Canad. Math. Bull. Vol. **35** (4) 1992, pp. 560–568

AN EASY PROOF FOR SOME CLASSICAL THEOREMS IN PLANE GEOMETRY

C. THAS

ABSTRACT. The main result of this paper is a theorem about three conics in the complex or the real complexified projective plane. Is this theorem new? We have never seen it anywhere before. But since the golden age of projective geometry so much has been published about conics that it is unlikely that no one noticed this result. On the other hand, why does it not appear in the literature? Anyway, it seems interesting to "repeat" this property, because several theorems in connection with straight lines and (or) conics in projective, affine or euclidean planes are in fact special cases of this theorem. We give a few classical examples: the theorems of Pappus-Pascal, Desargues, Pascal (or its converse), the Brocard points, the point of Miquel. Finally, we have never seen in the literature a proof of these theorems using the same short method ($\Omega_a^b\Omega_c^a\Omega_b^c = I$: see the proof of the main theorem).

1. **The main theorem** $\mathcal{P}$ is the notation for the complexified real projective plane or for the complex projective plane. "Complexified real" means that we also consider imaginary points in this plane; for instance, any real line, which is not a tangent line of a real conic, will intersect the conic in two points (real points or conjugate complex points).

For points we use italics $a, b, c, \ldots$ and capitals for straight lines. The conic containing the five points a, b, c, d, e or tangent to the five lines A, B, C, D, E is denoted by $K(abcde)$ or $K(A\ B\ C\ D\ E)$, respectively. Moreover $K(a\ b\ c\ d\ D)$, with $d \in D$, is the conic through a, b, c, d and tangent to the line D at the point d, and so on. For instance, if we use later the notation $K(s\ b\ c\ B\ C)$, and $c \in B$, $b \in C$ then it is the conic through s, tangent to the line B at the point c and tangent to the line C at the point b.

MAIN THEOREM (FIGURE A). *Consider in $\mathcal{P}$ three mutually different points a, b, c and through a three mutually different straight lines A_1, A_2, A_3, through b three mutually different lines B_1, B_2, B_3, through c three mutually different lines C_1, C_2, C_3. Put $A_i \cap B_i = c_i$, $B_i \cap C_i = a_i$, $C_i \cap A_i = b_i$, $i = 1, 2, 3$. Then the three conics $K(a_1\, a_2\, a_3\, b\, c)$, $K(b_1\, b_2\, b_3\, c\, a)$, $K(c_1\, c_2\, c_3\, a\, b)$ have three points in common.*

PROOF. First consider the most general case: special cases, where for instance one or more conics are degenerate, are treated later on. Look at the following projectivities:

$$\Omega_a^b : a(A_1, A_2, A_3, \ldots) \overline{\wedge}\, b(B_1, B_2, B_3, \ldots)$$
$$\Omega_b^c : b(B_1, B_2, B_3, \ldots) \overline{\wedge}\, c(C_1, C_2, C_3, \ldots)$$
$$\Omega_c^a : c(C_1, C_2, C_3, \ldots) \overline{\wedge}\, a(A_1, A_2, A_3, \ldots).$$

Received by the editors September 17, 1990; revised July 24, 1991.
AMS subject classification: 51A05 (51M05).

The locus of the common points of corresponding lines of Ω_b^c, Ω_c^a, Ω_a^b are the conics $K(a_1\ a_2\ a_3\ b\ c)$, $K(b_1\ b_2\ b_3\ c\ a)$ and $K(c_1\ c_2\ c_3\ a\ b)$, respectively. But $\Omega_a^b\Omega_c^a\Omega_b^c$ is a projectivity with three mutually different invariant lines B_1, B_2, B_3 and thus it is the identity transformation. As a corollary, any common point ($\neq a, b, c$) of two of the three conics belongs also to the third conic and this completes the proof.

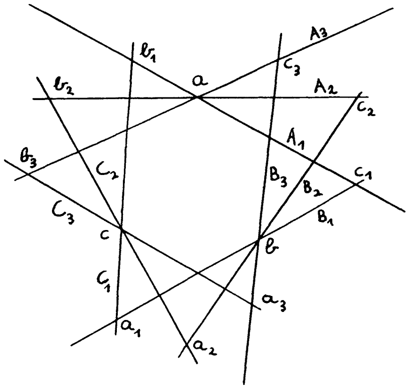

FIGURE A

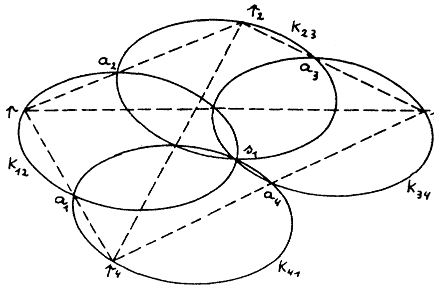

FIGURE B

Next, we give a kind of converse:

PROPOSITION (FIGURE B). *Consider* $n+3$ *points* $a_1, \ldots, a_n, s_1, s_2, s_3$, $n \geq 3$, *in the projective plane* $\mathcal{P}$ *and the conics* $K_{12}(s_1\ s_2\ s_3\ a_1\ a_2)$, $K_{23}(s_1\ s_2\ s_3\ a_2\ a_3)$, $\ldots$, $K_{n-1,n}(s_1 s_2 s_3 a_{n-1} a_n)$, $K_{n1}(s_1 s_2 s_3 a_n a_1)$. *For any point* p *of* K_{12}, *we put* $pa_2 \cap K_{23} = \{a_2, p_2\}$, $p_2 a_3 \cap K_{34} = \{a_3, p_3\}$, $\ldots$, $p_{n-2}a_{n-1} \cap K_{n-1,n} = \{a_{n-1}, p_{n-1}\}$, $p_{n-1}a_n \cap K_{n1} = \{a_n, p_n\}$. *Then the points* p_n, a_1 *and* p *are collinear.*

PROOF. Consider the following projectivities:

$$\Omega_1^2 : a_1(a_1s_1, a_1s_2, a_1s_3, \ldots) \,\overline{\wedge}\, a_2(a_2s_1, a_2s_2, a_2s_3, \ldots)$$
$$\Omega_2^3 : a_2(a_2s_1, a_2s_2, a_2s_3, \ldots) \,\overline{\wedge}\, a_3(a_3s_1, a_3s_2, a_3s_3, \ldots)$$
$$\ldots$$
$$\Omega_{n-1}^n : a_{n-1}(a_{n-1}s_1, a_{n-1}s_2, a_{n-1}s_3, \ldots) \,\overline{\wedge}\, a_n(a_ns_1, a_ns_2, a_ns_3, \ldots)$$
$$\Omega_n^1 : a_n(a_ns_1, a_ns_2, a_ns_3, \ldots) \,\overline{\wedge}\, a_1(a_1s_1, a_1s_2, a_1s_3, \ldots).$$

The conic K_{12} ($K_{23}, \ldots, K_{n-1,n}, K_{n1}$, respectively) is the locus of the common points of corresponding lines of the projectivity Ω_1^2 ($\Omega_2^3, \ldots, \Omega_{n-1}^n, \Omega_n^1$, respectively) and $\Omega_1^2\Omega_n^1\Omega_{n-1}^n \cdots \Omega_2^3$ is an identity transformation. For the line a_2p we have

$$a_2p \xrightarrow{\Omega_2^3} a_3p_2 \xrightarrow{\Omega_3^4} a_4p_3 \xrightarrow{\Omega_4^5} \cdots \xrightarrow{\Omega_{n-1}^n} a_np_{n-1} \xrightarrow{\Omega_n^1} a_1p_n \xrightarrow{\Omega_1^2} a_2p$$

which means that p_n, a_1 and p are collinear. This completes the proof.

In Figure b the four conics K_{12}, K_{23}, K_{34}, K_{41} have, apart from s_1 two common imaginary (conjugate complex) points s_2, s_3 (consider the same figure in the extended complexified real affine plane and s_2, s_3 are the common points at infinity of the four ellipses).

2. **Special cases in the projective plane** Notations: $ab = C$, $bc = A$ and $ca = B$.

COROLLARY 1. *Suppose that in the main theorem $a_3 = b_3 = c_3 = s$. Then the conics $K(a_1\ a_2\ s\ b\ c)$, $K(b_1\ b_2\ s\ c\ a)$ and $K(c_1\ c_2\ s\ a\ b)$ have, apart from s, two common points.*

COROLLARY 2. *Suppose that $a_2 = b_2 = c_2 = s_1$ and $a_3 = b_3 = c_3 = s_2$. Then the conics $K(a_1\ s_1\ s_2\ b\ c)$, $K(b_1\ s_1\ s_2\ c\ a)$ and $K(c_1\ s_1\ s_2\ a\ b)$ have, apart from s_1 and s_2, a common point.*

COROLLARY 3. *Suppose that $A_3 = C$, $B_3 = A$ and $C_3 = B$. Then the conics $K(a_1\ a_2\ b\ c\ B)$, $K(b_1\ b_2\ c\ a\ C)$ and $K(c_1\ c_2\ a\ b\ A)$ have three common points.*

COROLLARY 4. *Suppose that in Corollary 3 $a_1 = b_1 = c_1 = s_1$ and $a_2 = b_2 = c_2 = s_2$. Then the conics $K(s_1\ s_2\ b\ c\ B)$, $K(s_1\ s_2\ c\ a\ C)$ and $K(s_1\ s_2\ a\ b\ A)$ have, apart from s_1 and s_2, a common point.*

COROLLARY 5. *Suppose that in the main theorem $B_2 = C_3 = A$, $C_2 = A_3 = B$ and $A_2 = B_3 = C$. Then the conics $K(a_1\ b\ c\ B\ C)$, $K(b_1\ c\ a\ C\ A)$ and $K(c_1\ a\ b\ A\ B)$ have three common points.*

COROLLARY 6. *Suppose that in Corollary 5 $a_1 = b_1 = c_1 = s$. Then the conics $K(s\ b\ c\ B\ C)$, $K(s\ c\ a\ C\ A)$ and $K(s\ a\ b\ A\ B)$ have, apart from s, two common points.*

Next we consider some cases where one or more of the conics are degenerate.

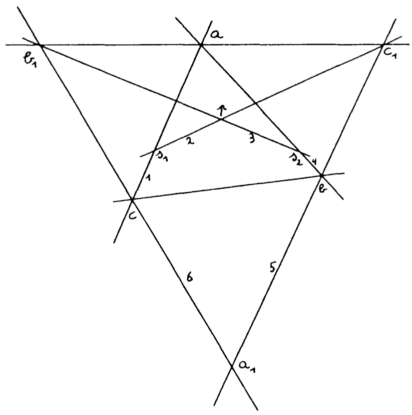

FIGURE C

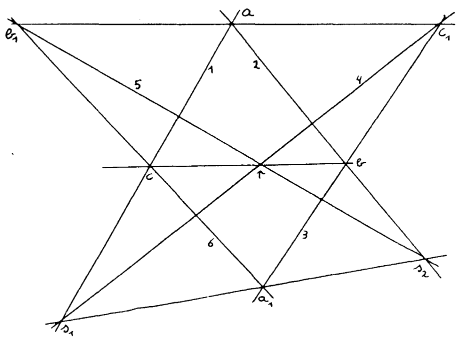

FIGURE D

COROLLARY 7 (FIGURE C). *Suppose that in Corollary 2 $s_1 \in ac$ ($s_1 \neq a, c$) and $s_2 \in ab$ ($s_2 \neq a, b$). In this case the conics $K(b_1\ s_1\ s_2\ c\ a)$ and $K(c_1\ s_1\ s_2\ a\ b)$ are degenerate.*

https://doi.org/10.4153/CMB-1992-073-8 Published online by Cambridge University Press

The components are (ac, b_1s_2) and (ab, c_1s_1), respectively. The conic $K(a_1\ s_1\ s_2\ b\ c)$ contains the fourth common point p (apart from s_1, s_2, a) of the first two conics. Remark that this is the converse of the theorem of ***Pascal****: denote the lines by $1, 2, \ldots, 6$ as in Figure c and $1 \cap 4$, $2 \cap 5$, $3 \cap 6$ are collinear.*

COROLLARY 8 (FIGURE D). *Suppose that in Corollary 7 the points s_1 and s_2 are such that $p \in bc$. Then $K(a_1\ s_1\ s_2\ b\ c)$ is also degenerate and s_1, s_2, a_1 are collinear. This is the theorem of* ***Pappus-Pascal****.*

COROLLARY 9 (FIGURE E). *Suppose that in the main theorem a, b, c are collinear and that $A_3 = B_3 = C_3 = abc$. Then the three conics are degenerate and the common point $a_1a_2 \cap b_1b_2$ is also a point of c_1c_2. This is* ***Desargues****' theorem (if the corresponding sides of the triangles $a_1b_1c_1$ and $a_2b_2c_2$ intersect on a straight line then the lines connecting corresponding vertices are concurrent). Recall that Desargues' theorem and its converse are dual theorems.*

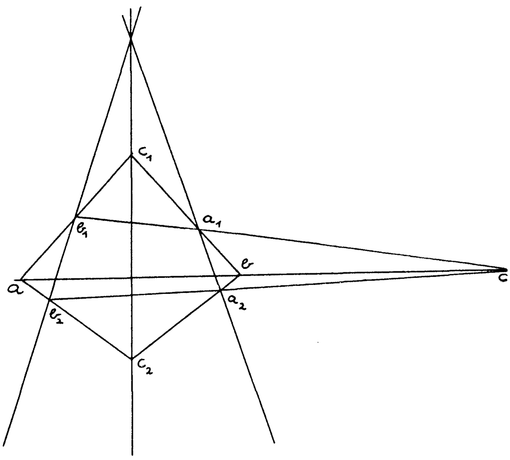

FIGURE E

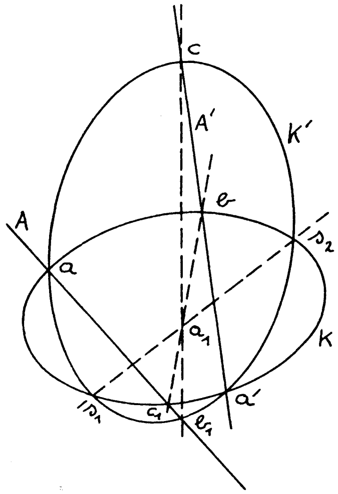

FIGURE F

COROLLARY 10 (FIGURE F). *Consider two conics K and K' which intersect at the points a, a', s_1, s_2 and consider two straight lines: A through a and A' through a' such that $A \cap K = \{a, c_1\}$, $A \cap K' = \{a, b_1\}$, $A' \cap K = \{a', b\}$, $A' \cap K' = \{a', c\}$. Then the lines s_1s_2, c_1b and b_1c are concurrent.*

PROOF. Put $a_1 = bc_1 \cap cb_1$. The conic $K(a_1\ s_1\ s_2\ b\ c)$ contains the fourth common point a' of $K = K(c_1\ s_1\ s_2\ a\ b)$ and $K' = K(b_1\ s_1\ s_2\ a\ c)$. But a', b, c are collinear and therefore $K(a_1\ s_1\ s_2\ b\ c)$ is degenerate, with components bc and s_1s_2. Thus the line s_1s_2 must contain the point a_1.

The following theorem is not a special case of the main theorem, but a corollary (and also an extension) of Corollary 10. In the literature it is sometimes called the theorem of the three conics ([3]).

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THEOREM (FIGURE G). *If three conics have two common points s_1, s_2 then the lines connecting the other common points of the conics (taken in pairs) are concurrent.*

PROOF. B_2 (B_3, respectively) is the line connecting the common points different from s_1, s_2 of the conics K_1 and K_3 (of K_1 and K_2, respectively). Put $B_2 \cap B_3 = s$. Consider a straight line S_1 through s such that $S_1 \cap K_1 = \{p_1, p_2\}$. The line $p_1 s_1$ intersects K_2 and K_3 in q_1 and r_1 respectively and the line $p_2 s_2$ intersects K_2 and K_3 in q_2 and r_2 respectively. Because of Corollary 10, the lines $q_1 q_2$ and $r_1 r_2$ both contain the points s and thus s is a point of the line B_1 which connects the common point, different from s_1 and s_2, of K_2 and K_3. This completes the proof.

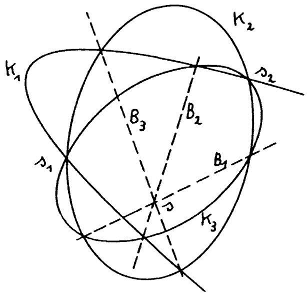

FIGURE G

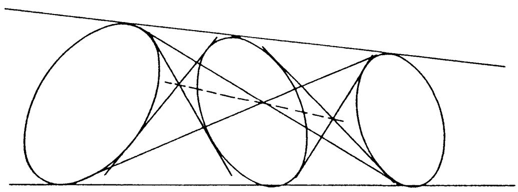

FIGURE H

Remark that Corollary 10, Pascal's theorem and the theorem of Pappus-Pascal are special cases of the theorem of the three conics: one, two or the three conics are degenerate.

3. The dual theorems

THE DUAL MAIN THEOREM. *Consider in $\mathcal{P}$ three distinct straight lines A, B, C and on A three distinct points a_1, a_2, a_3, on B three distinct points b_1, b_2, b_3, and on C three distinct points c_1, c_2, c_3. Put $a_i b_i = C_i$, $b_i c_i = A_i$ and $c_i a_i = B_i$, $i = 1, 2, 3$. Then the conics $K(A_1\, A_2\, A_3\, B\, C)$, $K(B_1\, B_2\, B_3\, C\, A)$ and $K(C_1\, C_2\, C_3\, A\, B)$ have three common tangent lines.*

Of course any result of Section 2 has a dual, but we only give the "theorem of the three tangential conics": if three conics have two common tangent lines, then the other tangent lines of the conics, taken in pairs, intersect in collinear points (Figure h). This theorem is not a special case of the dual main theorem, but the special cases obtained from this theorem where one, two or the three tangential conics are degenerate actually are special cases of the dual main theorem. If two of these conics are degenerate, we get the theorem of **Brianchon** and if the three conics are degenerate, we find again the theorem of **Pappus-Pascal**.

4. Special cases in the affine and Euclidean plane We work in the extended real affine or Euclidean plane, *i.e.* we have a line at infinity. We also consider imaginary points in these planes: for instance all circles intersect the line at infinity in the same pair of conjugate complex points, the two *circular points* of the Euclidean plane.

Through any point of the Euclidean plane not on the line at infinity there are two *isotropic lines*, the lines connecting the point with the circular points.

COROLLARY 11. *We consider two triangles abc and $a'b'c'$ in the Euclidean plane such that $a' \in bc, a' \neq b, c$; $b' \in ac, b' \neq a, c$; $c' \in ab, c' \neq a, b$. Then the circumscribed circles of the triangles $ab'c'$, $bc'a'$ and $ca'b'$ have a point in common. Sometimes this is called the theorem of* ***Miquel****.*

PROOF. This is Corollary 2 where s_1, s_2 are the circular points of the Euclidean plane.

REMARK. This last result is often given as an application of the "theorem of the ninth point": all cubics through eight fixed points pass through a ninth fixed point. Consider the degenerate cubics (circle through a, b', c'; straight line $a'bc$), (circle through b, c', a'; straight line $b'ca$), (circle through c, a', b'; straight line $c'ab$): these curves have the six points a, b, c, a', b', c' and the two circular points in common, so they have another common point.

Remark also that the theorem of the three conics has a very short proof if we use the theorem of the ninth point: consider in Figure g the degenerate cubics (K_1, B_1), (K_2, B_2), (K_3, B_3), ...

COROLLARY 12. *Suppose that A, B, C, D are straight lines in the Euclidean plane, different from the line at infinity, no three of which are concurrent and no two lines are parallel. Then the circumscribed circles of the triangles ABC, ACD, BCD and ABD have a common point. This is the point of* ***Miquel****.*

PROOF. Consider Corollary 2 where s_1, s_2 are the circular points and where a, b, c are collinear. Use this corollary twice and you obtain this result at once.

COROLLARY 13. *Consider in the Euclidean plane a triangle abc. The circles through a and with tangent line A at b, through b and with tangent line B at c, through c and with tangent line C at a have a common point. This is one of the* ***Brocard*** *points of the triangle abc. The other Brocard point is the common point of the circle through a with tangent line A at c, the circle through b with tangent line B at a and the circle through c with tangent line C at b.*

PROOF. This is Corollary 4 where s_1, s_2 are the circular points of the plane.

COROLLARY 14. *Suppose that ABCD is a complete quadrilateral in the affine plane (i.e. A, B, C, D are lines no three of which are concurrent) and that A, B, C, D are different from the line at infinity. The intersection points of the sides A, B, C, D are denoted as follows: $A \cap C = a$, $A \cap B = b$, $B \cap C = f$, $B \cap D = e$, $C \cap D = d$, $D \cap A = c$. We suppose that no two of the lines A, B, C, D are parallel. The diagonals are the lines ae, bd, cf.*

Then the parabola with tangent lines ae, bd, A, D, the parabola with tangent lines bd, cf, A, C and the parabola with tangent lines cf, ae, A, B have a common tangent line (different from the line at infinity).

PROOF. Use the dual of Corollary 2 in the affine plane. Consider the following projectivities:
Ω_1: line $ae \longrightarrow$ line bd; $a \longrightarrow b$, $e \longrightarrow d$, point at infinity of $ae \longrightarrow$ point at infinity of bd;
Ω_2: line $bd \longrightarrow$ line cf; $b \longrightarrow c$, $d \longrightarrow f$, point at infinity of $bd \longrightarrow$ point at infinity of cf;
Ω_3: line $cf \longrightarrow$ line ae; $c \longrightarrow a$, $f \longrightarrow e$, point at infinity of $cf \longrightarrow$ point at infinity of ae.
Then $\Omega_3\Omega_2\Omega_1$ is the identity transformation on ae and this gives at once the result.

REMARK. The midpoints m_1, m_2, m_3 of the diagonals ae, bd, cf of the complete quadrilateral are collinear: this gives the **Newton**-line of the quadrilateral. Let the points at infinity of ae, bd, cf be denoted by n_1, n_2, n_3, respectively. We have $(aem_1n_1) = (bdm_2n_2) = (cfm_3n_3) = -1$. This means that the common tangent line of the three parabolas in the foregoing corollary is precisely this **Newton**-line. But from this corollary it does not follow that m_1, m_2, m_3 are collinear. Therefore we prove this now. Consider the point r at infinity of the parabola with tangent lines A, B, C, D. Because of the theorem of Desargues-Sturm we know that there is a second conic through r with tangent lines A, B, C, D and for the tangent line N at r of this conic we have: (ra, re, N, line at infinity)=(rb, rd, N, line at infinity)=(rc, rf, N, line at infinity)= -1. This means that $N \cap ae = m_1$, $N \cap bd = m_2$, $N \cap cf = m_3$ or that N is the **Newton**-line.

We conclude this section with a corollary of the dual of our proposition in the affine plane.

COROLLARY 15. *In the affine plane we consider two non-singular ellipses or hyperbolas K_1 and K_2, in general position. The common tangent lines of these conics are A, B, C, D and p is any point of D, while P_1, P_2 are the second tangent lines of K_1 and K_2 through p. If P_1', P_2' are the tangent lines of K_1 and K_2 which are parallel with P_1, P_2, then A, B, C, P_1', P_2' are tangent lines of a parabola.*

PROOF. The conic $K(ABCP_1'P_2')$ is denoted by K_3. Consider the point $P_2 \cap P_2' = q$. This is a point at infinity. Through q we have a second tangent line P_3 of K_3 and we put $P_3 \cap P_1' = r$. Because of the dual of the proposition we know that the second tangent line of K_1 through r is the tangent line P_1, *i.e.* r is a point at infinity, which means that P_3 is the line at infinity and thus K_3 is a parabola. This completes the proof.

5. A combination of the main theorem and its dual Consider the most general case for the main theorem. This means that a, b, c (and a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3, respectively) are not collinear and that a, b, c, a_1, a_2, a_3, b_1, b_2, b_3, c_1, c_2, c_3 are twelve mutually different points. It is easy to see that in this case the conics $K(a_1\ a_2\ a_3\ b\ c)$, $K(b_1\ b_2\ b_3\ c\ a)$ and $K(c_1\ c_2\ c_3\ a\ b)$ are not degenerate. They have three common points s_1, s_2, s_3 and we assume that these points are mutually different (*i.e.* they form a triangle).

Remark that in Figure a the triangles $a_1a_2a_3$ and $b_1b_2b_3$ are in perspective. This is also the case for $b_1b_2b_3$ and $c_1c_2c_3$ and for $c_1c_2c_3$ and $a_1a_2a_3$. We have:

THEOREM. *Consider the configuration of the main theorem. The points* $a_1a_2 \cap b_1b_2$, $a_2a_3 \cap b_2b_3$ *and* $a_3a_1 \cap b_3b_1$ *are collinear on a line which is denoted by* C'. *The analogous lines for the triangles* $b_1b_2b_3$ *and* $c_1c_2c_3$, *and the triangles* $c_1c_2c_3$ *and* $a_1a_2a_3$ *are denoted by* A' *and* B', *respectively. If* s_1, s_2, s_3 *are the common points of the conics* $K(a_1\ a_2\ a_3\ b\ c)$, $K(b_1\ b_2\ b_3\ c\ a)$ *and* $K(c_1\ c_2\ c_3\ a\ b)$ *then* s_1s_2, s_2s_3, s_3s_1 *are common tangent lines of the conics* $K(a_1a_2\ a_2a_3\ a_3a_1\ B'\ C')$, $K(b_1b_2\ b_2b_3\ b_3b_1\ C'\ A')$ *and* $K(c_1c_2\ c_2c_3\ c_3c_1\ A'\ B')$.

PROOF. The last three conics have of course three common tangent lines: this is the dual main theorem. We only have to prove that these tangent lines are the lines s_1s_2, s_2s_3, s_3s_1. But therefore we need a lemma. We first recall the following property: if two triangles are inscribed in a same non-degenerate conic then they are also circumscribed about a same conic.

LEMMA. *Consider two conics* K_1 *and* K_2 *which are not degenerate with mutually different common points* s, t, s_1, s_2 *and two triangles* $a_1b_1c_1$, $a_2b_2c_2$ *such that* $a_1, b_1, c_1 \in K_1$ *and* $a_2, b_2, c_2 \in K_2$. *Moreover we suppose that* a_1a_2, b_1b_2, c_1c_2 *are concurrent with common point* s. *Then the line of perspectivity of the two triangles (theorem of Desargues) is the fourth common tangent line of the conics* K_1' *and* K_2' *which have the sides of the triangles* $a_1b_1c_1$, ts_1s_2 *and* $a_2b_2c_2$, ts_1s_2 *as tangent lines.*

PROOF. Consider for instance the line a_1sa_2 and through a_1 the two lines a_1b_1, a_1c_1; through s the two lines sb_1, sc_1; through a_2 the two lines a_2b_2, a_2c_2. From a special case of the main theorem we have at once that the conics K_1, K_2 and $K(a_1\ a_2\ t\ a_1b_1 \cap a_2b_2\ a_1c_1 \cap a_2c_2)$ have besides t two other common points, which are of course s_1 and s_2. Thus the triangles with vertices t, s_1, s_2 and a_1, $a_1b_1 \cap a_2b_2$, $a_1c_1 \cap a_2c_2$ are inscribed in a conic and therefore also circumscribed about a conic. From this it follows that the line of perspectivity is a tangent line of K_1'. In the same way we find that this line is a tangent line of K_2'. This completes the proof of the lemma.

In order to complete the proof of the theorem it is clearly sufficient to use this lemma twice.

We conclude this paper with a corollary in the Euclidean plane:

COROLLARY 16. *Consider in the Euclidean plane two circles* K_1 *and* K_2 *with common points* s *and* t *(different from the circular points). Suppose that* $a_ib_ic_i$ *is an inscribed triangle of* K_i, $i = 1, 2$, *such that* a_1a_2, b_1b_2, c_1c_2 *contain the point* s. *The line through* $c' = a_1b_1 \cap a_2b_2$, $a' = b_1c_1 \cap b_2c_2$ *and* $b' = c_1a_1 \cap c_2a_2$ *is denoted by* S. *We have:*

1. *The line* S *is the fourth common tangent line of the parabolas* P_1 *and* P_2 *which have both* t *as focal point and which respectively have the sides of the triangles* $a_1b_1c_1$, $a_2b_2c_2$ *as tangent lines (the three other common tangent lines are the line at infinity and the isotropic lines through* t*).*
2. *The points* a', b', c_1, c_2, t *(*b', c', a_1, a_2, t *and* c', a', b_1, b_2, t*, respectively) are points of a circle.*
3. *The feet of the perpendicular lines through* t *onto the sides of the triangles* $a_1b_1c_1$ *and* $a_2b_2c_2$ *are points of straight lines which are denoted by* S_1 *and* S_2*, respectively*

(lines of Simspon-Wallace). We have that $S_1 \cap S_2$ is the foot of the perpendicular line through t onto the line S.

PROOF. 1. This is a special case of the lemma, where s_1 and s_2 are the circular points of the Euclidean plane.

2. This is in fact a straightforward special case of the main theorem (see the proof of the lemma).

3. Recall that the foot of the perpendicular line through the focal point of a parabola onto a tangent line of the parabola is a point of the tangent line at the top of the parabola. From this it follows that S_1 and S_2 are the tangent lines at the tops of the parabolas P_1 and P_2. This completes the proof.

REMARK. It is easy to verify that if $s = t$, *i.e.* K_1 and K_2 are tangent, then the line S is the line at infinity, while the parabolas P_1 and P_2 have the same point at infinity. The lines S_1 and S_2 are parallel in this case and the perpendicular line through $s = t$ on these lines is the common axis of P_1 and P_2.

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Department of mathematics
State University of Ghent
Krijgslaan, 281
B-9000 Ghent
Belgium