without unity, does $|R|$ have a factor of the form p^4? The following example gives a negative answer.

Example 6: Let $\mathbb{Z}_2[x]/\langle x^2 \rangle = \{0, 1, x, 1 + x\}$. Let $X = \{0, x\}$ and $A = \{0, 1\}$. Consider the ring $R_1 = \begin{bmatrix} 0 & X \\ X & A \end{bmatrix}$. The right ideal $\begin{bmatrix} 0 & 0 \\ X & A \end{bmatrix}$ is not an ideal. So R_1 is not right duo. The minimal right ideals $\begin{bmatrix} 0 & 0 \\ X & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & X \\ 0 & 0 \end{bmatrix}$ are ideals, so R_1 is strongly right bounded. Similarly, R_1 is also strongly left bounded but not left duo. One notes that $|R| = 8$. Let $R_2 = \begin{bmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & 0 \end{bmatrix}$, which is right duo but not strongly left bounded. Now $R = R_1 \oplus R_2$ is a BT-ring with $2^3 \cdot 3^2$ elements.

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In the above example, neither R_1 nor R_2 is a BT-ring but $R = R_1 \oplus R_2$ is a BT-ring. However, if $R = R_1 \oplus R_2$ is a finite unitary BT-ring, then at least one of R_1 and R_2 must be a BT-ring. This follows from [**5**, Proposition 3] which states that a finite unitary ring is right duo if and only if it is left duo. □

We conclude this paper with the following questions: Does there exist a (unitary) BT-ring with 16 (32) elements that is not isomorphic to the BT-ring R of Example 1 [**3**, Example 8]?

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