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THE ABSOLUTE S_k-MEASURE OF TOTALLY POSITIVE ALGEBRAIC INTEGERS

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Abstract

Let α be a totally positive algebraic integer of degree d, with conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$. The absolute S_k-measure of α is defined by $s_k(\alpha) = d^{-1}\sum_{i=1}^{d}\alpha_i^k$. We compute the lower bounds v_k of $s_k(\alpha)$ for each integer in the range $2 \le k \le 15$ and give a conjecture on the results for integers $k > 15$. Then we derive the lower bounds of $s_k(\alpha)$ for all real numbers $k > 2$. Our computation is based on an improvement in the application of the LLL algorithm and analysis of the polynomials in the explicit auxiliary functions.

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1. Introduction

Let α be a totally positive algebraic integer of degree d, that is, its conjugates $\alpha_1 = \alpha, \alpha_2, \ldots, \alpha_d$ are all positive real numbers, while its minimal polynomial is $P(x) = a_0x^d + a_1x^{d-1} + \cdots + a_{d-1}x + a_d$, where $a_0 = 1$ and $a_i \in \mathbb{Z}$ $(1 \le i \le d)$. For $k > 0$, we define the S_k*-measure* of α by

$$S_k(\alpha) = S_k(P) = \sum_{i=1}^{d}\alpha_i^k,$$

and the *absolute* S_k*-measure* of α by $s_k(\alpha) = s_k(P) = S_k(\alpha)/d$. It follows from the arithmetic–geometric inequality that $s_k(\alpha) > 1$ unless $\alpha = 1$. Let $\mathcal{T}_k$ be the spectrum of $s_k(\alpha)$, that is,

$$\mathcal{T}_k = \{s_k(\alpha) \mid \alpha \ne 1 \text{ is a totally positive algebraic integer}\}.$$

1.1. The absolute trace of totally positive algebraic integers. When $k = 1$, $S_1(\alpha) = \mathrm{Tr}(\alpha)$ is the usual *trace* of α and $\mathrm{tr}(\alpha) = \mathrm{Tr}(\alpha)/d$ denotes the *absolute trace* of α. The Schur–Siegel–Smyth trace problem [3] is to find the smallest limit point of $\mathcal{T}_1$.

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That is, *given* $\rho < 2$, *show that all but finitely many totally positive algebraic integers* α *satisfy* $\mathrm{tr}(\alpha) > \rho$.

One way of trying to solve this problem is to study the lower bound of $\mathrm{tr}(\alpha)$. Schur [17] solved this for $\rho = \sqrt{e}$ with an exceptional polynomial P_3 (all the exceptional polynomials denoted by P_j in this paper can be found in Table 2). Siegel [18] solved it for $\rho = 1.7336\ldots$ with another exceptional polynomial P_7. Smyth [20] solved it for $\rho = 1.7719\ldots$ with two new exceptional polynomials P_9 and P_{10}. The method of explicit auxiliary functions established by Smyth [20] has been used by many authors to study this problem, but no more exceptional polynomials have been found. In [22], the authors solved it for $\rho = 1.793145\ldots$.

On the other hand, Serre [1] showed that the method of Smyth [20] does not produce such a value for any ρ larger than $1.8983021\ldots$. Applying this result, Smith [19] recently proved that the smallest limit point of $\mathcal{T}_1$ is less than 1.8984, and his ongoing computational work suggests that it could be decreased to 1.81.

1.2. The lower bounds of $s_k(\alpha)$ for integers $k \geq 2$. More generally, no exact value of the smallest limit point of $\mathcal{T}_k$ is known for any $k > 0$. Studying its lower and upper bounds is closely related to the study of the spectrum $\mathcal{M}_p$ of the measure defined by

$$M_p(\alpha) = \left(\frac{1}{d}\sum_{i=1}^{d} |\alpha_i|^p\right)^{1/p}$$

for $p > 0$, where α varies over all totally real algebraic integers (all of whose conjugates are real numbers) of degree d, because $(M_{2p}(\alpha))^2 = (s_p(\alpha^2))^{1/p}$ for α totally real. Smyth [21] carried out a detailed analysis of $\mathcal{M}_p$. His results can be used to analyse the structure of $\mathcal{T}_k$. For instance, it follows from [21] that the set $\mathcal{T}_2$ consists of five isolated points $s_2(P_1), s_2(P_3), s_2(P_7), s_2(P_{12}), s_2(P_{13})$ in the interval $(1, \upsilon_2)$ (where $\upsilon_2 = 5.19610\ldots$), is everywhere dense in $(6, \infty)$ and is undetermined in $(\upsilon_2, 6)$; and the set $\mathcal{T}_3$ consists of four isolated points $s_3(P_1), s_3(P_3), s_3(P_7), s_3(P_{12})$ in $(1, \upsilon_3)$ (where $\upsilon_3 = 16.26481\ldots$), is everywhere dense in $(20, \infty)$ and is undetermined in $(\upsilon_3, 20)$. Similar results for $\mathcal{T}_k$ for $k = 4, 5, 6, 9, 12, 15$ can also be deduced from [21]. Moreover, It follows that the upper bound for the limit point of $\mathcal{T}_k$ for each $k > 1$ is $D(k) = d(2k)^{2k}$, where $d(p) = \lim_{n\to\infty} M_p(2\cos(2\pi/n))$ for $p > 2$.

In [11], Liang and Wu improved the value of υ_2 to $5.31935\ldots$, the value of υ_3 to $17.56765\ldots$, and found a new isolated point $s_2(P_{10})$ in $(1, \upsilon_2)$ and three new isolated points $s_3(P_{13}), s_3(P_{14}), s_3(P_{15})$ in $(1, \upsilon_3)$. In [9], Flammang improved the value of υ_2 to $5.32176\ldots$. She also showed that the method of auxiliary functions does not give a value for any υ_2 larger than $5.895237\ldots$. This result, combined with [19, Corollary 5.5], can be used to show that the smallest limit point of $\mathcal{T}_2$ is less than 5.8953. In [8], Flammang improved the value of υ_3 to $17.56827\ldots$.

In this work, we compute lower bounds υ_k for $s_k(\alpha)$ for each integer in the range $2 \leq k \leq 15$, and go further to study the results for integers $k > 15$. Then we derive lower bounds for $s_k(\alpha)$ for all real numbers $k > 2$.

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TABLE 1. The values of υ_k, υ'_k and indices of exceptional polynomials in Theorem 1.1.

k	υ_k	υ'_k	Indices of exceptional polynomials
2	5.32716	5.32176	$\underline{1}$; $\underline{3}$; $\underline{7}$; $\underline{12}$; $\underline{13}$; $\underline{10}$
3	17.6201	17.5682	$\underline{1}$; $\underline{3}$; $\underline{7}$; $\underline{12}$; $\underline{13}$; $\underline{14}$; $\underline{15}$
4	61.0588	55.7906	$\underline{1}$; $\underline{3}$; $\underline{7}$; $\underline{12}$; $\underline{13}$; 14; 15
5	218.024	201.516	$\underline{1}$; $\underline{3}$; $\underline{7}$; $\underline{12}$; $\underline{13}$; 14; 15; 8; 16
6	794.576	712.650	$\underline{1}$; $\underline{3}$; $\underline{7}$; $\underline{12}$; $\underline{13}$; 14; 15; $\underline{8}$; 16; 2; 4
7	2933.78	–	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4
8	10941.5	–	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4
9	41089.5	32752.3	$\underline{1}$; $\underline{3}$; $\underline{7}$; $\underline{12}$; $\underline{13}$; 14; 15; $\underline{8}$; 16; $\underline{2}$; $\underline{4}$; 17
10	155162	–	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17
11	588582	–	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17
12	2242242	1511035	$\underline{1}$; $\underline{3}$; $\underline{7}$; $\underline{12}$; 13; 14; 15; $\underline{8}$; 16; $\underline{2}$; $\underline{4}$; 17; 18
13	8574378	–	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18
14	32901254	–	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18
15	126620232	84162493	$\underline{1}$; $\underline{3}$; $\underline{7}$; $\underline{12}$; $\underline{13}$; 14; 15; $\underline{8}$; 16; $\underline{2}$; $\underline{4}$; 17; 18; 6

THEOREM 1.1. *If $\alpha \neq 1$ is a totally positive algebraic integer, then for each integer in the range $2 \leq k \leq 15$ we have $s_k(\alpha) \geq \upsilon_k$ with finitely many exceptions, where $\upsilon_2, \ldots, \upsilon_{15}$ are given in Table 1.*

If $s_k(\alpha) < \upsilon_k$, that is, α is an exception, then $s_k(\alpha)$ is an isolated point in $\mathcal{T}_k$. The exceptional polynomials, whose indices are listed in Table 1, can be read off from Table 2. In Table 1, the values υ'_k denote the previous results [8, 9, 21] for the lower bounds of $s_k(\alpha)$. The exceptional polynomials with underlined indices are found from previous research. The polynomials marked with asterisks in Table 2 will appear later in this paper as the predicted exceptional polynomials for $k > 15$.

We observe the staircase distribution of the exceptional polynomials in Table 1 and propose the following conjecture.

CONJECTURE 1.2. For an algebraic integer α, if $s_k(\alpha)$ is an isolated point in $\mathcal{T}_k$ for $k > 2$, then $s_{k+1}(\alpha)$ is also an isolated point in $\mathcal{T}_{k+1}$.

Let $\omega_k = \upsilon_k^{1/k}$. We give a function $g(x)$ to approximate the values of ω_k for $2 \leq k \leq 15$, and to predict the behaviour of υ_k for integers $k > 15$:

$$g(x) = 1 + u_0\left(\frac{\log(u_1 \log(x))}{u_2 \log(x)}\right)^{2/x}$$

for $x \geq 2$, where the parameters $u_0 = 2.96$, $u_1 = 4.22$ and $u_2 = 3.51$ are optimised according to the values of υ_k for integers $2 \leq k \leq 15$.

PROPOSITION 1.3. *If $\alpha \neq 1$ is a totally positive algebraic integer, then for each integer in the range $2 \leq k \leq 15$, with finitely many exceptions, we have $s_k(\alpha) \geq g^k(k)$.*

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TABLE 2. The exceptional polynomials P_j.

j	d	Coefficients	j	d	Coefficients
1	1	−2 1	13	6	1 −21 70 −84 45 −11 1
2	1	−3 1	14	8	1 −36 210 −462 495 −286 91 −15 1
3	2	1 −3 1	15	9	−1 45 −330 924 −1287 1001 −455 120 −17 1
4	2	2 −4 1	16	11	−1 66 −715 3003 −6435 8008 −6188 3060 −969 190
5	2	1 −4 1 *			−21 1
6	2	5 −5 1	17	14	1 −105 1820 −12376 43758 −92378 125970 −116280
7	3	−1 6 −5 1			74613 −33649 10626 −2300 325 −27 1
8	3	−1 9 −6 1	18	15	−1 120 −2380 18564 −75582 184756 −293930 319770
9	4	1 −7 13 −7 1			−245157 134596 −53130 14950 −2925 378 −29 1
10	4	1 −8 14 −7 1	19	18	1 −171 4845 −54264 319770 −1144066 2704156
11	4	1 −24 26 −9 1 *			−4457400 5311735 −4686825 3108105 −1560780
12	5	−1 15 −35 28 −9 1			593775 −169911 35960 −5456 561 −35 1 *

TABLE 3. The values of $g^k(k)$ and polynomials with $s_k(P_j) < g^k(k)$ for $16 \le k \le 25$.

k	$g^k(k)$	Indices of polynomials
16	489867601	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18; 6
17	1899197603	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18; 6
18	7374496229	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18; 6
19	28673594453	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18; 6
20	111622140680	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18; 6; 19
21	434989673867	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18; 6; 19
22	1696753682790	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18; 6; 19
23	6624116609508	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18; 6; 19;
24	25880365332155	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18; 6; 19; 11
25	101184713328898	1; 3; 7; 12; 13; 14; 15; 8; 16; 2; 4; 17; 18; 6; 19; 11; 5

CONJECTURE 1.4. If $\alpha \neq 1$ is a totally positive algebraic integer, then for all integers $k > 15$, with finitely many exceptions, we have $s_k(\alpha) \ge g^k(k)$.

With Conjecture 1.4, for each integer $k > 15$, we give an estimate of υ_k with $g^k(k)$ and some of the exceptions satisfying $s_k(\alpha) < g^k(k)$. For instance, for $16 \le k \le 25$, we list in Table 3 the values of $g^k(k)$ and indices of the polynomials P_j satisfying $s_k(P_j) < g^k(k)$. Note that all these polynomials conform to Conjecture 1.2.

Let $d'(k) = D(k)^{1/k}$. In Figure 1, the behaviours of ω_k, $g(k)$ and $d'(k)$ are represented graphically. One can prove that $g(k)$ tends to $1 + u_0$ as k tends to infinity. Although this bound is not very sharp for k sufficiently large, it gives an explicit approximate

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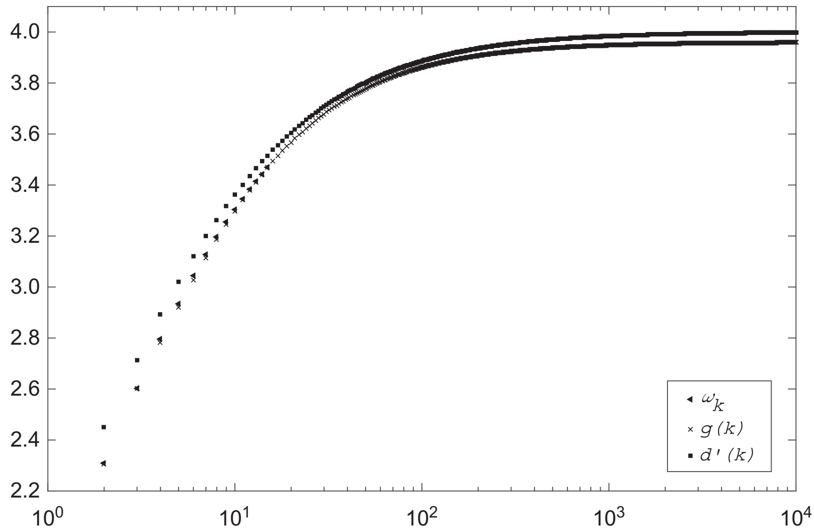

FIGURE 1. The values of ω_k for $2 \le k \le 15$ and the asymptotic behaviour of $g(k)$ and $d'(k)$ for $k > 0$.

expression of υ_k for the first time. The values of u_0, u_1 and u_2 in $g(x)$ can be modified if the results on υ_k for integers $2 \le k \le 15$ are improved.

1.3. The lower bounds of $s_k(\alpha)$ for all real numbers $k > 2$. As an application of Theorem 1.1, we give the corresponding results for the lower bounds of $s_k(\alpha)$ for all $k > 2$, which supersede the results in [21].

COROLLARY 1.5. *If $\alpha \neq 1$ is a totally positive algebraic integer, then for all real numbers $k > 2$ we have $s_k(\alpha) > \upsilon_{[k]}$, with finitely many exceptions.*

The values of $\upsilon_{[k]}$ (where $[k]$ is the integer part of k) in Corollary 1.5 are read off from Table 1, and the isolated points of $\mathcal{T}_k$ in $(1, \upsilon_{[k]})$ are found with the aid of Table 1 (for more details see [21]). For instance, for $k = 3.5$, there are two elements $s_{3.5}(P_1)$ and $s_{3.5}(P_3)$ of $\mathcal{T}_{3.5}$ in $(1, 17.6201)$. The proof of Corollary 1.5 refers to the proof of [21, Theorem 2].

This paper is organised as follows. In Section 2 we explain how to use the explicit auxiliary functions to compute the lower bounds of $s_k(\alpha)$ for totally positive algebraic integers, and give the relation between the auxiliary function and integer transfinite diameter. In Section 3 we show our improvement in finding polynomials used in the explicit auxiliary functions. In Section 4, we give some numerical results.

2. The method of auxiliary functions

2.1. The explicit auxiliary function. Auxiliary functions of this type have been used by many authors for the computation of different measures of totally positive

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algebraic integers. We take

$$f(x,\mathbf{c}) = \psi(x) - \sum_{j=1}^{J} c_j \log|Q_j(x)| \quad \text{for } x \in I, \tag{2.1}$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_J)$, the c_j are positive real numbers, the Q_j are nonzero polynomials in $\mathbb{Z}[x]$ and I is a real interval. For instance, if $\psi(x) = \log(x+1)$, the auxiliary function (2.1) can be applied for the lower bound of the absolute length $R(\alpha) = L(\alpha)^{1/d}$, where $L(\alpha) = \sum_{i=0}^{d} |a_i|$ (for more details, see [5, 6, 14]); and $\psi(x) = \log(\max\{1, x\})$ for the lower bound of the absolute Mahler measure $\Omega(\alpha) = M(\alpha)^{1/d}$, where $M(\alpha) = |a_0| \prod_{i=1}^{d} \max(1, |\alpha_i|)$ (for more details, see [5, 7, 14]).

To prove Theorem 1, we take $\psi(x) = x^k$ and $I = (0, +\infty)$ (see [9, 13, 15, 24]) for each integer in the range $2 \le k \le 15$. That is,

$$f_k(x,\mathbf{c}) = x^k - \sum_{j=1}^{J} c_j \log|Q_j(x)| \quad \text{for } x > 0. \tag{2.2}$$

Let m_k be the minimum of $f_k(x,\mathbf{c})$ for $x > 0$. Then $\sum_{i=1}^{d} f_k(\alpha_i, \mathbf{c}) \ge dm_k$, that is,

$$S_k(\alpha) \ge dm_k + \sum_{j=1}^{J} c_j \log|\mathrm{Res}(P, Q_j)|,$$

where $\mathrm{Res}(P, Q_j) = \prod_{i=1}^{d} Q_j(\alpha_i)$ is the resultant of P and Q_j. If P does not divide any Q_j, it follows that $\mathrm{Res}(P, Q_j)$ is a nonzero integer for all $1 \le j \le J$ and $s_k(\alpha) \ge m_k$. Hence we have to solve an optimisation problem to determine

$$m_k(\mathbf{c}) = \max_{\mathbf{c}} \min_{x>0} f_k(x, \mathbf{c}).$$

2.2. The relation between the auxiliary function and integer transfinite diameter. Let K be a compact subset of $\mathbb{C}$. If ϕ is a positive function defined on K, the ϕ-generalised integer transfinite diameter [2] of K is defined by

$$t_{\mathbb{Z},\,\phi}(K) = \liminf_{\substack{h \ge 1 \\ h \to \infty}} \inf_{\substack{H \in \mathbb{Z}[x] \\ \deg H = h}} \sup_{x \in K} (|H(x)|^{1/h} \phi(x)).$$

In the auxiliary function (2.1), if we replace the positive real constants c_j by rational numbers for $1 \le j \le J$, we obtain

$$f(x,\mathbf{c}) = \psi(x) - \frac{t}{h} \log|H(x)|,$$

where the polynomial $H \in \mathbb{Z}[x]$ is of degree h and t is a positive real number. We want to determine a function $f(x,\mathbf{c})$ whose minimum m in I is as large as possible. Thus we need to seek a polynomial $H \in \mathbb{Z}[x]$ such that

$$\sup_{x \in I} |H(x)|^{t/h} e^{-\psi(x)} \le e^{-m}.$$

Now, if we suppose that t is fixed, say $t = 1$, we need to get an effective upper bound on the weighted integer transfinite diameter with the weight $\phi(x) = e^{-\psi(x)}$ and the compact set $K = I$. It is sufficient to find an explicit polynomial $H \in \mathbb{Z}[x]$ and then use the sequence of the successive powers of H.

3. Finding polynomials used in auxiliary functions

3.1. An improvement in the application of the LLL algorithm. The main point is to make a good choice of the polynomials to be used in the explicit auxiliary functions. In 2003, based on the LLL algorithm (the lattice reduction algorithm) [10], the third author [23] developed an algorithm to search the polynomials in (2.1) systematically. Before that, the polynomials were found heuristically. In 2009, Flammang [4] developed an algorithm called the recursive algorithm on the basis of the method in [23].

With the third author's algorithm [23], we consider the auxiliary function (2.1). We start with the polynomial x, get the best c_1 and take $t = c_1$. Supposing that we have some polynomials $Q_1, Q_2, \ldots, Q_J$, we optimise the numbers $c_1, c_2, \ldots, c_J$ with the semi-infinite linear programming method that was introduced into number theory by Smyth [21]. This gives us a new number t and we continue by induction to get $J + 1$ polynomials. That is, we have

$$F = \prod_{j=1}^{J} Q_j^{c_j}$$

of degree t (for noninteger t, we can multiply it by a large enough integer to make it integral), and we seek a polynomial $Q \in \mathbb{Z}[x]$ of degree q such that

$$\sup_{x \in I} |F(x)Q(x)|^{1/(t+q)} e^{-\psi(x)} \leq e^{-m}.$$

We want the quantity $\sup_{x\in I} |F(x)Q(x)| e^{-\psi(x)(t+q)}$ as small as possible. We apply the LLL algorithm to the linear forms

$$|F(x_i)Q(x_i)| e^{-\psi(x_i)(t+q)}.$$

The x_i are control points in the interval I, chosen as the points where the function f has local minima.

The LLL algorithm tends to give polynomials of small degree with small Euclidean norm, which are always 'good' candidates for the set $\{Q_1, Q_2, \ldots, Q_J\}$. But sometimes we need polynomials with larger degree, which is a challenge to the previous method. Besides, as the absolute value of $\psi(x)$ in I increases, it becomes intractable to search for Q_j with the LLL algorithm. This requires an improvement in the application of the LLL algorithm.

In the auxiliary function (2.2), we observe that some irreducible polynomials with small absolute S_k-measure always appear repeatedly as factors of polynomials given by the LLL algorithm, and tend to have high powers. To make the LLL algorithm

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produce more factors that differ from the existing polynomials Q_j, we introduce the perturbations $\delta_0, \delta_1, \ldots, \delta_J$, where $\delta_j \geq 0$ are real numbers. That is, let

$$\mathcal{F} = \prod_{j=1}^{J} Q_j^{c_j+\delta_j}$$

be of degree t'. We apply the LLL algorithm to the linear forms

$$|\mathcal{F}(x_i)Q(x_i)|e^{-x_i^k(t'+q)\delta_0} \tag{3.1}$$

in the interval $I_0 = (0, A)$, where A is large enough.

For different sets $\{\delta_0, \delta_1, \delta_2, \ldots, \delta_J\}$, the LLL algorithm will produce some different polynomials. The δ_j ($j = 0, \ldots, J$) are chosen so that the LLL algorithm gives 'good' polynomials. We note that for a fixed k in (3.1), in the set $\{\delta_0, \delta_1, \ldots, \delta_J\}$ that gives 'good' polynomials, there are only a small number of nonzero elements, the corresponding Q_j to which always have small absolute S_k-measure. In fact, for $k \geq 2$, all the perturbations of exceptional polynomials P_j are nonzero.

In the program applying the LLL algorithm, we make some modifications so that it produces a large number of candidates for Q_j in less time. This improvement in the application of the LLL algorithm helps us improve the lower bounds of $s_k(\alpha)$. It can be used to find polynomials for any explicit auxiliary function with the form (2.1).

3.2. The Kronecker polynomials. Since all the exceptional polynomials should be used in the auxiliary functions, we analyse the characteristics of the known exceptions. That may help us find more candidates for the set $\{Q_1, Q_2, \ldots, Q_J\}$.

It is clear that the absolute S_k-measure of a totally positive algebraic integer is related to the distribution of its conjugates. An important measure of an algebraic integer α associated with its conjugates is $\overline{|\alpha|} = \max_{1\leq i\leq d} |\alpha_i|$, the house of α. We observe that all the known exceptions for $2 \leq k \leq 15$ satisfy $\overline{|\alpha|} < 4$. This is an extension of the observations of McAuley [12] and Smyth [21]. In fact, for fixed α, $s_k(\alpha)^{1/k}$ is an increasing function of k [21], and $s_k(\alpha)^{1/k}$ tends to $\overline{|\alpha|}$ as k tends to infinity [12]. Note that $d'(k)$ tends to 4 as k tends to infinity. This suggests that the minimal polynomials of totally positive algebraic integers with $\overline{|\alpha|} < 4$ may be useful in the auxiliary function (2.2).

Kronecker showed that the polynomials whose zeros all lie in $[0, 4]$ are precisely

$$K_m = \prod_{\substack{(l,m)=1\\ 0\leq l\leq m/2}} \left(x - \left(2 + 2\cos\frac{2\pi l}{m}\right)\right)$$

(the so-called Kronecker polynomials [16]) for every positive integer m. A cyclotomic polynomial $\Phi_n(z)$ of degree $\varphi(n)$ (where $\varphi(n)$ is the Euler function) produces a Kronecker polynomial of degree $\varphi(n)/2$ by the change of variable $x = z + 1/z + 2$. This transformation produces 158 Kronecker polynomials of degree less than or equal to 40. We test them in the auxiliary function (2.2) for each integer $2 \leq k \leq 15$, and reserve the ones with nonzero c_j. This idea is very effective, especially for k relatively large.

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TABLE 4. Values of l_k, m_k, n_k and $\lceil\alpha\rceil_k$ for $2 \le k \le 15$.

k	l_k	m_k	n_k	$\lceil\alpha\rceil_k$	k	l_k	m_k	n_k	$\lceil\alpha\rceil_k$
2	133	43	14	4.585557	9	117	51	38	4.136069
3	106	34	16	4.380522	10	112	55	44	4.129987
4	127	37	19	4.289224	11	107	56	48	4.129987
5	142	42	25	4.289224	12	106	56	48	4.129987
6	136	43	28	4.191787	13	108	59	51	4.104735
7	122	43	31	4.191787	14	98	57	49	4.104735
8	123	48	37	4.191787	15	97	60	52	4.104735

We give the number of the Kronecker polynomials used in our calculation in Table 4 in the following section.

4. Numerical results and analyses

When k is large enough, the computation of υ_k becomes intractable, so we stop at $k = 16$. Thanks to the improvement in the application of the LLL algorithm, we find a large number of irreducible polynomials used in the auxiliary function (2.2). For instance, we find 72 new polynomials for $k = 2$, 60 new polynomials for $k = 3$, and improve the values of υ_2 and υ_3. Besides, for k relatively large, the Kronecker polynomials mentioned in Section 3.2 work very well. The number of all the different irreducible polynomials used in (2.2) is 490 (available from the corresponding author).

In Table 4, for each integer in the range $2 \le k \le 15$, we give the number l_k of irreducible polynomials used for computing υ_k. Among them are m_k monic irreducible polynomials with all zeros positive (the so-called totally positive polynomials) and n_k Kronecker polynomials. The biggest house of the m_k totally positive polynomials is denoted by $\lceil\alpha\rceil_k$. With increasing k, the proportion of totally positive polynomials and that of Kronecker polynomials increase, while the biggest house of the available totally positive polynomials decreases. It is an interesting question whether all the available totally positive polynomials used in the auxiliary function (2.2) should be Kronecker polynomials to compute the lower bounds of $s_k(\alpha)$ for k large enough. It is also interesting to investigate the relation between the house and the absolute S_k-measure of totally positive algebraic integers.

If $k > 0$ is a real number, lower bounds υ_k of $s_k(\alpha)$ can also be calculated. For example, for $k = 2.5$, using 114 irreducible polynomials found by the improved algorithm, we improve the previous result $\upsilon_{2.5} = 9.0509$ (with isolated points $s_{2.5}(P_1), s_{2.5}(P_3), s_{2.5}(P_7), s_{2.5}(P_{12})$ in $(1, \upsilon_{2.5})$) [21] to $\upsilon_{2.5} = 9.6101$ and find two new isolated points $s_{2.5}(P_{13})$ and $s_{2.5}(P_{14})$. Theoretically, all the results in [21] on the lower bounds of $s_k(\alpha)$ for $k > 0$ can be improved with our method.

All the computations are done using the Pascal programming language and Pari/GP.

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