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ON CYCLIC SUBGROUPS OF FINITE GROUPS

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In [3] Laffey has shown that if Z is a cyclic subgroup of a finite subgroup G, then either a nontrivial subgroup of Z is normal in the Fitting subgroup $F(G)$ or there exists a g in G such that $Z^g \cap Z = 1$. In this note we offer a simple proof of the following generalisation of that result:

Theorem. *Let G be a finite group and X and Y cyclic subgroups of G. Then there exists a g in G such that $X^g \cap Y \trianglelefteq F(G)$.*

Remark. Note that X and Y need not to be conjugate subgroups of G. Our notation is standard (see [2]). A subgroup H is said to be a component of G if $[H, H] = H, H$ is subnormal in G and $H/Z(H)$ is a simple group. The product of all components of G is called $E(G)$. Let $F^*(G) = F(G) \cdot E(G)$. Then $C_G(F^*(G)) \subseteq F^*(G)$ (see page 165 of [1]),

Proof of Theorem. We choose G to be a minimal counterexample and X, Y chosen so that $\max(|X|, |Y|)$ is of minimal order. Consequently,

(I) We may assume $|X| = |Y|$ and if X_p is a S_p-subgroup of X then there exists a g in G such that $X_p^g \in \mathrm{Syl}_p(Y)$.

(II) The group X is of square-free order.

Suppose p^2 divides $|X|$. Then by induction, we may assume (by replacing Y by a suitable conjugate) that $L = X \cap N = Y \cap N \neq 1$, where $N = \Omega_1(Z(O_p(G)))$. First assume $L \trianglelefteq G$. Since $C_G(L) \trianglelefteq G$ and X, Y are contained in $C_G(L)$, we have by induction that $G = C_G(L)$.

But by induction on G/L, we have a contradiction. So $L \ntrianglelefteq G$. Let $g \in G$ such that $X^g \cap Y$ is a p'-group. Set $X_O^g = O_{p'}(X^g)$ and $T = C_G(N)X_O^g$. Since $N \subseteq F(T)$ and $(|N|, |X_O^g|) = 1$, we have $F(T) \subseteq C_G(N)$. Hence $F(T) = F(C_G(N)) = F(G)$. By induction, there exists a $h \in C_G(N)$ such that $X_O^{gh} \cap (Y \cap T) \trianglelefteq F(T) = F(G)$. Since $L^h = L$, we have $X^{gh} \cap Y = X_O^{gh} \cap Y = X_O^{gh} \cap (Y \cap T) \trianglelefteq F(G)$, and we have a contradiction. This proves that X is of square-free order.

(III) $G = F^*(G)X = F^*(G)Y$.

Let $G_O = F^*(G)X$. Since $C_G(F^*(G)) \subseteq F^*(G)$ and $F(G_O) \supseteq F(G)$, we have $Z(F(G_O)) \subseteq Z(F(G))$. If $G_O \subset G$, then by induction there exists a $g \in G_O$ with $X^g \cap (Y \cap G_O) \trianglelefteq F(G_O)$. i.e. $X^g \cap Y \cap G_O \subseteq Z(F(G_O)) \subseteq Z(F(G))$. Since $X^g \cap Y = X^g \cap Y \cap G_O$ we are done.

(IV) If $1 \neq N \subseteq \Omega_1(Z(O_p(G)))$ with $N \trianglelefteq G$, then $F(G/N) \supset F(G)/N$. Suppose

$F(G/N)=F(G)/N$. Then by induction we may assume $L=O_p(X\cap Y)N\trianglelefteq F(G)$ but $O_p(X\cap Y)\ntrianglelefteq F(G)$. Since $G=F^*(G)X$, we have $L\trianglelefteq G$. Let $g\in G-N_G(O_p(X\cap Y))$ and $T=C_G(L)X^g$. Since $O_{p'}(F(G))\subseteq C_G(L)$ we have $O_{p'}(F(G))\subseteq O_{p'}(F(T))$. But since $O_{p'}(F(T))\subseteq C_G(L)$ and $F(C_G(L)\subseteq F(G)$, we have $O_{p'}(F(T))=O_{p'}(F(G))$. Now by induction, there is a $t\in T$ such that $X^{gt}\cap(T\cap Y)\trianglelefteq F(T)$. Since t centralises $O_p(X\cap Y)$, we have $X^{gt}\cap(T\cap Y)$ is a p'-group. Hence $X^{gt}\cap Y\trianglelefteq O_{p'}(F(G))$, a contradiction.

(V) $O_p(F(G))$ is elementary abelian, $F(G)\cap E(G)=1$ and $X\cap F(G)=Y\cap F(G)=1$.

Suppose $O_p(F(G))$ is not elementary abelian. Setting $N=\phi(O_p(G))\cap\Omega_1(Z(O_p(G))$, we have $F(G/N)=F(G)/N$ contradicting (IV). Similarly, if $F(G)\cap E(G)\neq 1$, let $N=\Omega_1(O_p(F(G)\cap E(G)))$ and again we have a contradiction by (IV). The last assertion follows by induction.

(VI) Either $F^*(G)=O_p(G)$ or $F^*(G)=E(G)$ is a minimal normal subgroup of G.

Suppose the contrary. We have the following cases:

(i) $F(G)=1$ and $1\neq M_1, M_2\trianglelefteq G$ such that $E(G)=M_1\times M_2$.

(ii) $E(G)=1$ and set $M_1=O_p(G)$ and $M_2=O_{p'}(F(G))$.

(iii) Set $M_1=F(G)\neq 1$ and $M_2=E(G)\neq 1$.

Now let $X_O=C_X(M_2)$ and $Y_O=C_Y(M_2)$. Note that by (*I*), $|X_O|=|Y_O|$. As $M_1X_O\trianglelefteq G$, we have $F^*(M_1X_O)=M_1$, and hence there exists a $f\in M_1$ with $X_O^f\cap Y_O=1$. Set $X=X_O\times X_{OO}$, $Y=Y_O\times Y_{OO}$ and $T=M_2X_{OO}^f$. Pick $m\in M_2$ such that $X_{OO}^f\cap Y_{OO}^M\trianglelefteq F(T)$. In cases (i) and (iii), $F(T)\subseteq C_G(M_2)$ and so $X_{OO}^f\cap Y_{OO}^m=1$. In case (ii) $F(T)\supseteq M_2$ and by (V), $(X_{OO}^f\cap Y_{OO}^m)\cap M_2=1$ and hence $X_{OO}^f\cap Y_{OO}^m=1$. Since $Y_O^m=Y_O$ and $|X_O|, |X_{OO}|$ are coprime numbers, we have $X^f\cap Y^m=(X_O^f\cap Y_O^m)\times(X_{OO}^f\cap Y_{OO}^m)=1$, a contradiction.

(VII) Final contradiction.

If $F^*(G)=O_p(G)$, we have by the Schur–Zassenhaus theorem [**2**, Thm. 2.1, p. 221] that $X^g=Y$ for some $g\in G$. By Maschke's theorem, [**2**, Thm. 3.1, p. 66], let $O_p(G)=V_1V_2\ldots V_r$ where V_i are minimal X-invariant subgroups. Let $1\neq v_i\in V_i$, then $X\cap X^{v_1v_2\cdots v_r}=1$, a contradiction. Hence by (VI), we have $F^*(G)=E(G)$ is a minimal normal subgroup of G. Let $X_p\in \mathrm{Syl}_p(X)$. Then by (I), there exists a g_p in G such that $X^{g_p}\in\mathrm{Syl}_p(Y)$. Since G is a counterexample, for any g in G there exists a prime p such that $p\mid |X^g\cap Y|$. Hence $X_p^g=X_p^{g_p}$ and $g\in N_G(X_p)g_p$. So we have $G=\bigcup_{p\in\pi(X)} N_G(X_p)g_p$.

Let $N_G(X_q)=\max(|N_G(X_p)|\mid p\in\pi(X))$. Then $|G\colon N_G(X_q)|<\pi(X)<r=\max(\pi(X))$. Hence $1\neq X_r\subseteq\bigcap_{g\in G} N_G(X_q)^g$ and so $E(G)\subseteq N_G(X_q)$, which is impossible. This final contradiction proves the theorem.

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