GENERALIZATION OF THE FIBONACCI SEQUENCE TO n DIMENSIONS

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Introduction. We introduce certain $n \times n$ matrices with integral elements that constitute a free semigroup with identity and generate the n-dimensional unimodular group. In terms of these matrices we define a certain sequence of n-dimensional vectors, which we show is the natural generalization to n dimensions of the Fibonacci sequence. Connections between the generalized Fibonacci sequences and certain Jacobi polynomials are found. The various basic identities concerning the Fibonacci numbers are shown to have natural extensions to n dimensions, and in some cases the proofs are rendered quite brief by the use of known theorems on matrices.

The results presented in this paper were obtained in the course of an attempt to generalize the notion of continued fractions to higher dimensions. Other results of that attempt will be presented in another paper.

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1. A free monoid that generates the unimodular group. Throughout this paper, $F(S)$ denotes the free monoid over the set S. The elements of $F(S)$, called words in the alphabet S, are represented as finite (possibly empty) sequences of elements of the set S. The operation of $F(S)$ is concatenation; $W_1 * W_2$ denotes the word resulting from the concatenation of the words W_1 and W_2. The empty word is denoted by $\emptyset$. If W is a word of length 1 consisting of the single element $P \in S$, we write $W = P$.

We always take the set S to be the set of elements of one of the symmetric groups S_n ($n = 1, 2, \ldots$). The elements of S_n may be represented as permutations of the set $\{1, \ldots, n\}$. Throughout this paper, E denotes the identity permutation, and T denotes the permutation defined by $T(j) = n + 1 - j$ ($j = 1, \ldots, n$). For all P_1, P_2, P in S_n, $P_1 P_2$ is defined by $(P_1 P_2)(j) = P_1(P_2(j))$ and P^{-1} denotes the inverse of P.

We use $\mathfrak{M}_n^+$ to denote the n-dimensional unimodular group, whose elements are the $n \times n$ matrices M with integral elements and determinant 1. The $n \times n$ identity matrix is denoted by I_n. To denote the element in the ith row and jth column of a matrix M, we always write $(M; i, j)$, thus avoiding elaborate subscript notations later.

With each word $W \in F(S_n)$ we associate a matrix $M(W) \in \mathfrak{M}_n^+$ in the following way:

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(1) If $P \in S_n$, then $(M(P); i, j) = \begin{cases} 1, & \text{if } P^{-1}(i) \leqslant P^{-1}(j), \\ 0, & \text{if } P^{-1}(i) > P^{-1}(j). \end{cases}$

(2) If $W_1, W_2 \in F(S_n)$, then $M(W_1 * W_2) = M(W_1)M(W_2)$. From (2) it follows that $M(\emptyset) = I_n$. From (1) we see that $M(E)$ is upper triangular with

$$(M(E); i, j) = \begin{cases} 1, & \text{if } i \leqslant j, \\ 0, & \text{if } i > j. \end{cases}$$

One also sees from (1) that

$$(M(P); P(i), P(j)) = (M(E); i, j).$$

Since $\det M(E) = 1$ and since $M(P)$ is obtained from $M(E)$ by applying the permutation P both to the rows and to the columns, $\det M(P) = 1$, and hence, by (2), $\det M(W) = 1$ for all $W \in F(S_n)$.

For the inverse matrices we have

$$((M(E))^{-1}; i, j) = \begin{cases} 1, & \text{if } j = i, \\ -1, & \text{if } j = i+1, \\ 0, & \text{if } j \neq i \text{ and } j \neq i+1. \end{cases}$$

$$((M(P))^{-1}; P(i), P(j)) = ((M(E))^{-1}; i, j).$$

Let Z^n denote the set of ordered n-tuples of integers. R^n denotes the set of ordered n-tuples of real numbers. For $u \in Z^n$ (or R^n), we write

$$u = (u(1), \ldots, u(n)).$$

For $u, v \in Z^n$ (or R^n), we write $u \geqslant v$ to express that

$$u(i) \geqslant v(i) \qquad (i = 1, \ldots, n).$$

If $u \geqslant v$ and $u \neq v$, we write $u > v$. Denoting the ith row of a matrix M by $\rho(M; i)$ and the jth column of M by $\kappa(M; j)$, we may write

$$\rho(M; i) = ((M; i, 1), \ldots, (M; i, n))$$

and

$$\kappa(M; j) = ((M; 1, j), \ldots, (M; n, j)).$$

If $W = PV$, $P \in S_n$, $V \in F(S_n)$, then

$$\rho(M(W); i) = \sum \rho(M(v); k),$$

where the sum extends over all k such that $P^{-1}(i) \leqslant P^{-1}(k)$. Since the elements of $M(V)$ are non-negative and since each row of $M(V)$ has at least one positive element, the rows of $M(W)$ satisfy

(3) $\qquad \rho(M(W); P(1)) > \rho(M(W); P(2)) > \ldots > \rho(M(W); P(n)).$

Similarly, the columns are seen to satisfy

(4) $\quad \kappa(M(W); P(n)) > \kappa(M(W); P(n-1)) > \ldots > \kappa(M(W); P(1)).$

THEOREM 1. *The mapping that associates with each $W \in F(S_n)$ the matrix $M(W)$ is a semigroup isomorphism from the free monoid $F(S_n)$ into the unimodular group $\mathfrak{M}_n$.*

Proof. From (2) above, it follows that the mapping is a homomorphism. To show that the mapping is one–one, we argue by induction on the length of the word W. Since the only word of length 0 is the empty word, it is vacuously true that distinct words of length not exceeding 0 have distinct matrices associated with them. Now let $k \geqslant 1$ and assume that distinct words of length not exceeding $k - 1$ have distinct matrices associated with them. Let $W = P * V$ be a word of length k and let U be a word of length not exceeding k such that $M(U) = M(W)$. Since $M(W) \neq I_n$, U has the form $Q * Y$, where $Q \in S_n$ and $Y \in F(S_n)$, and we may write

$$M(U) = M(Q)M(Y) = M(P)M(V) = M(W).$$

The rows of $M(U)$ are the same as those of $M(W)$, and using (3) above, we obtain $Q = P$. It follows that $M(Y) = M(V)$, and by our induction hypothesis $Y = V$. We now have

$$U = Q * Y = P * V = W,$$

and we may conclude that distinct words of length not exceeding k have distinct matrices associated with them.

THEOREM 2. *The matrices $M(W)$, $W \in F(S_n)$, generate the unimodular group $\mathfrak{M}_n^+$.*

Proof. It is known **(2, 6)** that $\mathfrak{M}_n^+$ is generated by the transvections $V(h, k)$ $(h = 1, \ldots, n;\ k = 1, \ldots, n;\ h \neq k)$. These are defined by putting $V(h, k) = 1$ if $i = j$ or if $i = h$ and $j = k$, and $V(h, k) = 0$ otherwise. For $n = 2$, the transvections are just the matrices $M(P)$, $P \in S_2$. For $n > 2$ it will be sufficient to express a single transvection, say $V(1, 2)$, in terms of the $M(W)$ and their inverses; by symmetry, every transvection $V(h, k)$ has a similar expression. If r, s, t are the numbers 1, 2, 3 in some order, let

$$P(r, s, t) \in S_n$$

be defined as follows:

$$\begin{aligned} P(r, s, t)(i) &= n + 1 - i \qquad \text{for } i = 1, \ldots, n - 3; \\ P(r, s, t)(n - 2) &= r; \\ P(r, s, t)(n - 1) &= s; \\ P(r, s, t)(n) &= t. \end{aligned}$$

Then a calculation shows that

$$M(P(3, 1, 2))[M(P(2, 3, 1))]^{-1}M(P(1, 2, 3))[M(P(1, 3, 2))]^{-1} = V(1, 2).$$

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Remark. Appel and Djorup (1) have recently proved that a group generated by a free subsemigroup need not be free. Our Theorems 1 and 2 provide another demonstration of this fact.

2. The n-dimensional analogue of the Fibonacci sequence.

THEOREM 3. *The n elements of each column $\kappa(M(W);j)$ $(j = 1, \ldots, n)$ of the matrix $M(W)$, $W \in F(S_n)$ are relatively prime non-negative integers. Conversely, every n-tuple of relatively prime non-negative integers occurs as a column of a matrix $M(W)$, $W \in F(S_n)$. Similar statements hold for rows.*

Proof. Since each $M(W)$ is a product of a finite number of the $M(P)$, $P \in S_n$, the elements of $M(W)$ are non-negative integers. Since $\det M(W) = 1$, the elements of any column are relatively prime.

To prove the converse, we proceed by induction on the greatest integer h in the given n-tuple of relatively prime non-negative integers. If $h = 1$, then the given n-tuple occurs as a column of some $M(P)$, $P \in S_n$. Now let $h \geqslant 1$ and suppose that every n-tuple of relatively prime non-negative integers whose greatest integer does not exceed h occurs as a column of a matrix $M(W)$, $W \in F(S_n)$, and let $(c(1), \ldots, c(n))$ be an n-tuple of relatively prime non-negative integers whose greatest integer is $h + 1$. Let $P \in S_n$ be such that $c(P(1)) \geqslant \ldots \geqslant c(P(n))$. Then $c(P(1)) = h + 1$ and the last non-zero integer in the list $(c(P(1)), \ldots, c(P(n)))$ does not exceed h. Thus the n integers

$$c(P(1)) - c(P(2)), \ldots, c(P(n-1)) - c(P(n)), c(P(n))$$

are relatively prime non-negative integers, and the greatest of them does not exceed h. By our induction assumption, there exists $V \in F(S_n)$ such that these integers are the elements of a column of $M(V)$. Then $(c(P(1)), \ldots, c(P(n)))$ is a column of the matrix $M(E)M(V) = M(E * V)$.

If $W \in F(S_n)$, $W = Q_1 * Q_2 * \ldots * Q_r$, $Q_j \in S_n$, define $PW \in F(S_n)$ by $PW = (PQ_1) * (PQ_2) * \ldots * (PQ_r)$. One then sees that

$$(M(PW); P(i), P(j)) = (M(W); i, j).$$

Therefore $(c(1), \ldots, c(n))$ is a column of the matrix $M(P(E * V))$.

It remains to be shown that similar statements hold for rows. Since $M(PT)$ is the transpose of $M(P)$, if $W = P_1 * \ldots * P_r$ and $Z = (P_r T) * \ldots * (P_1 T)$, then $M(Z)$ is the transpose of $M(W)$, and an n-tuple occurs as a row of $M(Z)$ if and only if it occurs as a column of $M(W)$. This completes the proof of the theorem.

DEFINITION. *An n-tuple of relatively prime non-negative integers is called a primitive n-tuple. By the depth of a primitive n-tuple we mean the length of the shortest word $W \in F(S_n)$ such that the matrix $M(W)$ contains the n-tuple as a column. An n-tuple v is called a maximal primitive n-tuple of depth d if v is primitive and of depth d and there is no primitive n-tuple u of depth d such that $u > v$.*

DEFINITION. *An n-tuple* $v = (v(1), \ldots, v(n))$ *is called positive if all of its components* $v(i)$ *are positive, descending if* $v(1) \geqslant \ldots \geqslant v(n)$, *and ascending if* $v(1) \leqslant \ldots \leqslant v(n)$.

THEOREM 4. *If* $v = (v(1), \ldots, v(n))$ *is a maximal primitive n-tuple of depth d and* $P \in S_n$, *then* $vP = (v(P(1)), \ldots, v(P(n)))$ *is also a maximal primitive n-tuple of depth d.*

For each d ($d = 0, 1, 2, \ldots$) *there is exactly one descending primitive n-tuple that is maximal of depth d. Denoting it by* $\phi_{n,d}$, *we have*

$$\phi_{n,0} = \kappa(M(E); 1), \qquad \phi_{n,1} = \kappa(M(E * T); n),$$
$$\phi_{n,2} = \kappa(M(E * T * E); 1), \qquad \phi_{n,3} = \kappa(M(E * T * E * T); n), \ldots .$$

Letting Q_n *denote the* $n \times n$ *matrix such that* $(Q_n; i, j) = 1$ *for* $i + j \leqslant n + 1$ *and* $(Q_n; i, j) = 0$ *for* $i + j > n + 1$, *we have* $Q^2 = M(E)M(T)$ *and*

$$\phi_{n,d+1} = Q_n \phi_{n,d}.$$

Proof. To prove the first assertion, we simply observe that if $u > vP$, then $uP^{-1} > v$.

The proof of the second assertion is by induction and is based on the following observations. (1) If u is a primitive n-tuple of depth $d + 1$, then there exist a primitive n-tuple v of depth d and $P \in S_n$ such that $u = M(P)v$. (2) If $u > v$, then for every $P \in S_n$ we have $M(P)u > M(P)v$, since all elements of $M(P)$ are 0's or 1's and one of the rows of $M(P)$ consists entirely of 1's. Therefore $M(P)v$ is not maximal of depth $d + 1$ unless v is maximal of depth d. (3) If v is positive, then $M(P)v$ is positive and no two of its components are equal. (4) There is only one maximal primitive n-tuple of depth 1 and it is positive (in fact, all of its components are 1's). (5) As a consequence of the preceding observations, any maximal primitive n-tuple of depth $d \geqslant 2$ is positive and has no two of its components equal.

For depth $d = 0$ the maximal descending primitive n-tuple

$$v = (v(1), \ldots, v(n))$$

is given by $v(1) = 1$, $v(2) = \ldots = v(n) = 0$. For depth $d = 1$ it is given by $v(1) = \ldots = v(n) = 1$. Now suppose that $d \geqslant 1$ and that for each depth not exceeding d there is exactly one maximal descending primitive n-tuple of that depth. Let v be a maximal descending primitive n-tuple of depth $d + 1$. Then we must have $v = M(P)u$ for some $P \in S_n$, where u is maximal of depth d. Then u is positive and in order that v be descending we must have $P = E$, $v = M(E)u$. Now it is easily seen that in order for v to be maximal, u must be ascending. By the induction hypothesis and the first assertion of the theorem, there is exactly one maximal ascending primitive n-tuple of depth d. Thus u is uniquely determined. Hence v is also uniquely determined and the proof of the second assertion is complete. In what follows, we shall frequently use ϕ_d instead of $\phi_{n,d}$ to denote the maximal descending primitive n-tuple of depth d. (We shall also write Q for Q_n, etc.).

If $\phi_d = (\phi_d(1), \ldots, \phi_d(n)) = \kappa(M(W); j)$, then the maximal ascending primitive n-tuple of depth d, obtained by reversing the components of ϕ_d, is equal to $\kappa(M(TW); T(j)) = \kappa(M(TW); n + 1 - j)$. Therefore

$$\phi_{d+1} = M(E)\kappa(M(TW); n + 1 - j) = \kappa(M(E * (TW)); n + 1 - j),$$

and proceeding step-by-step, we obtain

$$\phi_0 = \kappa(M(E); 1),$$

$$\phi_1 = \kappa(M(E * (TE)); n + 1 - 1) = \kappa(M(E * T); n),$$

$$\phi_2 = \kappa(M(E * (T(E * T)); n + 1 - n) = \kappa(M(E * T * E); 1),$$

$$\phi_3 = \kappa(M(E * (T(E * T * E))); n + 1 - 1) = \kappa(M(E * T * E * T); n),$$

and so on.

Since $\phi_{d+1} = M(E)\kappa(M(TW); n + 1 - j)$, the ith component of ϕ_{d+1} satisfies

$$\begin{aligned}
\phi_{d+1}(i) &= \sum_{k=1}^{n} (M(E); i, k)(M(TW); k, n + 1 - j) \\
&= \sum_{k=i}^{n} (M(TW); k, n + 1 - j) \\
&= \sum_{m=1}^{n+1-i} (M(TW); n + 1 - m, n + 1 - j) \\
&= \sum_{m=1}^{n+1-i} (M(TW); T(m), T(j)) = \sum_{m=1}^{n+1-i} (M(W); m, j) \\
&= \sum_{m=1}^{n+1-i} \phi_d(m) = \sum_{k=1}^{n} (Q; i, k)\phi_d(k).
\end{aligned}$$

Thus $\phi_{d+1} = Q\phi_d$.

Finally, we note that

$$\begin{aligned}
(Q^2; i, j) &= \sum_{k=1}^{n} (Q; i, k)(Q; k, j) \\
&= \sum_{k=1}^{\min(n+1-i, n+1-j)} 1 \\
&= \min(n + 1 - i, n + 1 - j),
\end{aligned}$$

while

$$\begin{aligned}
(M(E * T); i, j) &= \sum_{k=1}^{n} (M(E); i, k)(M(T); k, j) \\
&= \sum_{k=\max(i,j)}^{n} 1, \\
&= n + 1 - \max(i, j),
\end{aligned}$$

so that $Q^2 = M(E)M(T)$. The proof of Theorem 4 is now complete.

Table I lists the values of $\phi_{n,d}$ for $n = 1, 2, 3, 4$, and $d \leqslant 6$.

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TABLE I

	$d = 0$	$d = 1$	$d = 2$	$d = 3$	$d = 4$	$d = 5$	$d = 6$
$n = 1$	(1)	(1)	(1)	(1)	(1)	(1)	(1)
$n = 2$	$\begin{pmatrix}1\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\end{pmatrix}$	$\begin{pmatrix}2\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\end{pmatrix}$	$\begin{pmatrix}5\\3\end{pmatrix}$	$\begin{pmatrix}8\\5\end{pmatrix}$	$\begin{pmatrix}13\\8\end{pmatrix}$
$n = 3$	$\begin{pmatrix}1\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\end{pmatrix}$	$\begin{pmatrix}3\\2\\1\end{pmatrix}$	$\begin{pmatrix}6\\5\\3\end{pmatrix}$	$\begin{pmatrix}14\\11\\6\end{pmatrix}$	$\begin{pmatrix}31\\25\\14\end{pmatrix}$	$\begin{pmatrix}70\\56\\31\end{pmatrix}$
$n = 4$	$\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$	$\begin{pmatrix}1\\1\\1\\1\end{pmatrix}$	$\begin{pmatrix}4\\3\\2\\1\end{pmatrix}$	$\begin{pmatrix}10\\9\\7\\4\end{pmatrix}$	$\begin{pmatrix}30\\26\\19\\10\end{pmatrix}$	$\begin{pmatrix}85\\75\\56\\30\end{pmatrix}$	$\begin{pmatrix}246\\216\\160\\85\end{pmatrix}$

If $n = 2$, the components of $\phi_{n,d}$ are just the numbers in the Fibonacci sequence. Indeed, if the problem concerning the breeding of rabbits considered by Leonardo of Pisa in 1202 is stated so that it asks for the successive distributions of the total population of rabbit pairs into adult pairs and newborn pairs, then the answer to the problem is provided by the sequence $\phi_{2,d}$. The reader will not find it difficult to formulate analogous population distribution problems that are answered by the sequences $\phi_{n,d}$ with $n = 3, 4, \ldots$.

Since the days of Luca Pacioli and Kepler, considerable attention has been paid to the limit of the ratio of successive numbers of the Fibonacci sequence, which is $\frac{1}{2}(1 + \sqrt{5})$ and which may be described as the ratio of the diagonal of a regular pentagon to its side. In our notation, this is

$$\lim_{d\to\infty} (\phi_d(1)/\phi_d(2)),$$

with $n = 2$. That analogous descriptions of the limiting ratios are possible in all of the higher-dimensional cases $n = 3, 4, 5, \ldots$ as well is the substance of the following theorem.

THEOREM 5. *Let n be a positive integer. Let $(\phi_\infty(1), \ldots, \phi_\infty(n))$ be a descending n-tuple of positive real numbers that are proportional to the n different lengths of diagonals of a regular $(2n + 1)$-gon. Then the components of the n-tuples ϕ_d have limiting ratios given by*

$$\lim_{d\to\infty} (\phi_d(i)/\phi_d(j)) = \phi_\infty(i)/\phi_\infty(j).$$

Proof. We have $\phi_d = Q^d\phi_0$, where Q is the matrix described in Theorem 4, and we begin by finding the characteristic polynomial of that matrix. Letting $D_n(\lambda) = \det (Q_n - \lambda I_n)$, we find by direct calculation that

$$\begin{aligned} D_1(\lambda) &= 1 - \lambda, \\ D_2(\lambda) &= -1 - \lambda + \lambda^2, \\ D_3(\lambda) &= -1 + \lambda + 2\lambda^2 - \lambda^3, \\ D_4(\lambda) &= 1 + \lambda - 3\lambda^2 - 2\lambda^3 + \lambda^4. \end{aligned}$$

For $n \geqslant 3$ we can obtain a recursion formula for D_n. We begin by expressing

$$D_n(\lambda) = \begin{vmatrix} 1-\lambda & 1 & 1 & \dots & 1 & 1 & 1 \\ 1 & 1-\lambda & 1 & \dots & 1 & 1 & 0 \\ 1 & 1 & 1-\lambda & \dots & 1 & 0 & 0 \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot & \cdot \\ 1 & 1 & 1 & \dots & -\lambda & 0 & 0 \\ 1 & 1 & 0 & \dots & 0 & -\lambda & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & -\lambda \end{vmatrix}$$

as the sum of two determinants, whose first rows are $1, 1, 1, \dots, 1, 1, 1$ and $-\lambda, 0, 0, \dots, 0, 0, 0$ respectively, and whose other rows are the same as those of $D_n(\lambda)$. In the first of these determinants, we subtract the first column from each of the other columns. We then expand each of these determinants by the first row, obtaining

$$D_n(\lambda) = (-1)^{n-1} \begin{vmatrix} \lambda & 0 & \dots & 0 & 0 & 1 \\ 0 & \lambda & \dots & 0 & 1 & 1 \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ 0 & 0 & \dots & 1+\lambda & 1 & 1 \\ 0 & 1 & \dots & 1 & 1+\lambda & 1 \\ 1 & 1 & \dots & 1 & 1 & 1+\lambda \end{vmatrix}$$

$$-\lambda \begin{vmatrix} 1-\lambda & 1 & \dots & 1 & 1 & 0 \\ 1 & 1-\lambda & \dots & 1 & 0 & 0 \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot \\ 1 & 1 & \dots & -\lambda & 0 & 0 \\ 1 & 0 & \dots & 0 & -\lambda & 0 \\ 0 & 0 & \dots & 0 & 0 & -\lambda \end{vmatrix}.$$

Reflecting the first determinant with respect to its secondary diagonal and expanding the second determinant by its last row, we obtain

$$D_n(\lambda) = (-1)^{n-1}D_{n-1}(-\lambda) + \lambda^2 D_{n-2}(\lambda).$$

Taking $D_0(\lambda) = 1$, we make this recursion formula hold for $n \geqslant 2$.

We introduce

$$E_n(\lambda) = (-1)^{n(n+1)/2} D_n((-1)^n\lambda),$$

and the recursion formula takes the simpler form

$$E_n(\lambda) + E_{n-1}(\lambda) + \lambda^2 E_{n-2}(\lambda) = 0.$$

The solution of this recursion formula is known **(3; 7)**. It has the form

$$E_n(\lambda) = c_1 p^n + c_2 q^n,$$

where $p + q = -1$, $pq = \lambda^2$,

$$c_1 = (E_1 - q)/(p - q), \quad \text{and} \quad c_2 = (p - E_1)/(p - q).$$

From this one obtains

$$(p - q)E_n = -\lambda^2(p^{n-1} - q^{n-1}) + E_1(p^n - q^n). \tag{5}$$

The solution of the equations $p + q = -1$, $pq = \lambda^2$ is given by $p = \lambda e^{iy}$, $q = \lambda e^{-iy}$, where $2 \cos y = -1/\lambda$. We may now write

$$(\lambda \sin y)E_n = -\lambda^{n+1} \sin(n - 1)y + E_1(\lambda^n \sin ny).$$

Using $E_1/\lambda = (-1 - \lambda)/\lambda = 2 \cos y - 1$, we readily obtain the equation

$$E_n/\lambda^n = \frac{\sin(n + 1)y - \sin ny}{\sin y} = \frac{\cos(n + \frac{1}{2})y}{\cos \frac{1}{2}y}.$$

By shifting the angle, we can relate this to the Dirichlet kernel. In fact, we have

$$E_n/\lambda^n = \frac{(-1)^n \sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x},$$

where $y = x + \pi$ and $2 \cos x = 1/\lambda$.

Now let us introduce polynomials $Z_n(u)$ $(n = 0, 1, 2, \ldots)$ defined so that

$$Z_n(2 \cos x) = \frac{\sin(n + \frac{1}{2})x}{\sin \frac{1}{2}x}.$$

From the equation $E_n(\lambda) = (-1)^n\lambda^n Z_n(1/\lambda)$ we obtain

$$D_n(\lambda) = (-1)^{n(n+1/2)}\lambda^n Z_n((-1)^n\lambda^{-1}).$$

It now follows that the roots of $D_n(\lambda)$ are real and distinct. They are the numbers

$$\frac{(-1)^n}{2\cos(2k\pi/(2n + 1))} \qquad (k = 1, \ldots, n).$$

These roots may also be written

$$\frac{(-1)^{n+1}}{2\cos((2m - 1)\pi/(2n + 1))} \qquad (m = 1, \ldots, n).$$

The characteristic root of the matrix Q_n whose absolute value is greatest is given by

$$\lambda_1 = \frac{(-1)^{2m}}{2\cos(2m\pi/(4m + 1))} \qquad \text{if } n = 2m \text{ is even}$$

and by

$$\lambda_1 = \frac{(-1)^{2m-1}}{2\cos(2m\pi/(4m - 1))}, \qquad \text{if } n = 2m + 1 \text{ is odd}.$$

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In either case λ_1 is positive and equal to $1/2\cos(n\pi/(2n+1))$, which is just the ratio of the longest diagonal to the side in a regular $(2n+1)$-gon.

From any vertex of a regular $(2n+1)$-gon there emanate $2n$ diagonals; here we are counting the sides among the diagonals. Two of these diagonals have length $\phi_\infty(1)$ and they are the equal sides of an isosceles triangle whose base has length $\phi_\infty(n)$ and whose base angles are equal to $n\pi/(2n+1)$. It follows that

$$\phi_\infty(n) = 2\phi_\infty(1)\cos(n\pi/(2n+1)),$$

or, equivalently, that $\phi_\infty(1) = \lambda_1\phi_\infty(n)$.

Parallel to each diagonal of length $\phi_\infty(k)$ $(k = 1, \ldots, n-1)$ there is a diagonal of length $\phi_\infty(k+1)$. These two diagonals are the bases of an isosceles trapezoid whose non-parallel sides have length $\phi_\infty(n+1-k)$ and whose base angles are equal to $n\pi/(2n+1)$. It follows that

$$\phi_\infty(k) - \phi_\infty(k+1) = 2\phi_\infty(n+1-k)\cos(n\pi/(2n+1)),$$

or, equivalently, that

$$\phi_\infty(n+1-k) = \lambda_1(\phi_\infty(k) - \phi_\infty(k+1)) \qquad (k = 1, \ldots, n-1).$$

Summing, we now have

$$\sum_{j=1}^{n+1-k} \phi_\infty(j) = \lambda_1\, \phi_\infty(k) \qquad (k = 1, \ldots, n).$$

Thus it is established that $(\phi_\infty(1), \ldots, \phi_\infty(n))$ is a characteristic vector of the matrix Q_n associated with the characteristic value λ_1.

Calling this vector ϕ_∞, we now have only to show that the limiting ratios of the n-tuple ϕ_d as $d \to \infty$ are the same as the ratios of the corresponding components of ϕ_∞. Here we make use of an argument given by Birkhoff **(4)**.

Since the matrix Q_n is real and symmetric, we may take characteristic vectors associated with its n distinct characteristic roots to form an orthogonal basis of our n-dimensional vector space. We may take ϕ_∞ as one of the vectors of this basis. We now have $\phi_0 = A\phi_\infty + \psi$, where A is positive and ψ is orthogonal to ϕ_∞. Now, since all characteristic roots $\lambda \neq \lambda_1$ have absolute values less than that of λ_1, we have

$$(\lambda_1)^{-d}\phi_d = (\lambda_1)^{-d}Q^d\phi_0 = (\lambda_1)^{-d}Q^d(A\phi_\infty + \psi) = A\phi_\infty + o(1) \to A\phi_\infty,$$

as $d \to \infty$. The proof of the theorem is now complete.

Many further properties of the polynomials Z_n, and consequently also of the polynomials D_n, follow from the equation

$$Z_n(2u) = 2^{2n}(n!)^2((2n)!)^{-1}P_n^{(\frac{1}{2},-\frac{1}{2})}(u),$$

which relates the Z_n to the Jacobi polynomials $P_n^{(\frac{1}{2},-\frac{1}{2})}$. These polynomials are

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specifically mentioned in Szegö **(8)**. In particular, the polynomials Z_n satisfy the differential equation

$$d[(2-u)^{3/2}(2+u)^{1/2}Z'_n(u)]/du + n(n+1)(2-u)^{1/2}(2+u)^{-1/2}Z_n(u) = 0$$

and are orthogonal on the interval $[-2, 2]$ with respect to the weight function $w(u) = (2-u)^{1/2}(2+u)^{-1/2}$.

THEOREM 6. *Let*

$$c(r) = c_n(r) = 2\cos(r\pi/(2n+1)).$$

Then the characteristic vector of the matrix Q_n which is associated with the characteristic root $(-1)^{n+1}/c(2k-1)$ is given by

$$\begin{bmatrix} x(1) \\ x(2) \\ x(3) \\ \cdot \\ \cdot \\ \cdot \\ x(n) \end{bmatrix} = \begin{bmatrix} -\{c(2(2k-1)) + c(4(2k-1)) + c(6(2k-1)) + \ldots + c(2n(2k-1))\} \\ \{c(4(2k-1)) + c(6(2k-1)) + \ldots + c(2n(2k-1))\} \\ -\{c(6(2k-1)) + \ldots + c(2n(2k-1))\} \\ \cdot \\ \cdot \\ \cdot \\ (-1)^n c(2n(2k-1)) \end{bmatrix}.$$

If the permutations $T, G \in S_n$ are defined by

$$T(k) = n+1-k, \qquad (1 \leqslant k \leqslant n),$$

$$G(k) = \begin{cases} n+2-2k, & (2k < n+2), \\ 2k-n-1, & (2k \geqslant n+2), \end{cases}$$

then we have, for $r = 1, \ldots, n$,

$$x(G(r))/x(r) = c(T(r)(2G(k)-1)).$$

Proof. Let $K = 2k-1$. To establish the theorem we must show that

$$\sum_{j=1}^{m} x(j) = ((-1)^{n+1}/c(K))x(n+1-m) \qquad (m = 1, \ldots, n).$$

Since, by definition,

$$x(j) = (-1)^j \sum_{i=j}^{n} c(2iK) \qquad (j = 1, \ldots, n),$$

this can be written

$$c(K)\sum_{j=1}^{m}(-1)^j \sum_{i=j}^{n} c(2iK) = (-1)^{n+1}(-1)^{n+1-m}\sum_{i=n+1-m}^{n} c(2iK),$$

or, equivalently,

$$\sum_{j=1}^{m}\sum_{i=j}^{n}(-1)^{m-j}c(K)c(2iK) = \sum_{i=n+1-m}^{n} c(2iK).$$

Changing the order of summation, we obtain in the left member

$$\sum_{i=1}^{m}\sum_{j=1}^{i}(-1)^{m-j}c(K)c(2iK) + \sum_{i=m+1}^{n}\sum_{j=1}^{m}(-1)^{m-j}c(K)c(2iK).$$

Assume first that m is even. Then the second term vanishes. The first term becomes

$$(-1)^{m+1}[c(K)c(2K) + c(K)c(6K) + \ldots + c(K)c((2m-2)K)],$$

and, using the fact that $c(a)c(b) = c(a-b) + c(a+b)$, we may write it in the form

$$(-1)^{m+1}[c(K) + c(3K) + c(5K) + c(7K) + \ldots + c((2m-1)K)].$$

In the right member we have

$$\sum_{i=n+1-m}^{n} c(2iK).$$

Since $c(a) = -c[(2n+1)K - a]$, this may be written

$$-\sum_{i=n+1-m}^{n} c((2n+1-2i)K).$$

If we let $s = 2n+1-2i$, then the right member becomes

$$-\sum_{\substack{s \text{ odd} \\ 1\leqslant s\leqslant 2m-1}} c(sK).$$

Since $(-1)^m = 1$ when m is even, the two members now agree.

Now let m be odd. Then the left member may be written

$$[c(K)c(2K) + c(K)c(6K) + \ldots + c(K)c(2mK)]$$
$$+ [c(K)c((2m+2)K) + c(K)c((2m+4)K + \ldots + c(K)c(2nK)].$$

Again using the addition formula, we put this in the form

$$\sum_{\substack{s \text{ odd} \\ 1\leqslant s\leqslant 2n+1}} c(sK) + \sum_{\substack{s \text{ odd} \\ 2m+1\leqslant s\leqslant 2n-1}} c(sK).$$

As in the preceding case, the right member may be put into the form

$$-\sum_{\substack{s \text{ odd} \\ 1\leqslant s\leqslant 2m-1}} c(sK).$$

Since

$$Z_n(2\cos x) = \sin(n+\tfrac{1}{2})x/\sin\tfrac{1}{2}x = 1 + 2\sum_{m=1}^{n}\cos mx,$$

we have

$$Z_n(2\cos(2b\pi/(2n+1))) = 0 \qquad \text{if } b \not\equiv 0 \pmod{2n+1}.$$

It follows that

$$\sum_{m=1}^{n} c(2bm) = -1 \qquad \text{if } b \not\equiv 0 \pmod{2n+1}.$$

If $b = K$, we may write

$$\sum_{m=1}^{n} c(2mK) = -\sum_{m=1}^{n} c((2n+1-2m)K) = -1,$$

and from this we obtain

$$\sum_{\substack{s \text{ odd} \\ 1 \leqslant s \leqslant 2n-1}} c(sK) = 1 \qquad (k = 1, \ldots, n).$$

Using this result together with the fact that $c((2n+1)K) = -2$, we see that the left and right members are equal, and the proof of the first assertion is complete.

To prove the second assertion we must show that for each r $(r = 1, \ldots, n)$

$$(-1)^{G(r)} \sum_{i=G(r)}^{n} c(2iK) = c(T(r)(2G(k)-1))(-1)^r \sum_{i=r}^{n} c(2iK).$$

We begin by observing that

$$2G(k) - 1 = \pm(2n - 4k + 3) \qquad (k = 1, \ldots, n).$$

Since $c(-a) = c(a)$, this allows us to write

$$c(T(r)(2G(k)-1)) = c((n+1-r)(2n-4k+3)).$$

Now, since

$$(n+1-r)(2n-4k+3) = (n-r)(2n+1) + (1-k)(4n+2) + (2r-1)(2k-1),$$

we may replace $c((n+1-r)(2n-4k+3))$ by $(-1)^{n-r}\, c((2r-1)K)$, and our assertion becomes

$$(-1)^{G(r)} \sum_{i=G(r)}^{n} c(2iK) = (-1)^n c((2r-1)K) \sum_{i=r}^{n} c(2iK).$$

In the case $n + 2 > 2r$, we have $G(r) = n + 2 - 2r$ and the assertion reduces to

$$\sum_{i=n+2-2r}^{n} c(2iK) = \sum_{i=r}^{n} c((2r-1)K)c(2iK).$$

Using the addition formula, we may put the right member into the form

$$\sum_{i=r}^{n} c((2r+2i-1)K) + \sum_{i=r}^{n} c((2i-2r+1)K),$$

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and if we split the first term into three parts, the right member takes the form

$$\sum_{\substack{s \text{ odd} \\ 4r-1 \leqslant s \leqslant 2n-1}} c(sK) + c((2n+1)K) + \sum_{\substack{s \text{ odd} \\ 2n+3 \leqslant s \leqslant 2n+2r-1}} c(sK) + \sum_{\substack{s \text{ odd} \\ 1 \leqslant s \leqslant 2n-2r+1}} c(sK).$$

Using the fact that $c(2n + 1 + a) = c(2n + 1 - a)$, we may rewrite the third term as

$$\sum_{\substack{s \text{ odd} \\ 2n-2r+3 \leqslant s \leqslant 2n-1}} c(sK),$$

and, since $c((2n + 1)K) = -2$ and

$$\sum_{\substack{s \text{ odd} \\ 1 \leqslant s \leqslant 2n-1}} c(sK) = 1,$$

the right member reduces to

$$\sum_{\substack{s \text{ odd} \\ 4r-1 \leqslant s \leqslant 2n-1}} c(sK) - 1.$$

Using the fact that $c(a) = -c((2n + 1)K - a)$, we may replace the left member by

$$-\sum_{i=n+2-2r}^{n} c((2n + 1 - 2i)K),$$

and, after changing the index, this becomes

$$-\sum_{\substack{s \text{ odd} \\ 1 \leqslant s \leqslant 4r-3}} c(sK).$$

The desired result now follows and our assertion is proved if $n + 2 > 2r$.

In the case $n + 2 \leqslant 2r$, we have $G(r) = 2r - n - 1$ and the assertion to be proved becomes

$$(-1)^{2r-n-1} \sum_{i=2r-n-1}^{n} c(2iK) = (-1)^n c((2r - 1)K) \sum_{i=r}^{n} c(2iK).$$

This may be rewritten

$$\sum_{i=2r-n-1}^{n} c((2n + 1 - 2i)K) = \sum_{i=r}^{n} c((2r + 2i - 1)K) + \sum_{i=r}^{n} c((2i - 2r + 1)K).$$

Splitting the left member and changing indices, we put this in the form

$$\sum_{\substack{s \text{ odd} \\ 1 \leqslant s \leqslant 2n-2r+1}} c(sK) + \sum_{\substack{s \text{ odd} \\ 2n-2r+3 \leqslant s \leqslant 4n-4r+3}} c(sK)$$
$$= \sum_{\substack{s \text{ odd} \\ 4r-1 \leqslant s \leqslant 2n+2r-1}} c(sK) + \sum_{\substack{s \text{ odd} \\ 1 \leqslant s \leqslant 2n-2r+1}} c(sK).$$

Now the desired result follows from the fact that $c(4n + 2 - a) = c(a)$. This completes the proof of Theorem 6.

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Remark. The characteristic root $(-1)^{n+1}/c(2k-1)$ will be λ_1, the characteristic root whose absolute value is maximum if we take $G(k) = 1$. Looking at the formula

$$x(G(r))/x(r) = c(T(r)(2G(k) - 1))$$

for this value of k, we see that as r successively assumes the values $1, 2, \ldots, n-1, n$, the right member successively assumes the values $c(n)$, $c(n-1)$, $\ldots$, $c(2)$, $c(1)$. As r varies monotonically, the right member varies monotonically, but $G(r)$ does not vary in such a simple fashion. In fact, the permutation G is associated with a certain folding, which we may describe in geometric-intuitive terms as follows. Cut out of paper an isosceles triangle $P_0 P_n P'_n$ with equal sides $P_0 P_n$ and $P_0 P'_n$ and with the angle $P'_n P_0 P_n$ equal to $\pi/(2n+1)$. On the side $P_0 P_n$ mark points $P_1, P_2, \ldots, P_{n-1}$ satisfying the conditions that

(1) the distances $P_0 P_1$, $P_1 P_2$, $P_2 P_3$, $\ldots$, $P_{n-1} P_n$ are proportional to the numbers $\phi_\infty(1)$, $\phi_\infty(2)$, $\phi_\infty(3)$, $\ldots$, $\phi_\infty(n)$,

(2) $$P_0 P_1 < P_0 P_2 < P_0 P_3 < \ldots < P_0 P_n.$$

In similar fashion mark points P_1, P_2, $\ldots$, P'_{n-1} on the side $P_0 P'_n$. Fold the paper triangle (accordion style) backward on the edge $P_1 P'_1$, forward on the edge $P_2 P'_2$, backward on the edge $P_3 P'_3$, etc. If the resulting pleated object is flattened down into a plane figure, then the marked points all lie on a circle and are, in fact, the vertices of a regular $(2n+1)$-gon. The cyclic arrangement of the marked points on the circle is given by

$$P_0, P_2, P_4, P_6, \ldots, P_5, P_3, P_1, P'_1, \ldots, P'_4, P'_2.$$

One readily sees that the numbers $2, 4, 6, \ldots, 5, 3, 1$, the subscripts on the P's other than P_0, are just the numbers

$$T^{-1}GT(1), T^{-1}GT(2), T^{-1}GT(3), \ldots, T^{-1}GT(n-2), \\ T^{-1}GT(n-1), T^{-1}GT(n),$$

and in this way we gain some understanding of the role of the permutation G for the particular k chosen. Presumably, for other values of k similar pictures involving folding can be constructed.

Theorems 5 and 6 lead one to suspect that in higher dimensions $n = 3, 4, 5$, $\ldots$, the sequence $\phi_0, \phi_1, \phi_2, \ldots$ plays a role analogous to that played by the Fibonacci sequence in dimension $n = 2$. This suspicion is strengthened by the fact that the identities of Lucas and of Simson for the Fibonacci numbers have simple generalizations to higher dimensions that are expressed in terms of the sequence $\phi_0, \phi_1, \phi_2, \ldots$. If the numbers $f_0, f_1, f_2, \ldots$ of the Fibonacci sequence are defined by the recursion formula $f_0 = 0, f_1 = 1, f_k + f_{k+1} = f_{k+2}$, Lucas's identities read:

$$f_{2n} = f_{n-1} f_n + f_n f_{n+1}, \qquad f_{2n+1} = f_n^2 + f_{n+1}^2.$$

A slightly more general identity which is known is

$$f_m = f_{k+1} f_{m-k} + f_k f_{m-k-1}.$$

Simson's identity reads $f_{n-1} f_{n+1} - f_n^2 = (-1)^n$.

THEOREM 7 (*generalization of Lucas's identities*). *If m and n are non-negative integers, then $(\phi_m, \phi_k) = \phi_{m+k}(1)$, where (ϕ_m, ϕ_k) denotes the inner product $\sum_j \phi_m(j)\phi_k(j)$, $1 \leqslant j \leqslant n$.*

Proof. Since Q is a symmetric matrix, we have

$$(\phi_m, \phi_k) = (Q^m \phi_0, Q^k \phi_0) = (\phi_0, Q^{m+k}\phi_0) = (\phi_0, \phi_{m+k}) = \phi_{m+k}(1).$$

THEOREM 8 (*generalization of Simson's identity*). *The determinant of the $n \times n$ matrix whose columns (from left to right) are $\phi_k, \phi_{k+1}, \ldots, \phi_{k+n-1}$ is equal to $((-1)^{n(n-1)/2})^k$ if $n \not\equiv 3 \pmod 4$, and to $(-1)((-1)^{n(n-1)/2})^k$ if $n \equiv 3 \pmod 4$.*

Proof. Starting with the $n \times n$ matrix whose columns (from left to right) are $\phi_0, \phi_1, \ldots, \phi_{n-1}$ and applying elementary row and column transformations that do not alter the determinant, one readily obtains a matrix (a_{ij}) in which $a_{ij} = 1$ if j is odd and $i = (j+1)/2$, $a_{ij} = 1$ if j is even and $i = n + 1 - (j/2)$, and $a_{ij} = 0$ otherwise. One then easily calculates that the determinant of this matrix is 1 if $n \not\equiv 3 \pmod 4$ and -1 if $n \equiv 3 \pmod 4$. To obtain the matrix whose columns are $\phi_k, \phi_{k+1}, \ldots, \phi_{k+n-1}$ from the matrix whose columns are $\phi_0, \phi_1, \ldots, \phi_{n-1}$, one need only apply the matrix Q^k. The theorem then results from the observation that in dimension n the determinant of the matrix Q is $(-1)^{n(n-1)/2}$.

The next theorem gives a recursion formula satisfied by the n-tuples ϕ_d. For $n = 2$ this reduces to the familiar recursion formula for the Fibonacci numbers.

THEOREM 9. *The characteristic polynomial $D_n(\lambda)$ of the $n \times n$ matrix Q is given by*

$$D_n(\lambda) = \sum_{j=0}^{n} \binom{n - j + [\frac{1}{2}j]}{[\frac{1}{2}j]} (-1)^{(n-j)(n-j-1)/2} (-1)^j \lambda^j.$$

Consequently, the n-tuple ϕ_d satisfies the recursion formula

$$\sum_{j=0}^{n} \binom{n - j + [\frac{1}{2}j]}{[\frac{1}{2}j]} (-1)^{(n-j)(n-j-1)/2} (-1)^j \phi_{k+j} = 0$$

for $k = 0, 1, 2, \ldots$.

Proof. The formula for $D_n(\lambda)$ may be derived from equation (5) for $E_n(\lambda)$, used in proving Theorem 5, by expressing the quantities $(p^{n-1} - q^{n-1})/(p - q)$ and $(p^n - q^n)/(p - q)$ in terms of the elementary symmetric functions $p + q$ and pq, and the using $p + q = -1$ and $pq = \lambda^2$. Alternatively, one may verify that the formula for $D_n(\lambda)$ holds for $n = 0$ and $n = 1$, and that it satisfies the recursion formula for $D_n(\lambda)$ developed in the proof of Theorem 5.

Now, by the Cayley–Hamilton theorem, the matrix Q satisfies its own characteristic equation and we have $D_n(Q) = 0$. It follows that $D_n(Q)\phi_k = 0$ for $k = 0, 1, 2, \ldots$, and, using the fact that $Q^j\phi_k = \phi_{k+j}$, we obtain the recursion formula stated in the theorem.

From the existence of the recursion formula described in Theorem 9 we may deduce that in any dimension n, for each of the sequences $\phi_0(r), \phi_1(r), \ldots, \phi_d(r), \ldots$ $(r = 1, \ldots, n)$, there is a generating function that is a rational function whose denominator is a polynomial closely related to the polynomial $D_n(\lambda)$. In fact, we may write

$$\sum_{d=0}^{\infty} \phi_d(r)t^d = p_{n,r}(t)/t^n D_n(1/t),$$

where $p_{n,r}(t)$ is a polynomial of degree less than n, to be determined, and where $t^n D_n(1/t)$ may also be written in the form $(-1)^{n(n+1)/2} Z_n((-1)^n t)$. Careful inspection shows that the numerators $p_{n,r}(t)$ are given by the following rule. If $r = 1$, then

$$p_{n,r}(t) = (-1)^{(n-1)(n-2)/2} Z_{n-2}((-1)^n t).$$

if $2 \leqslant r \leqslant n$, then

$$p_{n,r}(t) = (-1)^{j(j-1)/2}(-1)^n t Z_j((-1)^n t),$$

where $j = n + 1 - 2r$ if $2 \leqslant r \leqslant [(n+1)/2]$ and $j = 2r - n - 2$ if

$$[(n+1)/2] < r \leqslant n.$$

Bearing in mind that the original source of the polynomials $Z_n(u)$ was the expansion of the Dirichlet kernel $\sin(n + \frac{1}{2})x/\sin \frac{1}{2}x$ in powers of $2\cos x$, we replace t by $(-1)^n 2\cos x$ and obtain expressions for the generating functions in the forms described in the following theorem.

THEOREM 10. *If $r = 1$, then*

$$\frac{(-1)^{(n-2)(n-3)/2} \sin((n-2) + \frac{1}{2})x}{(-1)^{n(n-1)/2} \sin(n + \frac{1}{2})x} = \sum_{d=0}^{\infty} \phi_d(1)((-1)^n 2\cos x)^d.$$

If $2 \leqslant r \leqslant n$, then

$$\frac{(-1)^{j(j-1)/2} \sin(j + \frac{1}{2})x}{(-1)^{n(n-1)/2} \sin(n + \frac{1}{2})x} = \sum_{d=0} \phi_{d+1}(r)((-1)^n 2\cos x)^d,$$

where

$$j = n + 1 - 2r \qquad \textit{if } 2 \leqslant r \leqslant [(n+1)/2]$$

and

$$j = 2r - n - 2 \qquad \textit{if } [(n+1)/2] < r \leqslant n.$$

In the familiar case $n = 2$, this yields a generating function for the Fibonacci sequence:

$$\sin(x/2)/[-\sin(5x/2)] = 1 + 1(2\cos x) + 2(2\cos x)^2 + 3(2\cos x)^3 + 5(2\cos x)^4 + \ldots,$$

the series converging for $2\pi/5 < x < 3\pi/5$.

Knowing the generating functions, we would not find it difficult to obtain formulas for the numbers $\phi_d(r)$ in terms of multinomial coefficients, which in dimension $n = 2$ would reduce to Lucas's formula expressing the Fibonacci numbers in terms of the binomial coefficients **(5)**. We shall not, however, develop these formulas here.

Since the denominator of each of the generating functions has the simple roots

$$t_k = (-1)^n 2\cos(2k\pi/(2n+1)) \qquad (k = 1, \ldots, n),$$

we can readily decompose the generating function into partial fractions and then expand each of them as a geometric series. It becomes clear that there exist constants $A_{r,1}, \ldots, A_{r,n}$ such that

$$\phi_d(r) = A_{r,1} t_1^d + A_{r,2} t_2^d + \ldots + A_{r,n} t_n^d$$

holds for all d. For $n = 2$ this procedure yields the familiar result concerning the Fibonacci numbers called Binet's formula (although it was known to Euler and Daniel Bernoulli):

$$f_k = (\tau^k - (-\tau)^{-k})/\sqrt{5}, \qquad \text{where } \tau = 2\cos(\pi/5).$$

Many other known identities concerning the Fibonacci numbers are ultimately based upon the identities considered in this paper; hence, they also can be extended to n dimensions.

Added in proof. Professor V. E. Hoggatt, Jr. of San Jose State College has recently called our attention to the 1963 Master's thesis of his student B. Junge, in which the matrix Q_n and its characteristic roots are studied and some divisibility properties of the polynomials $D_n(\lambda)$ are developed.

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