SUBNORMALITY AND GENERALIZED COMMUTATION RELATIONS OF FAMILIES OF OPERATORS†

by JERZY BARTŁOMIEJ STOCHEL

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1. Every family of subnormal operators in a Hilbert space fulfils the Halmos–Bram condition on a suitable dense subset of its domain **[2]**, **[3]**. In **[2]** and **[4]** it is shown that the generalized commutation relation implies the Halmos–Bram condition for one operator. In this paper it is proved that the generalized commutation relation implies the Halmos–Bram condition for infinite families of operators (in a special case Jorgensen proved it in a different way for finite families of operators, see **[2]**) and as an example of the application of this property it is shown that every family of generalized creation operators in the Bargmann space of an infinite order, indexed by mutually orthogonal vectors from l^2 is subnormal. See **[1]** for the definitions.

2. A family $\mathbb{S}=\{S_i : i\in I\}$ of densely defined linear operators in a complex Hilbert space H with the same domain $D(\mathbb{S}):=D(S_i)$ $(i\in I)$ is said to be a *subnormal family* if there are a Hilbert superspace K and a family $\mathbb{N}=\{N_i : i\in I\}$ of commuting normal operators in K such that $H\subset K$ and

$$S_if=N_if \qquad \text{for all } f\in D(\mathbb{S}), \qquad i\in I. \tag{1}$$

Denote by $\mathcal{F}$ the set of all functions $\alpha : I\to\mathbb{Z}_+$ equal to zero for all but a finite number of values $i\in I$, where $\mathbb{Z}_+$ denotes the set of all positive integers. In the sequel let the family $\mathbb{S}$ have a common invariant subspace M (i.e. M is a dense linear subspace of H such that $M\subset D(S_i)$, $S_i(M)\subset M$, $i\in I$) on which

$$S_iS_jf=S_jS_if \qquad \text{for all } f\in M, \qquad i,j\in I. \tag{2}$$

Then the family $\mathbb{S}$ satisfies (1) and

$$\mathbb{S}^{\alpha}f=\mathbb{N}^{\alpha}f \qquad \text{for all } f\in M, \qquad \alpha\in\mathcal{F}, \tag{3}$$

where $\mathbb{S}^{\alpha}f=\prod S_i^{\alpha(i)}f$ (the same about $\mathbb{N}^{\alpha}f$).

In **[3]** it was proved that such a family with $M=D(\mathbb{S})$ is subnormal if the following conditions are satisfied:

(4) the Halmos–Bram condition

$$\sum_{\alpha,\beta\in\mathcal{F}}\langle\mathbb{S}^{\alpha}f(\beta),\mathbb{S}^{\beta}f(\alpha)\rangle\geq 0$$

holds for every finitely supported function $f:\mathcal{F}\to M$.

(5) M is a linear subspace spanned by the set

$$\{\mathbb{S}^{\alpha}f : \alpha\in\mathcal{F}, f\in\bigcap\{\mathcal{A}(S_i): i\in I\}\},$$

where $\mathcal{A}(S_i):=\{f\in D : \limsup[(n!)^{-1}\|S_i^nf\|]^{1/n}<\infty\}$.

The properties above will be referred to as Theorem A.

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Now let $\mathbb{A}=\{A_i : i\in I\}$, $\mathbb{B}=\{B_i : i\in I\}$, $\mathbb{E}=\{E_{i,j} : i, j\in I\}$, $\mathbb{C}=\{C_{i,j} : i, j\in I\}$ be families of densely defined linear operators and M denote a linear subspace of H. We say that the pair of families $(\mathbb{A}, \mathbb{B})$ satisfies the *generalized commutation relation* (respectively *semicommutation relation*) on M with the family $\mathbb{E}$ (respectively $\mathbb{C}$), if M is invariant under $\mathbb{A}$, $\mathbb{B}$, $\mathbb{E}$ (respectively $\mathbb{C}$) and:

(6) $$[A_i, B_j]f = E_{i,j}^2 f([A, B]:=AB-BA), \qquad f\in M, \qquad i, j\in I$$
$$\langle E_{i,j}f, g\rangle = \langle f, E_{i,j}g\rangle, \qquad f, g\in M, \qquad i, j\in I,$$
(respectively $[A_i, B_j]f = C_{i,j}f$, $f\in M$, $i, j\in I$),

and

(7) $$A_iE_{k,l}f = E_{k,l}A_if, \qquad B_jE_{k,l}f = E_{k,l}B_jf, \qquad E_{i,j}E_{k,l}f = E_{k,l}E_{i,j}f$$
(respectively $A_iC_{k,l}f = C_{k,l}A_if, B_jC_{k,l}f = C_{k,l}B_jf$) $\quad f\in M, \quad i, j, k, l\in I$.

From this definition it follows immediately that

(8) $$C_{i,j}C_{k,l}f = C_{k,l}C_{i,j}f \qquad \text{for all } f\in M, \qquad i, j, k, l\in I.$$

If in addition the following condition is fulfilled

(9) $$E_{i,j}=0 \text{ (respectively } C_{i,j}=0) \text{ if } i\neq j \qquad (i, j\in I),$$

then we say that the pair of families $(\mathbb{A}, \mathbb{B})$ satisfies the strong generalized commutation relation (respectively semicommutation relation).

Next we make some remarks about notation. Let $\alpha, \beta\in(Z_+)^n$ and

$$\mathfrak{m}=\begin{pmatrix}\mathfrak{m}_{11} & \dots & \mathfrak{m}_{in}\\ . & \dots & .\\ \mathfrak{m}_{k1} & \dots & \mathfrak{m}_{kn}\end{pmatrix}\in\{(\mathbb{Z}_+)^n\}^k.$$

In the sequel we shall use the following notation:

(10) $$\mathfrak{m}_i:=(\mathfrak{m}_{i1}, \dots, \mathfrak{m}_{in})\in(\mathbb{Z}_+)^n, \qquad \mathfrak{m}^i:=(\mathfrak{m}_{1i}, \dots, \mathfrak{m}_{ki})\in(\mathbb{Z}_+)^k,$$

(11) $$w(\mathfrak{m}):=\sum_{i=1}^k \mathfrak{m}_i\in(\mathbb{Z}_+)^n, \qquad k(\mathfrak{m}):=\sum_{i=1}^n \mathfrak{m}^i\in(\mathbb{Z}_+)^k,$$

(12) $$\alpha\pm\beta:=(\alpha_1\pm\beta_1, \dots, \alpha_n\pm\beta_n), \qquad |\alpha|:=\sum_{i=1}^n\alpha_i, \qquad \alpha!:=\prod_{j=1}^n\alpha_j!,$$

(13) $$\mathfrak{m}!:=\prod_{i=1}^k\prod_{j=1}^n\mathfrak{m}_{ij}!.$$

(14) $$\binom{\alpha}{\beta}:=\prod\binom{\alpha_i}{\beta_i}, \quad \text{where} \quad \binom{\alpha_i}{\beta_i}:=\frac{\alpha_i!}{\beta_i!\,(\alpha_i-\beta_i)!} \quad \text{if} \quad \alpha_i\geq\beta_i$$
$$\text{or} \quad \binom{\alpha_i}{\beta_i}:=0 \quad \text{if} \quad \alpha_i<\beta_i,$$

(15) $R^0:=\mathrm{Id}$ and $R^{-s}:=0$ for $s>0$, where R denotes any linear operator in H.

3. Now we can state and prove our first result.

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THEOREM 1. *Let* $(\mathbb{A}, \mathbb{B})$ *be a pair of families of linear operators, which satisfy the generalized semicommutation relation on a linear space* M *with the family of operators* $\mathbb{C} = \{C_{i,j} : i, j \in I\}$, *where* $I = (\mathbb{Z}_+)^n$. *Then*

$$\mathbb{A}^\alpha \mathbb{B}^\beta f = \sum_{\mathfrak{m} \in \{(Z_+)^n\}^n} \frac{k(\mathfrak{m})!\, w(\mathfrak{m})!}{\mathfrak{m}!} \binom{\alpha}{k(\mathfrak{m})} \binom{\beta}{w(\mathfrak{m})} \mathbb{B}^{\beta - w(\mathfrak{m})} \mathbb{A}^{\alpha - k(\mathfrak{m})} \mathbb{C}^{\mathfrak{m}} f$$

for all $f \in M$, $\alpha, \beta \in (Z_+)^n$, *where* $\mathbb{C}^{\mathfrak{m}} := \prod C_{i,j}^{\mathfrak{m}_{ij}}$ (*see property* (8)) *and* $\mathbb{A}^\alpha$, $\mathbb{B}^\beta$ *are defined as in* (3).

We shall prove this theorem by induction. First we state and prove some lemmas.

LEMMA 1.

$$\binom{\alpha}{\beta} \cdot \binom{\alpha - \beta}{\gamma} = \binom{\alpha}{\beta + \gamma} \cdot \binom{\beta + \gamma}{\gamma},$$

for all $\alpha, \beta, \gamma \in (\mathbb{Z}_+)^n$.

LEMMA 2

$$\frac{k(\mathfrak{m})!\, w(\mathfrak{m})!}{\mathfrak{m}!} \cdot \binom{w\left(\begin{vmatrix} \mathfrak{m} \\ \alpha \end{vmatrix}\right)}{w(\mathfrak{m})} \cdot |\alpha|! = \frac{k\left(\begin{vmatrix} \mathfrak{m} \\ \alpha \end{vmatrix}\right)!\, w\left(\begin{vmatrix} \mathfrak{m} \\ \alpha \end{vmatrix}\right)!}{\begin{vmatrix} \mathfrak{m} \\ \alpha \end{vmatrix}!}$$

for all $\mathfrak{m} \in \{(\mathbb{Z}_+)^n\}^k$ *and* $\alpha \in (\mathbb{Z}_+)^n$, *where*

$$\begin{vmatrix} \mathfrak{m} \\ \alpha \end{vmatrix} := \begin{pmatrix} \mathfrak{m}_{11} & \dots & \mathfrak{m}_{1n} \\ . & \dots & . \\ \mathfrak{m}_{k1} & \dots & \mathfrak{m}_{kn} \\ \alpha_1 & \dots & \alpha_n \end{pmatrix}.$$

Proof. It is sufficient to note that

$$k\left(\begin{vmatrix} \mathfrak{m} \\ \alpha \end{vmatrix}\right) = (k(\mathfrak{m}), |\alpha|), \qquad w\left(\begin{vmatrix} \mathfrak{m} \\ \alpha \end{vmatrix}\right) = w(\mathfrak{m}) + \alpha \quad \text{and} \quad \begin{vmatrix} \mathfrak{m} \\ \alpha \end{vmatrix}! = \mathfrak{m}!\, \alpha!.$$

Then from a simple calculation we obtain our statement. QED

LEMMA 3.

$$\frac{k(\mathfrak{m})!\, w(\mathfrak{m})!}{\mathfrak{m}!} \cdot \binom{k(|\mathfrak{m}, \alpha|)}{\alpha} \cdot |\alpha|! = \frac{k(|\mathfrak{m}, \alpha|)!\, w(|\mathfrak{m}, \alpha|)!}{|\mathfrak{m}, \alpha|!}$$

for all $\alpha \in (\mathbb{Z}_+)^k$, *and* $\mathfrak{m} \in \{(Z_+)^n\}^k$, *where*

$$|\mathfrak{m}, \alpha| := \begin{pmatrix} \mathfrak{m}_{11} & \dots & \mathfrak{m}_{1n} & \alpha_1 \\ . & \dots & . & . \\ \mathfrak{m}_{k1} & \dots & \mathfrak{m}_{kn} & \alpha_k \end{pmatrix}.$$

Proof. Similarly as before, it is sufficient to note that

$$k(|\mathfrak{m}, \alpha|) = k(\mathfrak{m}) + \alpha, \qquad w(|\mathfrak{m}, \alpha|) = (w(\mathfrak{m}), |\alpha|) \quad \text{and} \quad |\mathfrak{m}, \alpha|! = \mathfrak{m}!\, \alpha!. \quad \text{QED}$$

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LEMMA 4. (See [4].) *Let $\{A\}$, $\{B\}$ be families of single linear operators which satisfy the generalized semicommutation relation on the linear space M with an operator C. Then the equation*

$$A^iB^jf = \sum_{k=0}^{\infty} k!\binom{j}{k}\binom{i}{k}B^{j-k}A^{i-k}C^kf \qquad \text{for all } f \in M, \qquad i, j \in \mathbb{Z}_+,$$

holds.

Now we return to the main proof. If $n = 1$, then $k(\mathrm{m}) = w(\mathrm{m}) = \mathrm{m}$ and the proof follows immediately from Lemma 4. So now we assume that our theorem is true for all pairs of n-element families of operators, which satisfy the generalized semicommutation relation on the linear space M. Next let $\mathbb{A} = (A_1, \ldots, A_{n+1})$, $\mathbb{B} = (B_1, \ldots, B_{n+1})$ be a pair of families of operators, which satisfies the generalized semicommutation relation on M with a family of operators $\mathbb{C} = \{C_{i,j} : i, j = 1, \ldots, n+1\}$. In the sequel $\gamma\,|n|$ and $\gamma\{n\}$ stand for the n-tuples $(\gamma_1, \ldots, \gamma_n)$ and $(0, \ldots, 0, \gamma_{n+1})$ respectively, where $\gamma \in (\mathbb{Z}_+)^{n+1}$. Let $\gamma \in (\mathbb{Z}_+)^{n+1}$ and $f \in M$. Then from the inductive assumption we obtain

$$\mathbb{A}_1^{\gamma|n|}\mathbb{B}_2^{\gamma\{n\}}f = \sum_{\mathrm{m}\in\{(\mathbb{Z}_+)^n\}^n} \frac{k(\mathrm{m})!\,w(\mathrm{m})!}{\mathrm{m}!}\binom{\gamma\,|n|}{k(\mathrm{m})}\binom{\gamma\{n\}}{w(\mathrm{m})}\mathbb{B}_2^{\gamma\{n\}-w(\mathrm{m})}\mathbb{A}_1^{\gamma|n|-k(\mathrm{m})}\mathbb{C}_1^{\mathrm{m}}f,$$

where

$$\mathbb{A}_1 := (A_1, \ldots, A_n), \qquad \mathbb{B}_2 := (B_2, \ldots, B_{n+1})$$

and

$$\mathbb{C}_1 := \{C_{i,j} : i = 1, \ldots, n \text{ and } j = 2, \ldots, n+1\}.$$

But

$$\binom{\gamma\{n\}}{w(\mathrm{m})} = 0,$$

if one of $\mathrm{m}^k \neq 0$ for $k = 1, \ldots, n-1$. Hence

$$\mathbb{A}_1^{\gamma|n|}\mathbb{B}_2^{\gamma\{n\}}f = \sum_{\rho\in(\mathbb{Z}_+)^n} |\rho|!\binom{\gamma\,|n|}{\rho}\binom{\gamma_{n+1}}{|\rho|}B_{n+1}^{\gamma_{n+1}-|\rho|}\mathbb{A}_1^{\gamma|n|-\rho} \times \{C_{i,n+1} : i = 1, \ldots, n\}^{\rho}f \qquad (f \in M,\ \gamma \in (\mathbb{Z}_+)^{n+1}). \tag{16}$$

Similarly we obtain:

$$\begin{aligned}\mathbb{A}_2^{\gamma\{n\}}\mathbb{B}_1^{\gamma|n|}f &= \sum_{\mathrm{m}\in\{(\mathbb{Z}_+)^n\}^n} \frac{k(\mathrm{m})!\,w(\mathrm{m})!}{\mathrm{m}!}\binom{\gamma\{n\}}{k(\mathrm{m})}\binom{\gamma\,|n|}{w(\mathrm{m})}\mathbb{B}_1^{\gamma|n|-w(\mathrm{m})}\mathbb{A}_2^{\gamma\{n\}-k(\mathrm{m})}\mathbb{C}_2^{\mathrm{m}}f \\ &= \sum_{\rho\in(\mathbb{Z}_+)^n} |\rho|!\binom{\gamma\,|n|}{\rho}\binom{\gamma_{n+1}}{|\rho|}\mathbb{B}_1^{\gamma|n|-\rho}\mathbb{A}_{n+1}^{\gamma_{n+1}-|\rho|}\{C_{n+1,j} : i = 1, \ldots, n\}^{\rho}f,\end{aligned} \tag{17}$$

for $\gamma \in (\mathbb{Z}_+)^{n+1}$ and $f \in M$, where $\mathbb{A}_2 := (A_2, \ldots, A_{n+1})$, $\mathbb{B}_1 := (B_1, \ldots, B_n)$ and $\mathbb{C}_2 := \{C_{i,j} : i = 2, \ldots, n+1 \text{ and } j = 1, \ldots, n\}$.

Now we return to the main calculation. From the inductive assumption, Lemmas 1–4 and the properties (7), (8), (16) and (17) we obtain:

$$\mathbb{A}^{\alpha}\mathbb{B}^{\beta}f = \mathbb{A}_1^{\alpha|n|}(\mathbb{A}_2^{\alpha\{n\}}\mathbb{B}_1^{\beta|n|})B_{n+1}^{\beta_{n+1}}f$$

$$= \mathbb{A}_1^{\alpha|n|}\left\{\sum_{\rho\in(\mathbf{Z}_+)^n} |\rho|!\binom{\beta\,|n|}{\rho}\binom{\alpha_{n+1}}{|\rho|}\mathbb{B}_1^{\beta|n|-\rho}A_{n+1}^{\alpha_{n+1}-|\rho|}\{C_{n+1,j}: i=1,\ldots,n\}^{\rho}\right\}B_{n+1}^{\beta_{n+1}}f$$

$$= \sum_{\rho\in(\mathbf{Z}_+)^n} |\rho|!\binom{\beta\,|n|}{\rho}\binom{\alpha_{n+1}}{|\rho|}\{\mathbb{A}_1^{\alpha|n|}\mathbb{B}_1^{\beta|n|-\rho}\}A_{n+1}^{\alpha_{n+1}-|\rho|}B_{n+1}^{\beta_{n+1}}\{C_{n+1,j}: j=1,\ldots,n\}^{\rho}f$$

$$= \sum_{\rho\in(\mathbf{Z}_+)^n} |\rho|!\binom{\beta\,|n|}{\rho}\binom{\alpha_{n+1}}{|\rho|}\left\{\sum_{\mathfrak{m}\in\{(\mathbf{Z}_+)^n\}^n}\frac{k(\mathfrak{m})!\,w(\mathfrak{m})!}{\mathfrak{m}!}\binom{\alpha\,|n|}{k(\mathfrak{m})}\binom{\beta\,|n|-\rho}{w(\mathfrak{m})}\right.$$
$$\left.\times\,\mathbb{B}_1^{\beta|n|-w(\mathfrak{m})-\rho}\mathbb{A}_1^{\alpha|n|-k(\mathfrak{m})}\{C_{i,j}: i,j=1,\ldots,n\}^{\mathfrak{m}}\right\}A_{n+1}^{\alpha_{n+1}-|\rho|}B_{n+1}^{\beta_{n+1}}$$
$$\times\{C_{n+1,j}: j=1,\ldots,n\}^{\rho}f \quad \text{(by Lemma 1)}$$

$$= \sum_{\rho\in(\mathbf{Z}_+)^n}\sum_{\mathfrak{m}\in\{(\mathbf{Z}_+)^n\}^n}|\rho|!\frac{k(\mathfrak{m})!\,w(\mathfrak{m})!}{\mathfrak{m}!}\binom{\alpha_{n+1}}{|\rho|}\binom{\alpha\,|n|}{k(\mathfrak{m})}\binom{\beta\,|n|}{w(\mathfrak{m})+\rho}\binom{w(\mathfrak{m})+\rho}{w(\mathfrak{m})}$$
$$\times\,\mathbb{B}_1^{\beta|n|-w(\mathfrak{m})-\rho}\mathbb{A}_1^{\alpha|n|-k(\mathfrak{m})}A_{n+1}^{\alpha_{n+1}-|\rho|}B_{n+1}^{\beta_{n+1}}\{C_{i,j}: i,j=1,\ldots,n\}^{\mathfrak{m}}$$
$$\times\{C_{n+1,j}: j=1,\ldots,n\}^{\rho}f \quad \text{(by Lemma 2)}$$

$$= \sum_{\mathfrak{n}\in\{(\mathbf{Z}_+)^n\}^{n+1}}\left\{\frac{k(\mathfrak{n})!\,w(\mathfrak{n})!}{\mathfrak{n}!}\binom{\alpha}{k(\mathfrak{n})}\binom{\beta\,|n|}{w(\mathfrak{n})}\mathbb{B}_1^{\beta|n|-w(\mathfrak{n})}\mathbb{A}_1^{(\alpha-k(\mathfrak{n}))|n|}\right\}$$
$$\times[A_{n+1}^{(\alpha-k(\mathfrak{n}))_{n+1}}B_{n+1}^{\beta_{n+1}}]\{C_{i,j}: i=1,\ldots,n+1\,, j=1,\ldots,n\}^{\mathfrak{n}}f \quad \left(\text{where } \mathfrak{n}:=\left|\begin{matrix}\mathfrak{m}\\ \rho\end{matrix}\right|\right)$$

$$= \sum_{\mathfrak{n}\in\{(\mathbf{Z}_+)^n\}^{n+1}}\{\,.\,.\,\}\left[\sum_{k\in\mathbf{Z}_+}k!\binom{(\alpha-k(\mathfrak{n}))_{n+1}}{k}\binom{\beta_{n+1}}{k}B_{n+1}^{\beta_{n+1}-k}A_{n+1}^{(\alpha-k(\mathfrak{n}))_{n+1}-k}C_{n+1,n+1}^{k}\right]$$
$$\times\{C_{i,j}: i=1,\ldots,n+1 \text{ and } j=1,\ldots,n\}^{\mathfrak{n}}f$$

$$= \sum_{\mathfrak{n}\in\{(\mathbf{Z}_+)^n\}^{n+1}}\left\{\frac{k(\mathfrak{n})!\,w(\mathfrak{n})!}{\mathfrak{n}!}\binom{\alpha}{k(\mathfrak{n})}\binom{\beta\,|n|}{w(\mathfrak{n})}\mathbb{B}_1^{\beta|n|-w(\mathfrak{n})}\right\}_{1^*}$$
$$\times\sum_{k\in\mathbf{Z}_+}\left\{k!\binom{(\alpha-k(\mathfrak{n}))_{n+1}}{k}\binom{\beta_{n+1}}{k}\right\}_{2^*}[\mathbb{A}_1^{(\alpha-k(\mathfrak{n}))|n|}B_{n+1}^{\beta_{n+1}-k}]$$
$$\times\{A_{n+1}^{(\alpha-k(\mathfrak{n}))_{n+1}-k}C_{n+1,n+1}^{k}\{C_{i,j}: i=1,\ldots,n+1 \text{ and } j=1,\ldots,n\}^{\mathfrak{n}}\}_{3^*}f \quad \text{(by (16))}$$

$$= \sum_{\mathfrak{n}\in\{(\mathbf{Z}_+)^n\}^{n+1}}\{1^*\}\sum_{k\in\mathbf{Z}_+}\{2^*\}\left[\sum_{\rho\in(\mathbf{Z}_+)^n}|\rho|!\binom{(\alpha-k(\mathfrak{n}))\,|n|}{\rho}\binom{\beta_{n+1}-k}{|\rho|}\right.$$
$$\left.\times\,B_{n+1}^{\beta_{n+1}-k-|\rho|}\mathbb{A}_1^{(\alpha-k(\mathfrak{n}))|n|-\rho}\{C_{i,n+1}: i=1,\ldots,n\}^{\rho}\right]\{3^*\}f$$

$$= \sum_{\mathfrak{n}\in\{(\mathbb{Z}_+)^n\}^{n+1}} \{1^*\} \sum_{k\in\mathbb{Z}_+} \sum_{\rho\in(\mathbb{Z}_+)^n} k!\,|\rho|! \binom{(\alpha-k(\mathfrak{n}))_{n+1}}{k} \binom{(\alpha-k(\mathfrak{n}))\,|n|}{\rho} \binom{\beta_{n+1}}{k}$$

$$\times \binom{\beta_{n+1}-k}{|\rho|} B_{n+1}^{\beta_{n+1}-(k+|\rho|)} \mathbb{A}_1^{(\alpha-k(\mathfrak{n}))|n|-\rho} A_{n+1}^{(\alpha-k(\mathfrak{n}))_{n+1}-k} \{C_{i,n+1} : i=1,\ldots,n\}^{\rho}$$

$$\times C_{n+1,n+1}^{k} \{C_{i,j} : i=1,\ldots,n+1 \text{ and } j=1,\ldots,n\}^{\mathfrak{n}} f \quad \text{(by Lemmas 1 and 2)}$$

$$= \sum_{\mathfrak{n}\in\{(\mathbb{Z}_+)^n\}^{n+1}} \{1^*\} \sum_{\tau\in(\mathbb{Z}_+)^{n+1}} |\tau|! \binom{\alpha-k(\mathfrak{n})}{\tau} \binom{\beta_{n+1}}{|\tau|} B_{n+1}^{\beta_{n+1}-|\tau|} \mathbb{A}^{(\alpha-k(\mathfrak{n}))-\tau} \quad (\text{where } \tau := |\rho, k|)$$

$$\times \{\{C_{i,n+1} : i=1,\ldots,n+1\}^{\tau} \{C_{i,j} : i=1,\ldots,n+1 \text{ and } j=1,\ldots,n\}^{\mathfrak{n}}\}_{4^*} f$$

$$= \sum_{\mathfrak{n}\in\{(\mathbb{Z}_+)^n\}^{n+1}} \sum_{\tau\in(\mathbb{Z}_+)^{n+1}} \frac{k(\mathfrak{n})!\,w(\mathfrak{n})!}{\mathfrak{n}!} |\tau|! \binom{\alpha}{k(\mathfrak{n})} \binom{\alpha-k(\mathfrak{n})}{\tau} \binom{\beta\,|n|}{w(\mathfrak{n})} \binom{\beta_{n+1}}{|\tau|}$$

$$\times \mathbb{B}_1^{\beta|n|-w(\mathfrak{n})} B_{n+1}^{\beta_{n+1}-\tau} \mathbb{A}^{\alpha-(k(\mathfrak{n})+\tau)} \{4^*\} f \quad \text{(by Lemmas 1 and 3)}$$

$$= \sum_{\mathfrak{w}\in\{(\mathbb{Z}_+)^{n+1}\}^{n+1}} \frac{k(\mathfrak{w})!\,w(\mathfrak{w})!}{\mathfrak{w}!} \binom{\alpha}{k(\mathfrak{w})} \binom{\beta}{w(\mathfrak{w})} \mathbb{B}^{\beta-w(\mathfrak{w})} \mathbb{A}^{\alpha-k(\mathfrak{w})} \mathbb{C}^{\mathfrak{w}} f,$$

where $\mathfrak{w} := |\mathfrak{n}, \tau|$ for any $\alpha, \beta \in (\mathbb{Z}_+)^{n+1}$.

4. In the sequel we shall consider pairs of the following families of operators: $(\mathbb{S}^*, \mathbb{S})$, where $\mathbb{S}^* = \{S_i^* : i \in I\}$. We will say that the family $\mathbb{S}$ satisfies the *(strong-)generalized (semi-)commutation relation* if the pair $(\mathbb{S}^*, \mathbb{S})$ satisfies the (strong-)generalized (semi-)commutation relation.

Similarly as in [4] from the generalized (semi-)commutation relation it follows that

(18) $$S_i^* C_{j,k} f = C_{j,k} S_i^* f \qquad \text{for all } f \in M, \qquad i, j, k \in I$$

(respectively $S_i^* E_{j,k} f = E_{j,k} S_i^* f$ for all $f \in M$, $i, j, k \in I$).

Naturally from the (strong-)generalized commutation relation follows the (strong-) generalized semicommutation relation, but the converse is not true.

EXAMPLE. Let H be a separable Hilbert space with the orthonormal basis $\{e_i : i \in \mathbb{Z}\}$. We define the operator S as follows:

$Se_i := a_i e_{i+1}$, where $a_i = 0$ if $i \geq 0$ and $a_i := |i|^{1/2}$ if $i < 0$. Clearly:

$$Ce_i := (S^*S - SS^*)e_i = (|a_i|^2 - |a_{i-1}|^2)e_i = \begin{cases} -e_i & \text{if } i \leq 0 \\ 0 & \text{if } i > 0 \end{cases}$$

and $CSe_i = SCe_i$. So the operator S fulfils the generalized semicommutation relation on $M = \mathrm{LIN}\{e_i : i \in \mathbb{Z}\}$, but does not fulfil the generalized commutation relation:

$$-1 = \langle Ce_i, e_i \rangle \neq \langle E^2 e_i, e_i \rangle = \langle Ee_i, Ee_i \rangle \geq 0 \quad \text{for} \quad i < 0.$$

In addition this operator is not subnormal, as the Halmos–Bram condition is not satisfied:

$$\sum_{i,j=0}^{1} \langle S^i f_j, S^j f_i \rangle = \|f_0\|^2 + 2\,\mathrm{Re}\langle Sf_0, f_1 \rangle + \|Sf_1\|^2 = 1 - 2a_{-1} + 0 = -1 < 0,$$

for $f_0 := -e_{-1}$ and $f_1 := e_0$.

Now we prove the main result of this paper.

THEOREM 2. *Let $\mathbb{S}$ be a family of linear operators in H that satisfies the strong generalized commutation relation on a linear spce M with the family $\mathbb{E}=\{E_{i,j}: i, j \in I\}$.*

(18) *If the family $\mathbb{S}$ satisfies the condition (2), then this family also satisfies the Halmos–Bram condition.*

(19) *If the family $\mathbb{S}$ satisfies the conditions (2) and (5) with $M = D(\mathbb{S})$, then $\mathbb{S}$ is subnormal.*

Proof. The second part of our theorem follows immediately from the first one and Theorem A. Now we prove the first part. Without loss of generality we can assume that $\mathbb{S}=(S_1, \ldots, S_n)$. Let $f:(\mathbb{Z}_+)^n \to M$ be a finitely supported function. Then from Theorem 1 and the condition (7), it follows that

$$\sum_{\alpha,\beta\in(\mathbb{Z}_+)^n} \langle \mathbb{S}^\beta f(\alpha), \mathbb{S}^\alpha f(\beta)\rangle = \sum_{\alpha,\,\beta\in(\mathbb{Z}_+)^n} \langle (\mathbb{S}^*)^\alpha \mathbb{S}^\beta f(\alpha), f(\beta)\rangle = \sum_{\mathfrak{m}\in\{(\mathbb{Z}_+)^n\}^n} \frac{k(\mathfrak{m})!\, w(\mathfrak{m})!}{\mathfrak{m}!}$$
$$\times\left\langle \sum_{\alpha\in(\mathbb{Z}_+)^n} \binom{\alpha}{k(\mathfrak{m})} (\mathbb{S}^*)^{\alpha-k(\mathfrak{m})} E^{\mathfrak{m}} f(\alpha),\ \sum_{\beta\in(\mathbb{Z}_+)^n} \binom{\beta}{w(\mathfrak{m})} (S^*)^{\beta-w(\mathfrak{m})} \mathbb{E}^{\mathfrak{m}} f(\beta)\right\rangle.$$

But $E_{i,j}=0$ for $i \neq j$ (see the condition (9)). Therefore $\mathbb{E}^{\mathfrak{m}} f = 0$ ($f \in M$) if $m_{ij} \neq 0$ for any pair $(i, j) \in \{1, \ldots, n\} \times \{1, \ldots, n\}$ such that $i \neq j$. Finally

$$\sum_{\alpha,\,\beta\in(\mathbb{Z}_+)^n} \langle \mathbb{S}^\beta f(\alpha), \mathbb{S}^\alpha f(\beta)\rangle = \sum_{\substack{\mathfrak{m}\in\{(\mathbb{Z}_+)^n\}^n \\ \mathfrak{m}\text{-diagonal}}} \frac{1}{\mathfrak{m}!} \left\| \sum_{\alpha\in(\mathbb{Z}_+)^n} \binom{\alpha}{\mathfrak{m}} (S^*)^{\alpha-\mathfrak{m}} \mathbb{E}^{\mathfrak{m}} f(\alpha)\right\|^2 \geq 0,$$

because $k(\mathfrak{m}) = w(\mathfrak{m}) = \mathfrak{m}$ for any diagonal matrix $\mathfrak{m}$. QED.

5. The result above can be applied to families of creation operators defined on Bargmann's Hilbert space of an infinite order **[1]**, **[5]**. Let B denote Bargmann's Hilbert space of an infinite order; i.e. the adequate subset of the set of all holomorphic functions on l^2, where l^2 denotes the Hilbert space of all square summable complex sequences. This Hilbert space has the following orthonormal basis: $\mathcal{M} := \{e_\tau : \tau \in J\}$, where $e_\tau(z) = (\tau!)^{-1/2} z^\tau$ ($z \in l^2$), J is the set of all sequences of nonnegative integers with only a finite number of nonzero entries and $z^\tau := \prod z_i^{\tau_i}$. (See **[1]**, **[5]**.) Bargmann defined the generalized creation operators as follows:

$$D(A_a^+) := \{f \in B : [z \to \langle z, a\rangle f(z)] \in B\}$$

and

$$(A^+ f)z := \langle z, a\rangle f(z) \qquad \text{for all } z \in l^2, \qquad f \in D(A_a^+).$$

THEOREM 3. *Let W denote the set of mutually orthogonal vectors from l^2. Then the family $\{A_a^+ : a \in W\}$ is subnormal.*

Proof. We know from **[5]** that the linear space:

$$\mathbb{C}[l^2] := \{\mathfrak{p}(\langle\cdot, x_1\rangle, \ldots, \langle\cdot, x_n\rangle) : \mathfrak{p}\text{-compl. polynomial}, x_i \in l^2\}$$

is invariant under A_a^+ ($a \in l^2$), the operators $\{A_a^+ : a \in W\}$ commute on $\mathbb{C}[l^2]$, $\langle A_a^+, A_b^+\rangle = \langle a, b\rangle \mathrm{Id}$ on $\mathbb{C}[l^2]$ for $a, b \in l^2$ and $A_a^+(L) \subset L \subset \mathcal{A}(A_a^+) \cap \mathbb{C}[l^2]$, where $L := \mathrm{LIN}(\bigcup \{(\mathbb{S})^\alpha \mathcal{M} : \alpha \in J\})$.

Now our statement follows directly from Theorem 2.

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INSTITUTE OF MINING AND METALLURGY
AL MICKIEWICZA 30
30-059 KRAKOW
POLAND