Complex Planetary Systems
Proceedings IAU Symposium No. 310, 2014
Z. Knežević & A. Lemaitre, eds.

doi:10.1017/S1743921314008047

Hill Stability in the Full 3-Body Problem

D. J. Scheeres

Abstract. Hill stability cannot be easily established in the classical 3-body problem with point masses, as sufficient energy for escape of one of the bodies can always be extracted from the gravitational potential energy. For the finite density, so-called Full 3-body problem the lower limits on the gravitational potential energy ensure that Hill stability can exist. For the equal mass Full 3-body problem this can be easily established, with the result that for any equal mass, finite density 3-body problem in or near a contact equilibrium, none of the components of the system can escape in the ensuing motion.

Keywords. celestial mechanics, methods: analytical, 3-body problem, Hill stability

1. Introduction

In Scheeres 2012 the Full N-body problem is introduced, distinguished from the traditional N-body problem in that each body has a finite density, and hence two bodies cannot come arbitrarily close to each other. The finite-sized bodies are assumed to be rigid, and only exhibit contact forces between each other of friction and coefficient of restitution. This means that relative equilibria must also include resting configurations, with all relative motion between components zeroed. This both results in new possible relative equilibrium configurations of an N-body system, with components resting on each other, and also places lower limits on the gravitational potential energy. This model has been developed to describe the relative mechanics and dynamics of self-gravitating rubble pile asteroids, where the relative forces between components are weak enough for the rigidity of the components to not be compromised. In Scheeres 2012 it is shown how this simple change provides a drastic modification of the stable states of the 2 and 3 body problems. For the 3-body problem under the restriction that the bodies are equal mass spheres these relative equilibria are shown in Fig. 1, and in addition to the classical Euler and Lagrange solutions also include the so-called Resting Euler, Resting Lagrange, Transitional, Transverse and Aligned. The energetic stability of these solutions are indicated on the figure, with light being stable and dark being unstable. In Fig. 2 we show the energy / angular momentum chart of these equilibria, which traces out the total (normalized) energy as a function of angular momentum for each relative equilibrium. Wherever a line stops or the ends of two lines meet each other indicates a bifurcation point where the equilibrium may not longer exist or may transition to a different type. We note that the Lagrange and Transverse configuration lines are not shown although they are indicated at the top of the chart, and that these bifurcate into existence once the Lagrange Resting equilibrium loses its stability. Similarly the Euler Resting configuration line is not shown, but comes into existence when the Euler Resting equilibrium loses its stability. The current paper explores one aspect of this problem, namely whether Hill stability can be proven to exist for such a Full 3-body problem.

2. Background

The main results are established by using the minimum energy function defined in Scheeres 2012, which can be derived from Sundman's Inequality:

$$\mathcal{E} = \frac{1}{2}\frac{H^2}{I_H} + \mathcal{U} \leqslant E \tag{2.1}$$

where E and H are the total energy and angular momentum of the system, including translational and rotational motion, I_H is the total moment of inertia of the system about the fixed total angular momentum vector of the system, and $\mathcal{U}$ is the gravitational potential energy of the system. The function $\mathcal{E}$ is intimately related to Smale's Amended Potential (Smale 1970) and can be used to discover all of the previously mentioned relative equilibria as outlined in Scheeres 2012.

If we assume that the three bodies are of equal mass and density the system can be normalized such that

$$I_H = 0.3 + \frac{1}{3}\left(r_{12}^2 + r_{23}^2 + r_{31}^2\right) \geqslant 1.3 \tag{2.2}$$

$$\mathcal{U} = -\left[\frac{1}{r_{12}} + \frac{1}{r_{23}} + \frac{1}{r_{31}}\right] \geqslant -3 \tag{2.3}$$

where r_{ij} denotes the distance between bodies i and j. For a finite density system, the distance between the bodies has a lower limit such that $r_{ij} \geqslant 1$. This implies that the potential energy and moment of inertia have lower limits, as indicated above. For a point mass 3-body problem the lower limit on relative distance is $r_{ij} \geqslant 0$, meaning that there is no lower limit on the potential energy for that case.

Finally, we define the stability concept of interest for this paper.

Definition: *A system is Hill Stable if* $r_{ij} < C < \infty$ *for all time, both future and past.*

3. Hill Stability of the equal mass Full 3-Body Problem

Hill stability tells us whether the components of a gravitating system can escape with respect to each other. In the two body problem, both full and point mass, this is simply established by considering the total energy of the system: If it is positive the system is not Hill stable in general, while if it is negative it is Hill stable. For multi-particle systems this simple result no longer holds. In general, for the point mass N-body problem it can be shown that if a system has a positive energy, then the system *is not* Hill stable, and that at least one body must escape (Pollard 1976). Thus, a Necessary Condition for Hill Stability in the N-body problem is that $E < 0$.

Thus, while Hill Stability can exist in the point mass 3-body problem, it cannot be rigorously shown based on simple inequalities, such as is possible in the restricted 3-body problem. At best, as discussed in Marchal and Saari 1975, constraints can be given for when exchanges cannot occur. This difficulty of establishing a sufficient condition for Hill Stability in the point mass 3-body problem can be understood by considering the minimum energy function $\mathcal{E} \leqslant E$. If one of the bodies escapes from the other two, say body 1, then $r_{12}, r_{31} \to \infty$ while $r_{23} < \infty$ (note, since $E < 0$, r_{23} cannot become arbitrarily large without violating Sundman's Inequality). Then $\mathcal{E} \to -1/r_{23} \leqslant E$, leading to $r_{23} \leqslant -1/E$. In the point mass problem the only restriction is that $r_{23} \geqslant 0$, thus this inequality is always satisfied, providing no definite restrictions.

https://doi.org/10.1017/S1743921314008047 Published online by Cambridge University Press

For the Full 3-body problem, with our constraint $r_{ij} \geqslant 1$, we note that the potential energy between each body, $\mathcal{U}_{ij} = -1/r_{ij}$ has a lower limit, $\mathcal{U}_{ij} \geqslant -1$. With this observation, we can present the following Theorem and Proof.

Theorem: *Consider an equal mass, finite density 3-body problem that has been normalized as in Eqns. 2.2 and 2.3. If $E < -1$ then it is Hill Stable.*

To prove, we first establish the result by contradiction and then show that systems that satisfy the constraint exist.

Proof. Given $E < -1$, assume that one of the bodies, say body 1, undergoes escape. Then $r_{12}, r_{31} \rightarrow \infty$, and $\mathcal{E} \rightarrow -1/r_{23}$. However, the Sundman inequality $\mathcal{E} \leqslant E < -1$ must still hold, leading to the inequality $r_{23} < 1$, a contradiction since in the normalized Full Body Problem we must have $r_{23} \geqslant 1$.

To show that systems with $E < -1$ can exist in the equal mass Full 3-body problem, consider Fig. 2, modified from Scheeres 2012. From this figure it is evident that all relative equilibrium configurations that have a resting component have an energy less than -1, for both stable and unstable configurations. Only orbital relative equilibria can have energies above this limit. Finally, note that the energy of the stable Aligned Configuration has an energy value < -1, for all finite distance between the components. □

Note that if a system is Hill stable, one cannot conclude that $E < -1$, making this only a sufficient condition. In fact, the particular solutions of the Lagrange and Euler relative equilibria are both Hill Stable (even though they are unstable to small variations in general), and both can exist at energies above -1.

The theorem leads directly to the following corollary.

Corollary: *All motions starting close enough to relative equilibrium in the Full 3-body problem that have resting elements are Hill Stable.*

Proof. The complete set of resting relative equilibria is presented in Fig. 2, and established through proof in Scheeres 2012. All of these equilibria have a total energy strictly less than -1. Thus, any deviation from a relative equilibrium can be made small enough such that the energy remains less than -1, and thus by the previous theorem are Hill stable. □

4. Discussion

The results presented in the above Theorem and Corollary have implications. Any equal mass 3-body system that undergoes rotational fission due to exogenous effects that increase its angular momentum over time will be Hill stable, and the components will not be able to escape from each other. This mirrors a similar result in Scheeres 2009 for the Full 2-body problem, where an equal mass resting configuration spun to fission is dynamically unstable, yet is Hill stable and cannot mutually escape. The implication (somewhat limited due to the strong model assumptions), is that an equal mass 'contact triple' asteroid will remain a contact triple, mirroring the more general results for two component bodies described in Jacobson & Scheeres 2011.

In analogy with the Full 2-body problem, these Hill stability results are not expected to exist in the more general problem where the bodies do not have equal mass. Thus, similar to the Full 2-body problem described in Scheeres 2009 there should exist limits on the relative masses of a Full 3-body problem for the relative equilibrium configurations to be Hill stable. Future research will probe this.

Equilibria in the Spherical, Full 3 Body Problem with Equal Size

Configuration	Name	Energetic Stability	Conditions
	Lagrange Resting	Stable	
	Euler Resting	Stable	For high enough *H*
	Aligned	Stable	Outer Solution
	Lagrange	Unstable	
	Euler	Unstable	
	Euler Resting	Unstable	For low *H*
	Aligned	Unstable	Inner Solution
	Transverse	Unstable	
	Transitional	Unstable	

Figure 1. Relative equilibria in the equal mass, finite density 3-body problem.

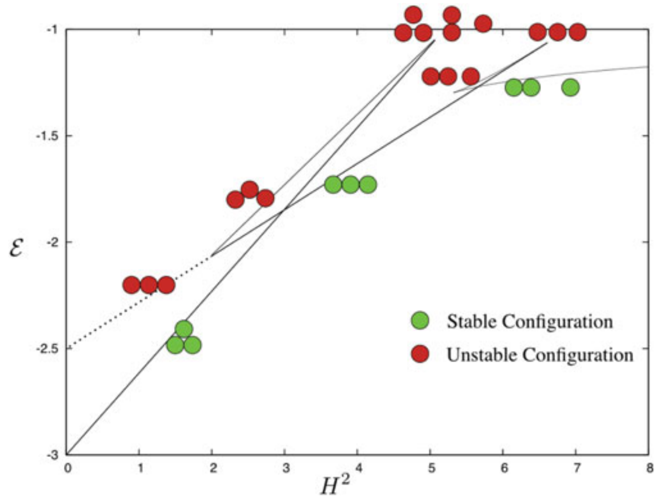

Figure 2. Chart showing the energy and angular momentum for all classes of relative equilibria in the equal mass Full 3-Body Problem. Cusps occur when the lines meet each other and indicate a bifurcation either creating or destroying relative equilibria.

References

S. A. Jacobson and D. J. Scheeres. *Icarus*, 214:161–178, July 2011.

C. Marchal and D. G. Saari. *Celestial mechanics*, 12(2):115–129, 1975.

H. Pollard. The Carus Mathematical Monographs, Providence: Mathematical Association of America, 1976.

D. J. Scheeres. *Celestial Mechanics and Dynamical Astronomy*, 104(1):103–128, 2009.

D. J. Scheeres. *Celestial Mechanics and Dynamical Astronomy*, 113(3):291–320, 2012.

S. Smale. *Inventiones mathematicae*, 10(4):305–331, 1970.

https://doi.org/10.1017/S1743921314008047 Published online by Cambridge University Press