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Analytic Besov spaces, approximation, and closed ideals

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In memory of the late Brahim Bouya (1977-2020)

Abstract. In this paper, we give a complete description of closed ideals of the Banach algebra $\mathcal{B}_p^s \cap \lambda_\alpha$, where $\mathcal{B}_p^s$ denotes the analytic Besov space and λ_α is the separable analytic Lipschitz space. Our result extends several previous results in Bahajji-El Idrissi and El-Fallah (2020, *Studia Mathematica* 255, 209–217), Bouya (2009, *Canadian Journal of Mathematics* 61, 282–298), and Shirokov (1982, *Izv. Ross. Akad. Nauk Ser. Mat.* 46, 1316–1332).

1 Introduction

Let $\mathbb{D}$ be the open unit disc of the complex plane $\mathbb{C}$, and let $\mathbb{T} := \partial\mathbb{D}$ be the unit circle. Let dA (resp. dm) be the normalized Lebesgue measure on $\mathbb{D}$ (resp. $\mathbb{T}$). The space of analytic functions on $\mathbb{D}$ is denoted by $\mathrm{Hol}(\mathbb{D})$.

The Hardy space H^p, $1 < p < \infty$, is the space of analytic functions f on $\mathbb{D}$ such that

$$\|f\|^p{}_{H^p} := \sup_{0\le r<1} \int_{\mathbb{T}} |f(r\zeta)|^p dm(\zeta) < \infty.$$

For $1 < p < \infty$ and $0 \le s < 1$, let $\mathcal{B}_p^s$ be the analytic Besov spaces given by

$$\mathcal{B}_p^s := \left\{ f \in \mathrm{Hol}(\mathbb{D}) : \mathcal{B}_p^s(f) := \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{(1-s)p-1} dA(z) < \infty \right\}.$$

It is well known that $\mathcal{B}_p^s$ is a subspace of the Hardy space H^p (see [2]).

Note that the classical Dirichlet space $\mathcal{D}$ corresponds to $p = 2$ and $s = 1/2$. In the standard notation, the weighted Dirichlet spaces $\mathcal{D}_{1-2s} = \mathcal{B}_2^s$ with $0 < s < 1/2$. Note also that $\mathrm{H}^2 = \mathcal{B}_2^0$. Various facts about Hardy and Dirichlet spaces can be found in [8, 10, 12].

The disc algebra $A(\mathbb{D})$ consists of continuous functions on $\overline{\mathbb{D}}$ that are analytic on $\mathbb{D}$. For $\alpha \in (0,1)$, the separable analytic Lipschitz algebra λ_α is given by

$$\lambda_\alpha := \{ f \in A(\mathbb{D}) : |f(z) - f(w)| = o(|z-w|^\alpha) \text{ as } |z-w| \text{ tends to } 0\}.$$

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Endowed with the norm,

$$\|f\|_\alpha = \|f\|_\infty + \sup_{z,w\in\mathbb{D},z\neq w} \frac{|f(z)-f(w)|}{|z-w|^\alpha},$$

where $\|f\|_\infty = \sup_{z\in\mathbb{D}} |f(z)|$, and λ_α is a Banach algebra.

The problem of the description of closed ideals of Banach algebras of analytic functions has been considered by several authors (see, for instance, [3, 7, 14–16, 19]). Let $X \subset A(\mathbb{D})$ be a Banach algebra, and let $\mathcal{I}$ be a nontrivial closed ideal of X. The inner factor of $\mathcal{I}$ will be denoted by $\theta_\mathcal{I}$. The zero set of $\mathcal{I}$, denoted by $E_\mathcal{I}$, is given by

$$E_\mathcal{I} = \{z \in \mathbb{T} : \ f(z) = 0, \quad \forall f \in \mathcal{I}\}.$$

We say that a closed ideal $\mathcal{I}$ of X is standard if

$$\mathcal{I} = \mathcal{J}(\theta_\mathcal{I}, E_\mathcal{I}),$$

where $\mathcal{J}(\theta_\mathcal{I}, E_\mathcal{I}) := \{f \in X : \ f_{|E_\mathcal{I}} = 0, \text{ and } f \in \theta_\mathcal{I} X\}$.

It is known that for the algebras $A(\mathbb{D})$, λ_α, and $\mathcal{B}_p^s$, where $p > 1$ and $\frac{1}{p} < s < 1$, all closed ideals are standard [16, 18, 19]. However, for the Banach algebra $\mathcal{B}_p^s \cap A(\mathbb{D})$, equipped with the canonical norm, it is still unknown if such result remains true, even for $p = 2$ and $s = 1/2$. This problem is related to the Brown–Shields conjecture (see [10, 11, 13]).

In the sequel, we consider the Banach algebra $\mathcal{B}_p^s \cap \lambda_\alpha$ equipped with the norm

$$\|f\|_{\mathcal{B}_p^s\cap\lambda_\alpha} := \mathcal{B}_p^s(f)^{1/p} + \|f\|_{\lambda_\alpha}, \qquad f \in \mathcal{B}_p^s \cap \lambda_\alpha.$$

In this paper, we prove that all closed ideals of the algebra $\mathcal{B}_p^s \cap \lambda_\alpha$ are standard. Namely, we have the following theorem.

Theorem 1.1 *Let $1 < p < \infty$, and let $0 < s < 1$. If $\mathcal{I}$ is a nontrivial closed ideal of $\mathcal{B}_p^s \cap \lambda_\alpha$, then*

$$\mathcal{I} = \{f \in \mathcal{B}_p^s \cap \lambda_\alpha : \ f_{|E_\mathcal{I}} = 0, \ \textit{and } f \in \theta_\mathcal{I}\mathcal{B}_p^s \cap \lambda_\alpha\}.$$

Note that the present result is only known for a limited range of indices, specifically for $p = 2$ and $s \in (0, 1/2]$ (see [3, 7]). More useful remarks are given in Section 4.

The nontrivial part of the proof of Theorem 1.1 is the inclusion $\mathcal{J}(\theta_\mathcal{I}, E_\mathcal{I}) \subset \mathcal{I}$. All the difficulties are overcome in two major steps.

(1) Establish that functions $g \in \mathcal{J}(\theta_\mathcal{I}, E_\mathcal{I})$ that decay rapidly to 0 as we approach $E_\mathcal{I}$, belong to $\mathcal{I}$. The set of such functions is denoted by $\mathcal{J}_0(\theta_\mathcal{I}, E_\mathcal{I})$. This step is achieved by a spectral synthesis theorem, which is proved with a careful analysis of the properties of the annihilator $\mathcal{I}^\perp$. We omit the proof here (more details can be found in [6, 19]).

(2) Prove that $\mathcal{J}_0(\theta_\mathcal{I}, E_\mathcal{I})$ is dense in $\mathcal{J}(\theta_\mathcal{I}, E_\mathcal{I})$.

The combination of these two steps gives the required inclusion and the consequences mentioned above.

Below, we proceed to prove the second point. Taking advantage of the method, based on cutoff functions, introduced in [3] and on an adequate expression of the

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norm of $\mathcal{B}_p^s \cap \lambda_\alpha$ due to Dyakonov and Boe (see [5, 9]), we provide an approximation theorem for this class of algebras. For more details, see Section 3.

Throughout the paper, we use the following notation:

- $A \lesssim B$ means that there is a constant C such that $A \le CB$.
- $A \asymp B$ means both $A \lesssim B$ and $B \lesssim A$.

2 Equivalent norms and cutoff functions

2.1 Equivalent norms

Given a function $f \in L^1(\mathbb{T})$, we denote by $P(f)$ the Poisson integral of f on $\mathbb{T}$,

$$P(f)(z) := \int_{\mathbb{T}} f(\zeta) d\mu_z(\zeta), \qquad (z \in \mathbb{D}),$$

where $d\mu_z(\zeta) := \frac{1-|z|^2}{|\zeta - z|^2} dm(\zeta)$.

For $f \in \mathrm{H}^1$ and $z \in \mathbb{D}$, write

$$\Psi(f,z) := P(|f|)(z) - |f(z)|,$$

and

$$\Phi(f,z) := \int_{\mathbb{T}} ||f(\zeta)| - P(|f|)(z)| d\mu_z(\zeta).$$

For $1 < p < \infty$ and $0 < s < 1$, the norm in $\mathcal{B}_p^s$ can be expressed only in terms of the modulus of functions. Namely, we have

(2.1)
$$\|f\|_{\mathcal{B}_p^s}^p \asymp |f(0)|^p + \int_0^1 \left\{ \int_{\mathbb{T}} (\Psi(f, r\zeta)^p + \Phi(f, r\zeta)^p)\, dm(\zeta) \right\} (1-r)^{-(ps+1)} dr.$$

In particular, if $2 \le p < \infty$ and $0 < s < 1/2$, then we get

$$\|f\|_{\mathcal{B}_p^s}^p \asymp |f(0)|^p + \int_0^1 \left\{ \int_{\mathbb{T}} \Psi(f^2, r\zeta)^{p/2} dm(\zeta) \right\} (1-r)^{-(ps+1)} dr.$$

These formulas were stated in [5, 9].

In what follows, we will use an equivalent norm in λ_α given in [3]. For any $f \in \lambda_\alpha$, we have

$$\|f\|_{\lambda_\alpha} = \|f\|_\infty + \sup_{\zeta_1, \zeta_2 \in \mathbb{T}, \zeta_1 \neq \zeta_2} \frac{||f(\zeta_1)| - |f(\zeta_2)||}{|\zeta_1 - \zeta_2|^\alpha} + \sup_{z \in \mathbb{D}} \frac{\Psi(f,z)}{(1-|z|)^\alpha}.$$

Let $f \in \mathrm{H}^1$ be an outer function, and let θ be an inner function. It is clear that, for $z \in \mathbb{D}$,

$$\Psi(f,z) = \Psi(\theta f, z) + |f(z)|(1 - |\theta(z)|) \le \Psi(\theta f, z), \tag{2.2}$$

and

$$\Phi(f,z) = \Phi(\theta f, z). \tag{2.3}$$

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It follows from (2.2) that

$$\|f\|_{\lambda_\alpha} \leq \|\theta f\|_{\lambda_\alpha}. \tag{2.4}$$

As a consequence of (2.2)–(2.4), the algebra $\mathcal{B}_p^s \cap \lambda_\alpha$ possesses the F-property. Namely, if $\theta f \in \mathcal{B}_p^s \cap \lambda_\alpha$, then

$$f \in \mathcal{B}_p^s \cap \lambda_\alpha \quad \text{and} \quad \|f\|_{\mathcal{B}_p^s \cap \lambda_\alpha} \lesssim \|\theta f\|_{\mathcal{B}_p^s \cap \lambda_\alpha}.$$

The involved constant depends only on s, p, and α.

2.2 Cutoff functions

Let f, g be two outer functions. Let $f \wedge g$, $f \vee g$ be the two outer functions associated with $|f| \wedge |g|(e^{it}) := \min(|f(e^{it})|, |g(e^{it})|)$ and $|f| \vee |g|(e^{it}) := \max(|f(e^{it})|, |g(e^{it})|)$, respectively. Namely, for $z \in \mathbb{D}$,

$$f \wedge g(z) = \exp\left(\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} \log(|f| \wedge |g|(e^{it}))dm(e^{it}) \right),$$

and

$$f \vee g(z) = \exp\left(\int_{\mathbb{T}} \frac{e^{it}+z}{e^{it}-z} \log(|f| \vee |g|(e^{it}))dm(e^{it}) \right).$$

The following inequalities were obtained in [3, 4], for $z \in \mathbb{D}$:

- $\Psi(f \wedge g, z) \leq \Psi(f, z) + \Psi(g, z)$.
- $\Psi(f \vee g, z) \leq \Psi(f, z) + \Psi(g, z)$.
- $\Psi(f \wedge f^\sigma, z) \leq \sigma^2 \Psi(f, z)$, $\sigma \geq 1$.

The main purpose of this section is to show that Φ satisfies also these inequalities. For this end, we will use the following identity several times:

$$\frac{1}{2}\Phi(f,z) = \int_{\Gamma(f)} (|f(\zeta)| - P(|f|)(z))d\mu_z(\zeta) = \int_{\Gamma^c(f)} (P(|f|)(z) - |f(\zeta)|)d\mu_z(\zeta), \tag{2.5}$$

with $\Gamma(f) := \{\zeta \in \mathbb{T} : |f(\zeta)| \geq P(|f|)(z)\}$ and $\Gamma^c(f) := \mathbb{T} \setminus \Gamma(f)$.

Theorem 2.1 *Let $f, g \in \mathrm{H}^1$ be two outer functions and $z \in \mathbb{D}$. Then, we have*

(i) $\Phi(f \wedge g, z) \leq \Phi(f, z) + \Phi(g, z)$,
(ii) $\Phi(f \vee g, z) \leq \Phi(f, z) + \Phi(g, z)$, *and*
(iii) $\Phi(f \wedge f^\sigma, z) \leq \sigma\Phi(f, z), \sigma \geq 1$.

As a consequence of the previous theorem and the formula (2.1), we obtain the following corollary.

Corollary 2.2 *Let $1 < p < \infty$, and let $0 < s < 1$. Let $f, g \in \mathcal{B}_p^s$ be two outer functions and $z \in \mathbb{D}$. Then, we have*

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(i) $\|f \wedge g\|_{\mathcal{B}_p^s} \lesssim \|f\|_{\mathcal{B}_p^s} + \|g\|_{\mathcal{B}_p^s}$,
(ii) $\|f \vee g\|_{\mathcal{B}_p^s} \lesssim \|f\|_{\mathcal{B}_p^s} + \|g\|_{\mathcal{B}_p^s}$, *and*
(iii) $\|f \wedge f^\sigma\|_{\mathcal{B}_p^s} \lesssim \sigma^2 \|f\|_{\mathcal{B}_p^s}$, $\sigma \geq 1$.

The involved constants depend only on s and p.

Proof Write $A := \{\zeta \in \mathbb{T} : |f(\zeta)| \geq |g(\zeta)|\}$ and $A^c := \{\zeta \in \mathbb{T} : |f(\zeta)| < |g(\zeta)|\}$. Let $z \in \mathbb{D}$.

(i) Note that

$$\Gamma^c(f \wedge g) \cap A \subset \{\zeta \in A : |g| \leq P(|g|)(z)\} = \Gamma^c(g) \cap A,$$

and

$$\Gamma^c(f \wedge g) \cap A^c \subset \{\zeta \in A^c : |f| \leq P(|f|)(z)\} = \Gamma^c(f) \cap A^c.$$

Thus, from the identity (2.5), we have

$$\begin{aligned}
\Phi(f \wedge g, z) &= 2\int_{\Gamma^c(f\wedge g)\cap A} (P(|f \wedge g|)(z) - |g|)\, d\mu_z \\
&\quad + 2\int_{\Gamma^c(f\wedge g)\cap A^c} (P(|f \wedge g|)(z) - |f|)\, d\mu_z \\
&\leq 2\int_{\Gamma^c(g)\cap A} (P(|g|)(z) - |g|) d\mu_z + 2\int_{\Gamma^c(f)\cap A^c} (P(|f|)(z) - |f|) d\mu_z \\
&\leq 2\int_{\Gamma^c(g)} (P(|g|)(z) - |g|) d\mu_z + 2\int_{\Gamma^c(f)} (P(|f|)(z) - |f|) d\mu_z \\
&= \Phi(g, z) + \Phi(f, z).
\end{aligned}$$

(ii) As above, we can see that

$$\Gamma(f \vee g) \cap A \subset \Gamma(f) \cap A \quad \text{and} \quad \Gamma(f \vee g) \cap A^c \subset \Gamma(g) \cap A^c.$$

By the same argument used in the proof of (i), we get (ii).

(iii) Let $\sigma \geq 1$. Suppose $P(|f|)(z) \geq 1$. On the one hand, from (2.5), we have

$$\begin{aligned}
\Phi(f \wedge f^\sigma, z) &= 2\int_{\Gamma^c(f\wedge f^\sigma)} (P(|f \wedge f^\sigma|)(z) - |f \wedge f^\sigma|) d\mu_z \\
&\leq 2\int_{\Gamma^c(f\wedge f^\sigma)\cap\{|f|<1\}} (P(|f|)(z) - |f^\sigma|) d\mu_z \\
&\quad + 2\int_{\Gamma^c(f\wedge f^\sigma)\cap\{|f|\geq 1\}} (P(|f|)(z) - |f|) d\mu_z \\
&\leq 2\sigma \int_{\Gamma^c(f\wedge f^\sigma)\cap\{|f|<1\}} (P(|f|)(z) - |f|) d\mu_z \\
&\quad + 2\int_{\Gamma^c(f\wedge f^\sigma)\cap\{|f|\geq 1\}} (P(|f|)(z) - |f|) d\mu_z.
\end{aligned}$$

The last inequality comes from the fact that $y - x^\sigma \leq \sigma(y - x)$ for $x \in [0,1]$ and $y \geq 1$. On the other hand, one can remark that

$$\Gamma^c(f \wedge f^\sigma) \cap \{|f| \geq 1\} \subset \Gamma^c(f) \cap \{|f| \geq 1\},$$

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and

$$\Gamma^c(f \wedge f^\sigma) \cap \{|f| < 1\} \subset \Gamma^c(f) \cap \{|f| < 1\}.$$

Hence, by considering these inclusions in the previous inequality, we obtain

$$\begin{aligned}\Phi(f \wedge f^\sigma, z) \le 2\sigma &\int_{\Gamma^c(f)\cap\{|f|<1\}} (P(|f|)(z) - |f|)d\mu_z \\ &+ 2\int_{\Gamma^c(f)\cap\{|f|\ge 1\}} (P(|f|)(z) - |f|)d\mu_z \\ &\le \sigma\Phi(f, z).\end{aligned}$$

This completes the proof in the case where $P(|f|)(z) \ge 1$. Now, consider the case $P(|f|)(z) < 1$. Remark that we have

$$\Gamma^c(f \wedge f^\sigma) \subset \{|f| < 1\}.$$

Here, we discuss two cases. First, we assume that $P(|f \wedge f^\sigma|)(z) \le [P(|f|)(z)]^\sigma$. We have $\Gamma^c(f \wedge f^\sigma) \subset \Gamma^c(f)$ and

$$[P(|f|)(z)]^\sigma - |f|^\sigma \le \sigma(P(|f|)(z) - |f|) \text{ on } \Gamma^c(f \wedge f^\sigma).$$

The inequality comes from the elementary inequality $x^\sigma - y^\sigma \le \sigma(x - y)$ for $0 \le y \le x \le 1$. Indeed,

$$\begin{aligned}\Phi(f \wedge f^\sigma, z) &= 2\int_{\Gamma^c(f\wedge f^\sigma)} (P(|f \wedge f^\sigma|)(z) - |f^\sigma|)d\mu_z \\ &\le 2\int_{\Gamma^c(f\wedge f^\sigma)} ([P(|f|)(z)]^\sigma - |f|^\sigma)\, d\mu_z \\ &\le 2\sigma\int_{\Gamma^c(f\wedge f^\sigma)} (P(|f|)(z) - |f|)d\mu_z \\ &\le \sigma\Phi(f, z).\end{aligned}$$

Finally, suppose that $[P(|f|)(z)]^\sigma \le P(|f \wedge f|^\sigma)(z)$. Note that $\Gamma(f \wedge f^\sigma) \subset \Gamma(f)$. Thus, by the identity (2.5), we have

$$\begin{aligned}\Phi(f \wedge f^\sigma, z) &= 2\int_{\Gamma(f\wedge f^\sigma)} (|f \wedge f^\sigma| - P(|f \wedge f^\sigma|)(z))d\mu_z \\ &\le 2\int_{\Gamma(f\wedge f^\sigma)} (|f \wedge f^\sigma| - [P(|f|)(z)]^\sigma)d\mu_z \\ &= 2\int_{\Gamma(f\wedge f^\sigma)\cap\{|f|\ge 1\}} (|f| - [P(|f|)(z)]^\sigma)d\mu_z \\ &\quad + 2\int_{\Gamma(f\wedge f^\sigma)\cap\{|f|<1\}} (|f|^\sigma - [P(|f|)(z)]^\sigma)d\mu_z \\ &\le 2\sigma\int_{\Gamma(f)\cap\{|f|\ge 1\}} (|f| - P(|f|)(z))d\mu_z \\ &\quad + 2\sigma\int_{\Gamma(f)\cap\{|f|<1\}} (|f| - P(|f|)(z))d\mu_z \\ &= \sigma\Phi(f, z).\end{aligned}$$

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The second inequality comes from the two elementary inequalities

$$y - x^\sigma \le \sigma(y - x), \quad (x, y) \in [0,1] \times [1, +\infty[,$$

and

$$x^\sigma - y^\sigma \le \sigma(x - y), \quad 0 \le y \le x \le 1.$$

The proof is complete. ∎

3 Approximation theorem

Let $p > 1$, and let $s, \alpha \in (0,1)$. The aim in this section is to provide an approximation theorem for $\mathcal{B}_p^s \cap \lambda_\alpha$.

Let $f \in \mathrm{H^p}$ be an outer function, and let θ be an inner function. It was mentioned in [1] that

$$\Psi(\theta(1 \wedge f), z) \le \Psi(\theta f, z), \quad z \in \mathbb{D}.$$

By combining (2.3) and Theorem 2.1, we can easily get

$$\Phi(\theta(1 \wedge f), z) \le \Phi(\theta f, z), \quad z \in \mathbb{D}.$$

By considering both of inequalities together with (2.1), we obtain

$$\|\theta(1 \wedge f)\|_{\mathcal{B}_p^s} \lesssim \|\theta f\|_{\mathcal{B}_p^s}.$$

Note that

$$||(1 \wedge f)(\zeta_1)| - |(1 \wedge f)(\zeta_2)|| \le ||f(\zeta_1)| - |f(\zeta_2)||, \qquad (\zeta_1, \zeta_2 \in \mathbb{T}).$$

Thus, we finally get

$$\|\theta(1 \wedge f)\|_{\lambda_\alpha} \le \|\theta f\|_{\lambda_\alpha}.$$

As a consequence of the above discussion, we obtain the following lemma.

Lemma 3.1 *Let f be an outer function, and let θ be an inner function such that $\theta f \in \mathcal{B}_p^s \cap \lambda_\alpha$. Then, $\theta(1 \wedge f) \in \mathcal{B}_p^s \cap \lambda_\alpha$ and*

$$\|\theta(1 \wedge f)\|_{\mathcal{B}_p^s \cap \lambda_\alpha} \lesssim \|\theta f\|_{\mathcal{B}_p^s \cap \lambda_\alpha},$$

where the involved constant depends only on α, s, and p.

Theorem 3.2 *Let $f \in \mathcal{B}_p^s \cap \lambda_\alpha$ be a function that vanishes on a closed subset E of $\mathbb{T}$. Then, given a constant $M > 0$, there exists a sequence $(f_n)_{n \ge 1}$ of $\mathcal{B}_p^s \cap \lambda_\alpha$ such that*

(1) $|f_n(z)| = O(\operatorname{dist}(z, E)^M)$, *for all $n \ge 1$, and*
(2) $\lim_{n \to +\infty} \|f_n f - f\|_{\mathcal{B}_p^s \cap \lambda_\alpha} = 0.$

Proof Let $f = \theta g \in \mathcal{B}_p^s \cap \lambda_\alpha \backslash \{0\}$, where θ and g are, respectively, the inner and outer factors of f. By assumption f vanishes on E. Since λ_α possesses the F-property,

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then $g \in \lambda_\alpha$. Thus, we have

$$|g(z)| = O(\mathrm{dist}(z, E)^\alpha).$$

Let $n \geq 1$, and for $\sigma = 1 + M/\alpha$, we put $f_n = 1 \wedge n^{\sigma-1} g^{\sigma-1}$. Clearly, we have

$$|f_n(z)| \leq n^{\sigma-1}|g(z)|^{\sigma-1} = O(\mathrm{dist}(z, E)^{\alpha(\sigma-1)}) = O(\mathrm{dist}(z, E)^M).$$

Using Lemma 3.1 and that $\mathcal{B}_p^s \cap \lambda_\alpha$ possesses the F-property, we obtain $f_n \in \mathcal{B}_p^s \cap \lambda_\alpha$. The sequence $f f_n = \theta(g \wedge n^{\sigma-1} g^{\sigma-1})$ converges uniformly to f on any compact subset of $\mathbb{D}$.

The sequence $(f f_n)$ converges to f in λ_α (see [3]). It remains to prove that $(f f_n)$ converges to f in $\mathcal{B}_p^s$. To this end, we write

$$\begin{aligned}
\Phi(f f_n, z) &= \Phi(\theta(g \wedge n^{\sigma-1} g^\sigma), z) \\
&= \Phi(g \wedge n^{\sigma-1} g^\sigma, z) \\
&= \frac{1}{n}\Phi(ng \wedge (ng)^\sigma, z) \\
&\leq \frac{\sigma}{n}\Phi(ng, z) = \sigma\Phi(g, z) = \sigma\Phi(f, z).
\end{aligned}$$

As a fact of matter, we know from [3] that

$$\Psi(f f_n, z) \leq \sigma^2 \Psi(f, z).$$

Hence, we get

$$\mathcal{B}_p^s(f f_n) \leq \sigma^2 \mathcal{B}_p^s(f). \tag{3.1}$$

Since $p > 1$, $\mathcal{B}_p^s$ is reflexive, and we obtain the desired result using the same argument stated in [3]. ∎

4 Some remarks

Let $\alpha \in (0, 1)$, and the analytic Lipschitz algebra Λ_α is defined by

$$\Lambda_\alpha := \{f \in A(\mathbb{D}) : |f(z) - f(w)| = O(|z - w|^\alpha) \text{ as } |z - w| \text{ tends to } 0\}.$$

A theorem of Hardy and Littlewood [8, 17] states that, $f \in \Lambda_\alpha$ if and only if

$$|f'(z)| = O((1 - |z|)^{\alpha-1}), \quad \text{as } |z| \to 1^-.$$

So, Λ_α endowed with the norm

$$\|f\|_{\Lambda_\alpha} = \|f\|_\infty + \sup_{z\in\mathbb{D}}(1 - |z|)^{1-\alpha}|f'(z)|$$

is a Banach algebra.

For $p > 1$ and $s \in (0, 1)$. We would like to know under which conditions $\mathcal{B}_p^s$ might be a Banach algebra. Note that if

$$\mathcal{B}_p^s \subset \Lambda_\alpha, \qquad \text{for some } \alpha \in (0, 1), \tag{4.1}$$

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then $\mathcal{B}_p^s$ becomes a Banach algebra. In this case, Theorem 1.1 gives a complete description of closed ideals of $\mathcal{B}_p^s$.

The purpose of the following result is to give a sufficient condition to ensure the inclusion (4.1).

Proposition 4.1 *For any $p > 1$ and $s \in (0,1)$, the following statements hold.*

(1) *If $ps < 1$, then $\Lambda_{\frac{1}{p}} \subset \mathcal{B}_p^s$.*

(2) *If $ps > 1$, then $\mathcal{B}_p^s \subset \Lambda_{(ps-1)/p}$.*

The following result is known, and we give the proof below for the sake of completeness.

Let $\beta_1, \beta_1 \in (0,1)$ such that $\beta_1 < \beta_2$. Thus, we have $\Lambda_{\beta_2} \subset \lambda_{\beta_1}$. Taking advantage of this remark and Proposition 4.1 to obtain the following.

Remark 4.2 For any $p > 1$ and $s \in (0,1)$, the following statements hold.

(i) If $ps < 1$ and $\alpha \in (s,1)$, then

$$\mathcal{B}_p^s \cap \lambda_\alpha = \lambda_\alpha.$$

In such situation, Theorem 1.1 gives us the description of closed ideals of λ_α stated already in [16].

(ii) If $ps > 1$ and $\alpha \in (0, s - \frac{1}{p}]$, then

$$\mathcal{B}_p^s \cap \lambda_\alpha = \mathcal{B}_p^s.$$

In particular, $\mathcal{B}_p^s$ is a Banach algebra. Furthermore, in this case, we recover Theorem 1 of [19] from Theorem 1.1.

Proof (1) Assume that $ps < 1$. Let $f \in \Lambda_{\frac{1}{p}}$, so we have

$$|f'(z)| = O((1-|z|)^{\frac{1}{p}-1}), \quad \text{as } |z| \to 1^-.$$

Thus, we get

$$|f'(z)|^p (1-|z|)^{p(1-s)-1} = O((1-|z|)^{-ps}), \quad \text{as } |z| \to 1^-.$$

In particular, $\mathcal{B}_p^s(f) = O(1)$. It follows that $\Lambda_{\frac{1}{p}} \subset \mathcal{B}_p^s$.

(2) Let $f \in \mathcal{B}_p^s$. Obviously, one can assume that $1/2 < |z| < 1$. The mean value property confirms that

$$f'(z) = \frac{4}{(1-|z|)^2} \int_{\mathbb{D}(z)} f'(w)\, dA(w),$$

where $\mathbb{D}(z) := \{w \in \mathbb{D} : |z-w| < \frac{1-|z|}{2}\}$. Thus, using Jensen's inequality, we have

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$$
\begin{aligned}
|f'(z)| &\leq \left(4 \int_{\mathbb{D}(z)} |f'(w)|^p \frac{dA(w)}{(1-|z|)^2}\right)^{1/p} \\
&\asymp (1-|z|)^{s-\frac{1}{p}-1} \left(4 \int_{\mathbb{D}(z)} |f'(w)|^p (1-|z|)^{p(1-s)-1} dA(w)\right)^{1/p} \\
&\lesssim \mathcal{B}_p^s(f)^{1/p} (1-|z|)^{(ps-1)/p-1}.
\end{aligned}
$$

Therefore, if $ps > 1$, we obtain the desired inclusion $\mathcal{B}_p^s \subset \Lambda_{(ps-1)/p}$. ■

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