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The Dual Pair $G_2 \times \mathrm{PU}_3(D)$ (p-Adic Case)

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Abstract. We study the correspondence of representations arising by restricting the minimal representation of the linear group of type E_7 and relative rank 4. The main tool is computations of the Jacquet modules of the minimal representation with respect to maximal parabolic subgroups of G_2 and $\mathrm{PU}_3(D)$.

Introduction

Let F be a p-adic field of residue characteristic not 2, and let D be the non-split quarternion algebra over F. Associated to D, there is a linear adjoint algebraic group $\underline{H_D}$ over F of type E_7, and relative rank 4. We shall let H_D denote the group of F-points of $\underline{H_D}$. There is a reductive dual pair

$$G_2 \times \mathrm{PU}_3(D) = G \times G' \subset H_D$$

Here, G_2 is split and $\mathrm{PU}_3(D)$ is an inner form of PGSp_6 of relative rank one. In this paper, we study the dual pair correspondence which arises from the restriction of the minimal representation Π of H_D to the dual pair $G_2 \times \mathrm{PU}_3(D)$. Recall that the minimal representation is the analogue of the Weil representation of the metaplectic group. As usual, if σ is an irreducible admissible representation of G_2, we let $\Theta(\sigma)$ denote the set of irreducible admissible representations σ' of $\mathrm{PU}_3(D)$, counted with multiplicities, such that $\sigma \otimes \sigma'$ is a quotient of Π. Similarly, we have the set $\Theta(\sigma')$. Then we determine the sets $\Theta(\sigma)$ and $\Theta(\sigma')$ when σ and σ' are non-cuspidal representations.

The techniques used in this work can already be found in [MS], where, among other things, the correspondence of tempered spherical representations was determined for the dual pair $G_2 \times \mathrm{PGSp}_6$ in the split adjoint group of type E_7. The point is that, to determine the correspondence of non-cuspidal representations, one reduces to the determination of the Jacquet modules of Π with respect to the maximal parabolic subgroups of G_2 and $\mathrm{PU}_3(D)$. There are essentially two steps involved in this. First, we determine the restriction of Π to certain maximal parabolic subgroups of H_D. Unlike the case of the Weil representation, where we have several different smooth models of the representation at hand, we compute the restriction just by using the fact that Π is minimal. The second step is essentially an orbit computation, involving the internal modules of the groups in question.

Let us describe the main results of the paper. Let π' be an irreducible (finite-dimensional) representation of $D^\times$. In [JL], Jacquet and Langlands associated to π' a square-integrable representation $\pi = \mathrm{JL}(\pi')$ of GL_2, an inner form of $D^\times$. Let Q be the minimal parabolic subgroup of $\mathrm{PU}_3(D)$. The Levi factor L of Q is given by:

$$L \cong D^\times \times D^\times / F^\times$$

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where $F^\times$ is embedded via $t \mapsto (t, t^{-1})$. Using unnormalized induction throughout the paper, let

$$\sigma' = \mathrm{Ind}_Q^{G'} \delta_Q^{1/2}(\pi' \times \pi')$$

where δ_Q is the modulus character of Q. Also, we let $Q_1 = L_1U_1$ (respectively $Q_2 = L_2U_2$) be the maximal parabolic subgroup of G_2 whose Levi factor is generated by the long root (respectively short root) of G_2, and let

$$\begin{cases} \sigma_1 = \mathrm{Ind}_{Q_1}^G \delta_{Q_1}^{1/2}\pi \\ \sigma_2 = \mathrm{Ind}_{Q_2}^G \delta_{Q_2}^{1/2}\pi. \end{cases}$$

For the ease of exposition, let us suppose that π is supercuspidal and both σ_1 and σ' are irreducible, which is true generically. Then we shall show:

$$\begin{cases} \Theta(\sigma_1) = \{\sigma'\} \\ \Theta(\sigma') = \{\sigma_1\} \end{cases}$$

and

$$\Theta(\sigma_2) = \varnothing.$$

Also, let St (respectively St$'$) be the Steinberg representation of G (respectively G'). Then

$$\Theta(\mathrm{St}') = \{\mathrm{St}\}.$$

Note that the above results are predicted by the natural inclusion of L-groups, that is, they respect Langlands functoriality. We refer the reader to [G], for more precise results, and a global variant of this correspondence.

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1 Representations of ℓ-Groups

In this first section, we establish some notation and discuss some basic facts on induced representations that are required later.

Let G be the F-points of a reductive algebraic group over F. Then recall that G is an ℓ-group, in the terminology of [BZ]. Let Alg(G) be the category of smooth representations of G, and let Irr(G) be the set of simple objects of Alg(G).

Recall that if $\mathrm{P} = \mathrm{MN}$ is a parabolic subgroup of G, then we have an exact functor

$$\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}} \colon \mathrm{Alg}(\mathrm{M}) \longrightarrow \mathrm{Alg}(\mathrm{G})$$

whose left and right adjoints are given by the functors of co-invariants (Jacquet functors). More precisely, let $\bar{\mathrm{N}}$ be the opposite unipotent radical. Then

$$\begin{cases} \mathrm{Hom}_{\mathrm{G}}\big(\pi, \mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\delta^{1/2}\sigma)\big) = \mathrm{Hom}_{\mathrm{M}}(\pi_{\mathrm{N}}, \delta^{1/2}\sigma) \\ \mathrm{Hom}_{\mathrm{G}}\big(\mathrm{Ind}_{\mathrm{P}}^{\mathrm{G}}(\delta^{1/2}\sigma), \pi\big) = \mathrm{Hom}_{\mathrm{M}}(\bar{\delta}^{1/2}\sigma, \pi_{\bar{\mathrm{N}}}) \end{cases} \tag{1.1}$$

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where δ and $\bar{\delta}$ are the moduli characters of P and $\bar{\mathrm{P}}$.

Finally, we record below an easy lemma:

Lemma 1.2 *Let* $0 \to V_1 \to V_2 \to V_3 \to 0$ *be an exact sequence of* G*-modules. If the center of* G *acts on* V_1 *and* V_3 *by different eigenvalues, then the sequence is split.*

2 Jacquet-Langlands Correspondence and Weil Representation

In this section, we recall the local Jacquet-Langlands correspondence from [JL]. First of all, there is a natural bijection between regular elliptic conjugacy classes of $\mathrm{GL}_2(F)$ and $D^\times$. For each π in $\mathrm{Irr}\big(\mathrm{GL}_2(F)\big)$, we write ch_π for its character, which is, by a well-known result of Harish-Chandra, a locally integrable function, locally constant on the set of all regular conjugacy classes. Then there exists a bijection $\pi \leftrightarrow \pi'$ between the set of all classes of irreducible essentially square integrable representations of $\mathrm{GL}_2(F)$ and the set of all classes of irreducible representations of $D^\times$ characterized by

$$-\,\mathrm{ch}_\pi = \mathrm{ch}_{\pi'}$$

on the set of regular elliptic conjugacy classes. Note that π is supercuspidal if the dimension of π is greater than one. Otherwise, π is a special representation in the terminology of [JL].

Let ψ be an additive (unitary) character ψ of F of conductor $\mathcal{O}_F$, the ring of integers of F. As in the introduction, D is the unique non-split quarternion algebra over the p-adic field F. Let Tr and N be the reduced trace and reduced norm on D. For $x \in D$, $\bar{x}$ denotes its conjugate with respect to the canonical anti-involution on D. The Jacquet-Langlands correspondence can be constructed using (a particular case of) the Weil representation W of $\mathrm{SL}_2(F)$. The space of W is the space of Schwarz function $S(D)$ on D, and the action of $\mathrm{SL}_2(F)$ is completely specified by:

$$\begin{cases} W\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\Phi(x) = |a|^2\Phi(ax) \\ W\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\Phi(x) = \psi\big(b\mathrm{N}(x)\big)\Phi(X) \\ W\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\Phi(x) = -\hat{\Phi}(x). \end{cases}$$

Here, $\hat{\Phi}$ denotes the Fourier transform of Φ with respect to the Haar measure on D determined by the character $\psi \circ \mathrm{Tr}$ of D.

Let $\tilde{R} = \mathrm{GL}_2(F) \times (D^\times \times D^\times / F^\times)$. The action of SL_2 on $S(D)$ defined above, extends to the action (also denoted by W) of

$$R = \{(g, \alpha, \beta) \in \tilde{R} : \det(g) = \mathrm{N}(\alpha\beta)\}.$$

To describe this action, it suffices to do so for elements of the form (g, α, β), with

$$g = \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}.$$

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Then the action is given by

$$W(g,\alpha,\beta)\Phi(x) = |a|\Phi(\bar{\alpha}x\beta).$$

Now let

$$\tilde{W} = \operatorname{ind}_R^{\tilde{R}} W.$$

For a general discussion of the representation $\tilde{W}$, we refer the reader to [Ro], where theta correspondences for similitude groups are treated at length. We now state the result of Jacquet and Langlands in terms of the action of $\tilde{R}$ on $\tilde{W}$:

Theorem 2.1 *Let π' be an irreducible representation of $D^\times$, and let $\tilde{\pi}$ be the contragredient of $\mathrm{JL}(\pi')$. Let $\Theta(\pi'\otimes\pi')$ be the set of irreducible admissible representations σ of $\mathrm{GL}_2(F)$ such that $\pi'\otimes\pi'\otimes\sigma$ is a quotient of $\tilde{W}$. Then*

$$\begin{cases}\Theta(\pi'\otimes\pi') = \{\tilde{\pi}\}\\ \Theta(\tilde{\pi}) = \{\pi'\otimes\pi'\}.\end{cases}$$

3 Groups and Dual Pairs

In this section, we describe the various groups that will be studied in this paper. In particular, we shall describe the group H_D, and the dual pair $G_2 \times \mathrm{PU}_3(D)$.

Let $J = J(D)$ be the Jordan algebra of 3-by-3 hermitian matrices with coefficients in D:

$$X = \begin{pmatrix} a & x & y\\ \bar{x} & b & z\\ \bar{y} & \bar{z} & c\end{pmatrix}.$$

Note that the algebra J has a natural decomposition

$$J = \oplus J_{ij} \quad (1 \le i \le j \le 3)$$

where J_{ij} consists of matrices X in J whose entries are 0 at all positions different from (i,j) and (j,i). In particular,

$$\begin{cases} J_{ii} \cong F\\ J_{ij} \cong D \quad \text{if } i < j.\end{cases}$$

Recall that J has a natural cubic form det, the determinant form, which in turn gives rise to a symmetric trilinear form such that $(X,X,X) = 6\det(X)$. The value (X,Y,Z) is given by

$$(3.1)\qquad \begin{aligned}(X,Y,Z) = {}& 2\,\mathrm{Tr}(XYZ) + \mathrm{Tr}(X)\,\mathrm{Tr}(Y)\,\mathrm{Tr}(Z)\\ &- \mathrm{Tr}(X)\,\mathrm{Tr}(YZ) - \mathrm{Tr}(Y)\,\mathrm{Tr}(XZ) - \mathrm{Tr}(Z)\,\mathrm{Tr}(XY).\end{aligned}$$

If X and Y are in J, then let $X \times Y \in J^*$ be the element such that

$$(X\times Y)(Z) = (X,Y,Z)$$

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for all $Z \in J$. Finally, we note that $(X, Y) = \mathrm{Tr}(XY)$ defines a symmetric bilinear form, which can be used to identify J and J^*.

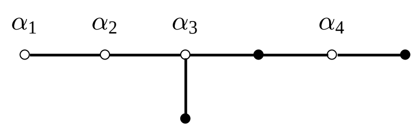
Let L_D be the algebraic group of linear transformation on J which preserves the determinant form. Then it is well-known that

$$L_D \cong \mathrm{SL}_3(D)/\mu_2$$

(as algebraic groups). The action of L_D on J is given, for $X \in J$, by:

$$X \mapsto gX\bar{g}^t$$

where $g \in \mathrm{SL}_3(D)$. Moreover $\mathrm{GL}_3(D)$ also acts on J by the same formula, and preserves the form det up to scaling. Note that L_D has center μ_3, and relative root system of type A_2. We let $\mathfrak{l}_D \cong \mathfrak{sl}_3(D)$ be its Lie algebra. Associated to D, there is a linear algebraic group H_D over F which is adjoint, of type E_7 and relative rank 4. The Satake diagram of H_D is:



The group H_D has relative root system F_4, and each short root space has dimension 4, and can be given the structure of D. Moreover [T, p. 66], H_D has unique (up to conjugacy) special maximal compact subgroup K, whose reductive quotient is a finite group of type ${}^2\mathrm{G}_m \times {}^2 E_6$, where ${}^2\mathrm{G}_m$ is the group of norm one elements in the quadratic extension K of the residue field F of F.

Now we describe the Lie algebra $\mathfrak{h}$ of H_D, following the construction in [Ru]. Notice that $\mathfrak{h}$ has a subalgebra $\mathfrak{h}_0 = \mathfrak{sl}_3 \oplus \mathfrak{sl}_3(D)$. This is obtained by covering the vertex α_2 in the Satake diagram. Via the adjoint action, $\mathfrak{h}$ decomposes as a $\mathfrak{h}_0$-module:

$$\mathfrak{h} = \mathfrak{sl}_3 \oplus \mathfrak{sl}_3(D) \oplus (V \otimes J) \oplus (V^* \otimes J^*) \tag{3.2}$$

where

$$\begin{cases} V = \langle e_1, e_2, e_3 \rangle \\ V^* = \langle e_1^*, e_2^*, e_3^* \rangle \end{cases}$$

are the standard representation of $\mathfrak{sl}_3(F)$ and its dual.

As for the bracket relations, most of them are obvious, except for the bracket between two elements of $V \otimes J$, and the bracket between an element of $V \otimes J$ and an element of $V^* \otimes J^*$. For the former, we have

$$[e_i \otimes X, e_j \otimes Y] = \pm e_k^* \otimes X \times Y$$

where $\pm$ is the sign of permutation (i, j, k) of $(1, 2, 3)$. For the latter, we refer the reader to [Ru] for details.

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Using the realization (3.2), we can describe the dual pair $G_2 \times \mathrm{PU}_3(D)$ very easily. Indeed, let e be a generic element of J, *i.e.*, $\det(e) \neq 0$, and let $G' \subset L_D$ be the subgroup which fixes e. Then G' is isomorphic to $\mathrm{PU}_3(D)$. Let G be the closed subgroup of H_D with Lie algebra:

$$\mathrm{g} = \mathrm{sl}_3 \oplus V \otimes e \oplus V^* \otimes e^*$$

where $2e^* = e \times e$. Then G is isomorphic to a split group of type G_2. It is easy to see that G and G' are mutual commutants in H_D, and so $G \times G'$ is a reductive dual pair in H_D. Moreover, the choice of e is not important, as all such e's are in the same orbit of $\mathrm{GL}_3(D)/\mu_2$. Hence, for definiteness, we fix the following choice of e:

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

This gives a fixed embedding:

$$G \times G' \hookrightarrow H_D.$$

Finally, we note that under the identification of J and J^* defined by $\mathrm{Tr}(XY)$, the element e corresponds to e^*.

4 Parabolic Subgroups

In this section, we describe various maximal parabolic subgroups of H_D. First we have the Heisenberg maximal parabolic subgroup $P_2 = M_2 \cdot N_2$, which corresponds to the vertex α_1 in the Satake diagram. Then N_2 is a Heisenberg group with center Z_2. Note that $\bar{N}_2/\bar{Z}_2$ is a representation of M_2, and we let Ω be the minimal non-trivial M_2-orbit in $\bar{N}_2/\bar{Z}_2$; it is the orbit generated by the highest weight vector. For a discussion of P_2, we refer the reader to [MS].

Now let $P_1 = M_1 \cdot N_1$ be the maximal parabolic subgroup of H_D corresponding to the vertex α_2. Then N_1 is a 3-step nilpotent group. On the level of Lie algebras, let

$$s_1 = \begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} \in \mathrm{sl}_3,$$

and $\mathrm{h}_1(i) = \{x \in \mathrm{h} : [s_1, x] = ix\}$. Then the Lie algebra $\mathrm{p}_1 = \mathrm{m}_1 \oplus \mathrm{n}_1$ of P_1 is given by

$$\begin{cases} \mathrm{m}_1 = \mathrm{h}_1(0) = \mathrm{gl}_2 \oplus \mathrm{l}_D \\ \mathrm{n}_1 = \oplus_{i \geq 0} \mathrm{h}_1(i), \end{cases}$$

where

$$\begin{cases} \mathrm{h}_1(1) = \langle e_1, e_2 \rangle \otimes J \\ \mathrm{h}_1(2) = \langle e_3^* \rangle \otimes J^* \cong \det \otimes J^* \\ \mathrm{h}_1(3) \cong \det \otimes \langle e_1, e_2 \rangle \subset \mathrm{sl}_3. \end{cases}$$

Here $\langle e_1, e_2\rangle$ is the standard representation of $L_1 \cong \mathrm{GL}_2$, and det is the usual determinant of 2-by-2 matrices. We let Ω_1 (respectively Ω_2) be the minimal non-trivial orbit of M_1 on $\mathrm{h}_1(-1)$ (respectively $\mathrm{h}_1(-2)$).

Consider now the intersection of $G \times G'$ with P_1. We have:

$$(G \times G') \cap P_1 = Q_1 \times G'$$

where $Q_1 = L_1 \cdot U_1$ is the maximal parabolic subgroup of $G = G_2$ whose Levi factor is generated by the long root. Then the Lie algebra u_1 of U_1 can be identified as the subspace of n_1, the Lie algebra of N_1, given by:

$$\begin{cases} \mathrm{u}_1(1) = \langle e_1, e_2\rangle \otimes \langle e\rangle \subset \mathrm{h}_1(1) \\ \mathrm{u}_1(2) = \langle e_3^* \otimes e^*\rangle \subset \mathrm{h}_1(2) \\ \mathrm{u}_1(3) = \mathrm{h}_1(3). \end{cases}$$

Let V_i be the orthogonal complement of $\mathrm{u}_1(i)$ in $\mathrm{h}_1(-i)$, for $i = 1, 2$. Then

$$\begin{cases} V_1 = \langle e_1^*, e_2^*\rangle \otimes J_0^* \\ V_2 = \det^* \otimes J_0^* \end{cases}$$

where J_0 is the subspace of J orthogonal to e^*, and J_0^* is the subspace of J^* orthogonal to e. We have

Lemma 4.1

1. *$\Omega_1 \cap V_1$ is the minimal non-trivial $L_1 \times G'$ orbit in V_1:*

$$\Omega_1 \cap V_1 = \{0 \neq v \otimes X \in \langle e_1, e_2\rangle \otimes J_0 : \mathrm{rank}(X) = 1\}.$$

2. *$\Omega_2 \cap V_2$ is the minimal non-trivial $L_1 \times G'$ orbit in V_2:*

$$\Omega_2 \cap V_2 = \{X \in J_0^* : \mathrm{rank}(X) = 1\}.$$

Now we do the same for another maximal parabolic subgroup $P = M \cdot N$ of H_D. Here, P corresponds to the vertex α_4, and N is a 2-step nilpotent group. The Lie algebra p of P can be described as follows. Let

$$s = \begin{pmatrix} 1 & & \\ & 0 & \\ & & -1 \end{pmatrix} \in \mathrm{sl}_3(D),$$

and define the spaces $\mathrm{h}(i)$ analogously as before. Then, for example,

$$\begin{cases} \mathrm{h}(1) = \mathrm{l}_D(1) \oplus V \otimes J_{12} \oplus V^* \otimes J_{23}^* \\ \mathrm{h}(2) = \mathrm{l}_D(2) \oplus V \otimes J_{11} \oplus V^* \otimes J_{33}^* \end{cases}$$

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where

$$\begin{cases} \mathrm{l}_D(1) \cong D \oplus D \\ \mathrm{l}_D(2) \cong D \end{cases}$$

are two summands of the nilpotent radical of the minimal parabolic subalgebra of $\mathrm{sl}_3(D)$ consisting of the upper-triangular matrices.

Now we have:

$$(G \times G') \cap P = G \times Q$$

where $Q = L \cdot U$ is the minimal parabolic subgroup of G'. The Lie algebra of U can be identified as:

$$\begin{cases} \mathrm{u}(1) = \left\{ \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix} : x + \bar{z} = 0 \right\} \subset \mathrm{l}_D(1) \\ \mathrm{u}(2) = \left\{ \begin{pmatrix} 0 & 0 & y \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} : \mathrm{Tr}(y) = 0 \right\} \subset \mathrm{l}_D(2). \end{cases}$$

The Levi factor L can be identified with $D^\times \times D^\times / F^\times$ where $F^\times$ is embedded into $D^\times \times D^\times$ via $t \mapsto (t, t^{-1})$. The adjoint action of (α, β) in L on $\mathrm{u}(1) \cong D$ and $\mathrm{u}(2) \cong D^0$ (where D^0 is the 3-dimensional space of traceless quaternions) is given by

$$\begin{cases} (\alpha, \beta) \colon x \mapsto \beta x \bar{\alpha} \\ (\alpha, \beta) \colon y \mapsto \mathrm{N}(\alpha\beta) y. \end{cases}$$

Note that with these identifications, the modulus character of Q is given by:

$$\delta_Q(\alpha, \beta) = |\mathrm{N}(\alpha \cdot \beta)|^5.$$

Let V_i' be the orthogonal complement of $\mathrm{u}(i)$ in $\mathrm{h}(-i)$, for $i = 1, 2$. Then, as a representation of $G \times L$, we have:

$$\begin{cases} V_1' = \text{Ø}^0 \otimes D \\ V_2' = \text{Ø}^0 \otimes \mathrm{N}. \end{cases}$$

Here, Ø^0 is the 7-dimensional space of traceless octonions, on which G acts, and the actions of $D^\times \times D^\times / F^\times$ on D and N are given respectively by:

$$\begin{cases} (\alpha, \beta) \colon x \mapsto \bar{\alpha}^{-1} x \beta^{-1} \\ (\alpha, \beta) \mapsto \mathrm{N}(\alpha\beta)^{-1}. \end{cases}$$

Lemma 4.2 *Let Ω' be the minimal M-orbit on* $\mathrm{h}(2)$*. Then $\Omega' \cap V_2'$ is the minimal $G \times L$-orbit in V_2'.*

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5 The Minimal Representation Π

In this section, we describe a particular representation Π of H_D, known as the minimal representation. Recall that for any irreducible admissible representation π of a reductive p-adic group H, the character $\mathrm{Tr}(\pi)$ of π is an invariant distribution, *i.e.*, a linear functional on the space of locally constant, compactly supported functions on H, which is invariant under conjugation. In a small neighborhood of 1, we can regard $\mathrm{Tr}(\pi)$ as a distribution in a neighborhood of 0 in the Lie algebra h of H. Then a well-known result of Harish-Chandra says that:

$$\mathrm{Tr}(\pi) = \sum_{\mathcal{O}} c_{\mathcal{O}} \hat{\mu}_{\mathcal{O}}$$

where the sum above is over the set of nilpotent orbits $\mathcal{O}$ in h and $\hat{\mu}_{\mathcal{O}}$ is the Fourier transform of the (suitably normalized) invariant measure on $\mathcal{O}$. The wave-front set $WF(\pi)$ of π is then the closure of the union of all those orbits $\mathcal{O}$ such that $c_{\mathcal{O}}$ is non-zero.

In the case of H_D, there is a unique minimal nilpotent orbit $\mathcal{O}_{\min}$. A representation π of H_D is said to be minimal if the only non-zero $c_{\mathcal{O}}$'s in the above character expansion are the ones corresponding to the minimal orbit and the trivial orbit. In other words, π is minimal if its wave-front set is the union of the trivial orbit and the minimal orbit. See [MS] for a justification of the term minimal.

Let K be the special maximal compact subgroup of H_D, and let K_1 be its pro-unipotent radical. Then, as we have mentioned, K/K_1 is a finite group of type ${}^2\mathrm{G}_m \times {}^2 E_6$. Now the minimal representation Π can also be characterized by the fact that it is the unique irreducible admissible representation of H_D such that Π^{K_1} is isomorphic to the minimal representation of the finite group K/K_1. This representation of K/K_1 is denoted by $\phi'_{2,4}$ in the notation of [Ca]. The representation $\phi'_{2,4}$ has dimension $q^{11} - q^8 + q^7 + q^5 - q^4 + q$, and is a unipotent principal series representation whose space of Borel-fixed vectors is 2-dimensional. Note in particular that Π is not K-spherical. Moreover, it is known that [Ru]

$$c_{\mathcal{O}_{\min}} = 1.$$

For the rest of this section, we review the results of Moeglin and Waldspurger [MW] on the values of coefficients of leading terms in the Harish-Chandra character formula. Let π be an irreducible representation of H. Let $\mathcal{O} \subset$ h be a nilpotent orbit such that if $\mathcal{O}'$ is an orbit with strictly bigger dimension than that of $\mathcal{O}$, then $c_{\mathcal{O}'} = 0$ in the character expansion of π. Pick an element f in $\mathcal{O}$, and let $s \in$ h be a semisimple element such that

$$[s, f] = -2f,$$

and such that $\mathrm{ad}(s)$ has integral eigenvalues. Existence of one such s is guaranteed by the Jacobson-Morozov theorem, but there are more choices as we shall see in our applications.

Abusing the notation, let

$$\mathrm{h}(i) = \{x \in \mathrm{h} \mid [s, x] = ix\}$$

and define

$$\begin{cases} \mathrm{n} = \oplus_{i>0}\mathrm{h}(i) \\ \bar{\mathrm{n}} = \oplus_{i<0}\mathrm{h}(i). \end{cases}$$

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Let N and $\bar{N}$ be the corresponding unipotent subgroups of H. Note that f is contained in $\mathfrak{h}(-2)$. We have two cases:

(I) $\mathfrak{h}(1) \equiv 0$. In this case the formula

$$\psi_f\big(\exp(x)\big) = \psi\big(\langle f, x\rangle\big) \tag{5.1}$$

defines a character of N. Here $\langle \, , \, \rangle$ is the Killing form on $\mathfrak{h}$, and ψ a non-trivial additive character of F. The main result of [MW] is

$$\dim \pi_{N,\psi_f} = \dim \mathrm{Hom}_N(\pi, \psi_f) = c_{\mathcal{O}}. \tag{5.2}$$

Here π_{N,ψ_f} is the maximal quotient of π on which N acts as a multiple of ψ_f [BZ, 2.30].

(II) $\mathfrak{h}(1) \neq 0$. Then Let $\mathfrak{n}' \subset \mathfrak{n}$ be the radical of the skew symmetric bilinear form

$$(x, y) := \langle [x, y], f\rangle, \tag{5.3}$$

where x and y are in $\mathfrak{n}$. Note that $\mathfrak{n}' \supseteq \oplus_{i>1}\mathfrak{h}(i)$ and the formula (5.1) defines a character Ψ_f of $N' = \exp \mathfrak{n}'$. Let N'' be the kernel of this character. Then N/N'' is a Heisenberg group. Let W_f be the smooth irreducible representation of N/N'' with central character ψ_f. In this case, the result of [MW] is that

$$\dim \mathrm{Hom}_N(\pi, W_f) = c_{\mathcal{O}}. \tag{5.4}$$

6 Jacquet Modules I

The bulk of the work of this paper is the computation of various Jacquet modules, which will be carried out in this and the next two sections. As in [MS], this computation will be based solely on the assumption that Π is minimal. While calculations in [MS] were based on (5.2) only, here we have to use both (5.2) and (5.4), due to more complicated structure of the maximal parabolic subgroups.

In this section, we compute Π_{U_1}, where U_1 is the unipotent radical of the maximal prabolic subgroup $Q_1 = L_1 \cdot U_1$ of G. But first, we need to determine the restriction of Π to the maximal parabolic subgroup P_1. Recall that N_1 is a 3-step nilpotent group:

$$\{1\} = N_1(4) \subset N_1(3) \subset N_1(2) \subset N_1(1) = N_1$$

with:

$$N_1(i)/N_1(i+1) \cong \mathfrak{h}_1(i)$$

as groups. Hence we have a filtration:

$$0 = \Pi_4 \subset \Pi_3 \subset \Pi_2 \subset \Pi_1 \subset \Pi_0 = \Pi$$

of P-modules such that:

$$E_i := \Pi_i/\Pi_{i+1} \cong (\Pi_i)_{N_1(i)}.$$

Our task is to describe the P-modules E_i.

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First, we know that $E_0 = \Pi_{N_1}$, and the structure of this M_1-module is easily computed by restricting the two-dimensional representation of the Iwahori Hecke algebra corresponding to Π to the Iwahori Hecke algebra of M_1. Furthermore, the center of M_1 coincides with the center of $L_1 \cong \mathrm{GL}_2$. It can be checked that the central character of π_{N_1} is $|z|^9$.

Next, we describe E_1. Note that $(E_1)_{N_1} = 0$, and $(E_1)_{N_1(2)} = E_1$ by definition. So we can regard E_1 as a representation of $N_1/N_1(2) \cong \mathrm{h}_1(1)$. Let $f_1 \in \mathrm{h}_1(-1)$. Then, as described by (5.1), f_1 defines a character ψ_{f_1} of $N_1/N_1(2)$, and by (5.2), we deduce that

$$\dim(E_1)_{N_1,\psi_{f_1}} = \begin{cases} 1 & \text{if } f \in \Omega_1 \\ 0 & \text{if not.} \end{cases} \tag{6.1}$$

Let f_1 in Ω_1. Let $M_{f_1} \subset M_1$ be the stabilizer of f_1 in M_1. Then M_{f_1} acts on the 1-dimensional space $\Pi_{N_1,\psi_{f_1}}$, via a character which we denote by δ_1. From (6.1) and arguing as in [MS, Thm. 6.1], we have

$$E_1 \cong \mathrm{ind}_{M_{f_1}N_1}^{P_1}(\delta_1 \otimes \psi_{f_1}) \cong \mathcal{C}_0^\infty(\Omega_1).$$

Finally, we consider E_2. By definition, $(E_2)_{N_1(3)} = E_2$, and $(E_2)_{N_1(2)} = 0$. Let f_2 be in $\mathrm{h}_1(-2)$. It defines a character ψ_{f_2} of $N_1(2)/N_1(3)$, and an irreducible representation W_{f_2} of N_1 as in Section 5. Then, by (5.4), we deduce that

$$(E_2)_{N_1(2),\psi_{f_1}} = \begin{cases} W_{f_2} & \text{if } f_2 \in \Omega_2 \\ 0 & \text{if not} \end{cases} \tag{6.2}$$

as N_1-modules. Now let f_2 in Ω_2. Let $M_{f_2} \subset M_1$ be the stabilizer of f_2 in M_1. Then M_{f_2} acts on W_{f_2}. Again, from (6.2) and arguing as in [MS, Thm. 6.1], we have

$$E_1 \cong \mathrm{ind}_{M_{f_2}N_1}^{P_1}(W_{f_2}) \cong \mathcal{C}_0^\infty(\Omega_2; W_{f_2})$$

where $\mathcal{C}_0^\infty(\Omega_2; W_{f_2})$ is the space of all locally constant, compactly supported sections of the P_1-equivariant vector bundle on Ω_2 whose fiber at f_2 is W_{f_2}.

Thus we have shown:

Proposition 6.3 *The module $\Pi_{N_1(3)}$ has a P_1-equivariant filtration, with successive quotients*

$$\Pi_{N_1}$$
$$\mathrm{ind}_{M_{f_1}N_1}^{P_1}(\gamma_1 \otimes \psi_{f_1})$$
$$\mathrm{ind}_{M_{f_2}N_1}^{P_1}(W_{f_2}).$$

Next we use this proposition to compute the Jacquet module Π_{U_1}. Using the fact that the Jacquet functor is exact, and that $U_1(3) = N_1(3)$ we see that we need to compute U_1-coinvariants for the subquotients in the proposition. Obviously, U_1 acts trivially on Π_{N_1}. Next,

$$\mathcal{C}_0^\infty(\Omega_1)_{U_1} = \mathcal{C}_0^\infty(V_1 \cap \Omega_1). \tag{6.4}$$

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Now recall that $V_1 \cap \Omega_1$ is the orbit of the highest weight vector in the irreducible $L_1 \times G'$-module $\langle e_1^*, e_2^* \rangle \otimes J_0^*$. It follows from (6.4) that

$$\operatorname{ind}_{M_{f_1} N_1}^{P_1} (\gamma_1 \otimes \psi_{f_1})_{U_1} = \operatorname{Ind}_{B_1 \times Q}^{L_1 \times G'} \big(\delta_1 \otimes \mathcal{C}_0^\infty (F^\times) \big)$$

where B_1 is a Borel subgroup of L_1 and $F^\times$ is the line in $V_1 \cap \Omega_1$ stabilized by $B_1 \times Q$.

Now we proceed to compute $(\operatorname{ind}_{M_{f_2} N_1}^{P_1} W_{f_2})_{U_1}$. First, we have

$$\mathcal{C}_0^\infty (\Omega_2; W_{f_2})_{U_1(2)} = \mathcal{C}_0^\infty (V_2 \cap \Omega_2; W_{f_2}) \tag{6.5}$$

where $V_2 \cap \Omega_2$ is the orbit of the highest weight vector in the irreducible $L_1 \times G'$-module $\langle e_3 \rangle \otimes J_0$. Identifying $\langle e_3 \rangle \otimes J_0$ with J_0, we now pick

$$f_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

To finish the calculation we have to calculate $(W_{f_2})_{U_1}$. In the terminology of Section 4, $N_{f_2} = N_1$, $N'_{f_2} = N_1(2)$, and the skew-linear form on $N_1 / N_1(2) \cong \langle e_1, e_2 \rangle \otimes J$ defined by the formula (5.3) is

$$(u \otimes X, v \otimes Y) = (u, v) \cdot (f_2, X, Y)$$

where (u, v) is the standard symplectic form on $\langle e_1, e_2 \rangle$. A direct computation using (3.1) shows that the kernel Δ of the bilinear form (f_2, X, Y) consists of the elements in the form

$$X = \begin{pmatrix} a & z & \bar{y} \\ \bar{z} & 0 & 0 \\ y & 0 & 0 \end{pmatrix}.$$

It follows that N''_{f_2} is the inverse image of the space $\langle e_1, e_2 \rangle \otimes \Delta \subset \mathrm{h}_1(1)$ under the projection map from N_1 to $N_1 / N_1(2) \cong \mathrm{h}_1(1)$. Let $\Delta^\perp$ be the complement of Δ in J consisting of all elements of the form

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b & x \\ 0 & \bar{x} & c \end{pmatrix}.$$

On these elements, the quadratic form (f_2, X, X) is given by

$$(f_2, X, X) = 2bc - 2\mathrm{N}(x).$$

It follows that the representation W_{f_2} is associated to the 12-dimensional symplectic space $\langle e_1, e_2 \rangle \otimes \Delta^\perp$. We fix a polarization of this space consisting of all elements of the form

$$e_1 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & x \\ 0 & \bar{x} & 0 \end{pmatrix} + e_1 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_1 \end{pmatrix} + e_2 \otimes \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & c_2 \end{pmatrix}.$$

We realize W_{f_2} on the space of locally constant, compactly supported functions $\Phi(x, c_1, c_2)$. The action of $U_1/U_1(2)$ is given by

$$\Pi(u)\Phi(x, c_1, c_2) = \psi(u_1c_1 + u_2c_2)\Phi(x, c_1, c_2),$$

where $u = (u_1e_1 + u_2e_2) \otimes e$ under the identification $U_1/U_1(2) \cong \langle e_1, e_2\rangle \otimes \langle e\rangle$. It follows that

$$(W_{f_2})_{U_1/U_1(2)} \cong \mathcal{C}_0^\infty(D) \cong W \tag{6.6}$$

where W is the Heisenberg representation associated to the symplectic space $\langle e_1, e_2\rangle \otimes D$. Now note that $L_1 \times L = \tilde{R}$ and $M_{f_2} \cap (L_1 \times L) = R$, which were introduced in Section 2. Furthermore, the action of M_{f_2} on W_{f_2} induces an action of R on W which, by the Schur Lemma, must be a twist by a character of R of the action described in the Section 2.

Lemma 6.7 *Any character of R is a restriction of a character of $\tilde{R}$ of the form $|\det|^t \otimes \chi$ where t is a real number and χ a unitary character of $D \times D/F^\times$.*

It follows from (6.5) and (6.6) that

$$(\mathrm{ind}_{M_{f_2}N_1}^{P_1} W_{f_2})_{U_1} \cong \mathrm{Ind}_{L_1\times Q}^{L_1\times G'} \delta_2 \otimes \tilde{W}$$

where δ_2 is a character of $L_1 \times L$ as in the Lemma.

We summarize what we have shown in the following proposition:

Proposition 6.8 *As a representation of $L_1 \times G'$, the module Π_{U_1} has a filtration with successive quotients*

$$\Pi_{N_1}$$
$$\mathrm{Ind}_{B_1\times Q}^{L_1\times G'} \delta_1 \otimes \mathcal{C}_0^\infty(F^\times)$$
$$\mathrm{Ind}_{L_1\times Q}^{L_1\times G'} \delta_2 \otimes \tilde{W},$$

where $\delta_2 = |\det|^{t_2} \otimes \chi_2$ is a character of $L_1 \times L$ as in Lemma 6.7. The central character of $L_1 \cong \mathrm{GL}_2$ on Π_{N_1} is $|z|^9$.

We shall show that $t_2 = 5$ and that the character χ_2 is trivial in Section 9, where we investigate the local correspondence.

7 Jacquet Modules II

In this section, we compute Π_{U_2}, where U_2 is the unipotent radical of the maximal parabolic subgroup $Q_2 = L_2 \cdot U_2$ of G. Since the computation is entirely similar to the case of split groups in [MS], we shall content ourselves with just stating the results. There is an exact sequence of P_2-modules

$$0 \longrightarrow \mathrm{ind}_{M_fN_2}^{P_2}(\gamma \otimes \psi_f) \longrightarrow \Pi_{Z_2} \longrightarrow \Pi_{N_2} \longrightarrow 0$$

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where f is in the orbit Ω. This then implies the following proposition.

Proposition 7.1 *As a representation of $L_2 \times G'$, the module Π_{U_2} has a filtration with successive quotients*

$$\Pi_{N_2}$$

$$\mathrm{Ind}_{B_2\times Q}^{L_2\times G'} \delta \otimes \mathcal{C}_0^\infty(F^\times).$$

The central characters of $L_2 \cong \mathrm{GL}_2$ on Π_{N_2} are $|z|^4$ and $|z|^6$, the latter corresponding to a one-dimensional summand of Π_{N_2}.

8 Jacquet Modules III

In this section, we compute the Jacquet module of Π with respect to the subgroup U of G'. Since the computation is similar to that in Section 6, we shall be brief.

As before, we first determine the restriction of Π to the maximal parabolic $P = M \cdot N$. Since now N is a 2-step nilpotent group

$$\{1\} = N(3) \subset N(2) \subset N(1) = N,$$

there is a filtration of P-modules:

$$0 = \Pi_3 \subset \Pi_2 \subset \Pi_1 \subset \Pi_0 = \Pi$$

with E_i defined as before. Hence, E_0 is again Π_N. As for E_1, we find that it is now equal to 0, since the minimal nilpotent orbit has empty intersection with $\mathfrak{h}(-1)$. Thus we only need to describe E_2, and as in Section 6 we find that

$$E_2 \cong \mathrm{ind}_{M_f N}^P W_f$$

where f is in the minimal orbit Ω'. We summarize the results without a detailed proof:

Proposition 8.1 *As a representation of $G \times L$, the module Π_U has a filtration with successive quotients:*

$$\Pi_N$$

$$\mathrm{Ind}_{P_1\times L}^{G\times L} \delta' \otimes \tilde{W}$$

where $\delta' = |\det|^{t'} \otimes \chi'$ is a character of $L_1 \times L$ as in Lemma 6.7. The central character of $L \cong D \times D/F^\times$, on Π_N is

$$(z_1, z_2) \mapsto |z_1 z_2|^6.$$

We shall show that $t' = 5$ and that the character χ' is trivial in Section 9, where we investigate the local correspondence.

9 Local Correspondence

In this section we prove the correspondence of representations discussed in the introduction, and also determine the characters δ_2 and δ' from the previous sections. Basically, the hard work has been done in the last three sections.

Let $\pi = \mathrm{JL}(\pi')$ be an irreducible square-integrable representation of GL_2 with unitary central character, and let

$$\pi(s) = \pi \otimes |\det|^s$$

where $s \in \mathrm{R}$. Consider the representations

$$\begin{cases} I_1(\pi, s) = \mathrm{Ind}_{P_1}^G \, \delta_{P_1}^{1/2} \pi(s) \\ I_2(\pi, s) = \mathrm{Ind}_{P_2}^G \, \delta_{P_2}^{1/2} \pi(s) \end{cases}$$

where

$$\begin{cases} \delta_{P_1} = |\det|^5 \\ \delta_{P_2} = |\det|^3. \end{cases}$$

If $s > 0$, then $I_i(\pi, s)$ has unique (Langlands) quotient, which we denote by $J_i(\pi, s)$. Similarly, let

$$\pi'(s) = \pi' \otimes |\mathrm{N}(\alpha\beta)|^s$$

and let

$$I(\pi', s) = \mathrm{Ind}_Q^{G'} \, \delta_Q^{1/2} \cdot \pi'(s) \otimes \pi'(s).$$

Again, if $s > 0$, then $I(\pi', s)$ has unique (Langlands) quotient, which we denote by $J(\pi', s)$.

The calculation of the characters δ' and δ_2 goes along with the proof of the following theorem.

Theorem 9.1 *Let $\pi = \mathrm{JL}(\pi')$ be an irreducible super-cuspidal representation of GL_2 with unitary central character, and let $s > 0$. Then*

$$\begin{cases} \Theta\big(J_1(\pi, s)\big) = \{J(\pi', s)\} \\ \Theta\big(J(\pi', s)\big) = \{J_1(\pi, s)\} \end{cases}$$

and

$$\Theta\big(J_2(\pi, s)\big) = \varnothing.$$

Let $1'$ be the trivial representation of G'. The one-dimensional summand of Π_{N_2} corresponding to the central character $|z|^6$ is isomorphic to $|\det|^3 \otimes 1'$, as $L_2 \times G'$-modules. It follows from the Frobenius reciprocity that $\Theta(1')$ contains a subquotient of $\mathrm{Ind}_{P_2}^G(|\det|^3)$. Since the subquotients of $\mathrm{Ind}_{P_2}^G(|\det|^3)$ are 1, the trivial representation of G, and $J_1(\mathrm{st}, 5/2)$ (where st is the Steinberg representation of GL_2) we have

$$\Theta(1') \cap \{1, J_1(\mathrm{st}, 5/2)\} \neq \varnothing.$$

Let σ be an irreducible representation in the intersection above. Since $1'$ is a submodule of $\mathrm{Ind}_Q^{G'} 1$,

$$\varnothing \neq \mathrm{Hom}_{G \times G'}(\Pi, \sigma \otimes \mathrm{Ind}_Q^{G'} 1) = \mathrm{Hom}_{G \times L}(\Pi_U, \sigma \otimes 1).$$

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Since the central character of Π_N is $|z_1z_2|^6$ by Proposition 8.1, and the central character of 1 is trivial, $\sigma \otimes 1$ must be a quotient of $\mathrm{Ind}_{P_1\times L}^{G\times L} \delta' \otimes \tilde{W}$ by Lemma 1.2. A calculation, using (1.1) and Theorem 2.1 shows that σ is a quotient of $\mathrm{Ind}_{P_1}^{G}(\mathrm{st} \otimes \delta')$. This is possible only if $\sigma = J_1(\mathrm{st}, 5/2)$, $\chi' = 1$ and $t' = 5$. Thus, we have determined the character δ' and have shown that

$$\Theta(1') = \{J_1(\mathrm{st}, 5/2)\}.$$

We are now ready to prove the following lemma.

Lemma 9.2 *Let* $\pi = \mathrm{JL}(\pi')$ *be an irreducible square-integrable representation of* GL_2*, and let* $s > 0$*. Then*

$$\begin{cases} \Theta\big(J_1(\pi, s)\big) \neq \varnothing \\ \Theta\big(J(\pi', s)\big) \subseteq \{J_1(\pi, s)\}. \end{cases}$$

Proof Let σ be in $\Theta\big(J(\pi', s)\big)$. Note that $J(\pi', s)$ is the unique submodule of $I(\tilde{\pi}', -s)$. By the Frobenius reciprocity,

$$\mathrm{Hom}_{G\times G'}\big(\Pi, \sigma \otimes I(\tilde{\pi}', -s)\big) = \mathrm{Hom}_{G\times L}\big(\Pi_U, \sigma \otimes \tilde{\pi}'(5/2 - s) \otimes \tilde{\pi}'(5/2 - s)\big).$$

Note that the central character of $\tilde{\pi}'(5/2 - s) \otimes \tilde{\pi}'(5/2 - s)$ is, up to a unitary character, equal to $|z_1z_2|^{5-2s}$. Since s is positive, it is different then the central character $|z_1z_2|^6$ of Π_N, it follows that

$$\mathrm{Hom}_{G\times L}\Big(\mathrm{Ind}_{P_1\times L}^{G\times L} \delta' \otimes \tilde{W}, \sigma \otimes \tilde{\pi}'\big((5/2 - s) \otimes \tilde{\pi}'(5/2 - s)\big)\Big) \neq \varnothing.$$

Again, a calculation, using (1.1) and Theorem 2.1 shows that σ must be $J_1(\pi, s)$. The lemma is proved. ■

For those s for which $I(\pi', s)$ is irreducible, we have in fact shown that $J_1(\pi, s) \otimes I(\pi', s)$ is a quotient of Π. Since $J_1(\pi, s)$ is unique submodule of $I_1(\tilde{\pi}, -s)$, it follows that

$$\varnothing \neq \mathrm{Hom}_{G\times G'}\big(\Pi, I_1(\tilde{\pi}, -s) \otimes I(\pi', s)\big) = \mathrm{Hom}_{L_1\times G'}\big(\Pi_{U_1}, \tilde{\pi}(5/2 - s) \otimes I(\pi', s)\big).$$

Again, the central characters show that $\tilde{\pi}(5/2 - s)$ cannot be a quotient of Π_{N_1}. Furthermore, if we take π to be supercuspidal, then $\tilde{\pi}(5/2 - s)$ cannot be a quotient of the middle term of the filtration of Π_{U_1} given by Proposition 6.8. Thus,

$$\mathrm{Hom}_{L_1\times G'}\big(\mathrm{Ind}_{L_1\times Q}^{L_1\times G'} \delta_2 \otimes \tilde{W}, \tilde{\pi}(5/2 - s) \otimes I(\pi', s)\big) \neq \varnothing$$

and this is possible only if $\chi_2 = 1$ and $t_2 = 5$. This determines δ_2, and the following analogue of Lemma 9.2 follows.

Lemma 9.3 *Let* $\pi = \mathrm{JL}(\pi')$ *be an irreducible super-cuspidal representation of* GL_2*, and let* $s > 0$*. Then*

$$\begin{cases} \Theta\big(J(\pi', s)\big) \neq \varnothing \\ \Theta\big(J_1(\pi, s)\big) \subseteq \{J(\pi', s)\}. \end{cases}$$

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The first part of the Theorem is now a simple combination of Lemma 9.2 and Lemma 9.3. For the second part note that $J_2(\pi, s)$ is unique submodule of $I_2(\tilde{\pi}, -s)$. Again, by the Frobenius reciprocity,

$$\mathrm{Hom}_G\big(\Pi, I_2(\tilde{\pi}, -s)\big) = \mathrm{Hom}_{L_2}\big(\Pi_{U_2}, \tilde{\pi}(3/2 - s)\big).$$

Since $\tilde{\pi}(3/2-s)$ is supercuspidal, it cannot be a quotient of the second term of the filtration of Π_{U_2} given by Proposition 7.1. Also, since the central character of $\tilde{\pi}(3/2 - s)$ is, up to a unitary character, equal to $|z|^{3-2s}$, it cannot be a quotient of Π_{N_2} if $s > 0$. This shows that $J_2(\pi, s)$ is not a quotient of Π. The theorem is proved.

We finish this paper by calculating $\Theta(\mathrm{St}')$ where St' is the Steinberg representation of G'. Since St' is a submodule of $I(1, 5/2)$, it follows as in the proof of Lemma 9.2 that

$$\begin{cases} \Theta(\mathrm{St}') \subseteq \{\mathrm{St}\} \\ \Theta(\mathrm{St}) \cap \{\mathrm{St}', J(1, 5/2)\} \neq \varnothing. \end{cases}$$

where St is the Steinberg representation of G. Since, by Lemma 9.2, $J(1, 5/2)$ cannot be paired with St, it follows that

$$\Theta(\mathrm{St}') = \{\mathrm{St}\}. \tag{9.4}$$

References

[1] J. Bernstein and A. Zelevinsky, *Representations of the group* $\mathrm{GL}_n(F)$ *where F is a non-archimedean local field.* Russian Math. Surveys **31**(1976), 1–68.
[2] R. Carter, *Finite Groups of Lie Type.* Wiley, 1985.
[3] W. T. Gan, *Exceptional Theta Correspondences.* Harvard Thesis, 1998.
[4] H. Jacquet and R. Langlands, *Automorphic Forms on* GL_2. Lecture Notes in Math. **114**(1970), Springer-Verlag.
[5] K. Magaard and G. Savin, *Exceptional* Θ*-correspondences I.* Compositio Math. **107**(1997), 1–35.
[6] C. Moeglin and J.-L. Waldspurger, *Modèles de Whittaker dégénérés pour des groupes p-adiques.* Math. Zeit. **196**(1987), 427–452.
[7] B. Roberts, *The Theta Correspondence for Similitudes.* Israel J. Math. **94**(1996), 285–317.
[8] K. Rumelhart, *Minimal Representation for Exceptional p-adic Groups.* Representation Theory **1**(1997), 133–181.
[9] J. Tits, *Classification of algebraic simple groups.* In: Algebraic groups and discontinuous groups, Proc. Sympos. Pure Math. **9**, Amer. Math. Soc., Providence, RI, 1966.

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