ON PARTIAL DIFFERENTIAL EQUATIONS IN A FIELD OF PRIME CHARACTERISTIC

ARNO JAEGER

In classical analysis ordinary differential equations and partial differential equations are distinct concepts, and the transition from one derivation to several partial derivations changes some of their properties distinctly. On the other hand, the algebraic theories of modified ordinary and partial differential equations **(5; 6)**, based on the differentiations in the sense of Hasse **(2)** and Schmidt **(3)** and the multidifferentiations in the sense of Jaeger **(4)**, turn out to be strikingly similar in the case of fields of prime number characteristic. However, the differential equations in fields of prime number characteristic in the usual, unmodified sense are special cases of the modified ones because of the relations between their respective operators; hence this similarity must also appear in the unmodified case. In the following, an easy explanation for this similarity is given, provided that only derivations of a separably generated algebraic function field F in n independent indeterminates over its ground field K are considered. For in this case it is shown that all partial differential equations can be replaced by ordinary differential equations in a suitable derivation D of F over K. This D is simply constructed in such a way that the set

$$(D, D^p, D^{p^2}, \dots , D^{p^{n-1}})$$

is a basis for the F-module of all derivations of F over K. An explicit example of such a "replacing derivation" is given in this paper.

Let F be a separably generated algebraic function field of n independent indeterminates of prime number characteristic $p > 0$ with a separating transcendence basis $x = (x_1, x_2, \dots , x_n)$ over its ground field K. Let D_i $(i = 1, 2, \dots , n)$ be the partial derivations of F over K uniquely defined **(1)** by $D_i(x_j) = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta. The linear mappings D_i of F into itself are commutative under multiplication; the abbreviation

$$D^\alpha = \prod_{i=1}^{n} D_i^{\alpha_i}$$

will be used for their general products where $\alpha = (\alpha_1, \alpha_2, \dots , \alpha_n)$ is a vector whose components are non-negative integers and

$$D_i^0 \qquad (i = 1, 2, \dots , n)$$

denotes the identity mapping 1. The multisequence $\{D^\alpha\}$ is called a *multi-differentiation of dimension n*; if $n = 1$ it is also called a *differentiation.*

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The formal expressions $A = \sum_\alpha a_\alpha D^\alpha$ ($a_\alpha \in F$) where the sum is extended over finitely many distinct vectors α whose components are non-negative integers are called *multidifferential operators in D* (or also *differential operators* if $n = 1$). They define mappings

$$y \to A(y) = \sum_\alpha a_\alpha D^\alpha(y)$$

of F into itself and form a ring $\Omega = \Omega(D, F)$ of mappings when addition is defined in the usual way and multiplication by $D_i y = D_i(y) D_i^0 + y D_i$ ($y \in F$; $i = 1, 2, \ldots, n$). The zero element of this ring is denoted by 0.

LEMMA 1. *The operators*

$$D_i^q \qquad (i = 1, 2, \ldots, n)$$

where $q = p^t$ and t is any positive integer, map every element of F onto zero.

Proof. The operator defines a derivation of F over K. It has the property that

$$D_i^q(x_j) = 0 \qquad (j = 1, 2, \ldots, n);$$

hence, its restriction to $K[x]$ is the trivial derivation (i.e. the zero operator). But the prolongation of a derivation of $K[x]$ to F is unique **(1)**, and hence this prolongation is a zero operator.

Let P be the two-sided ideal of Ω generated by

$$D_1^p, D_2^p, \ldots, D_n^p.$$

If $a_\alpha \neq 0$ and $D^\alpha \notin P$ the number $\sum \alpha_i$ is called the *order of $a_\alpha D^\alpha$*. Let $A = \sum_\alpha a_\alpha D^\alpha$ be a multidifferential operator which does not lie in P, then the minimum of the orders of the additive terms $a_\alpha D^\alpha$ for which $a_\alpha \neq 0$ and $D^\alpha \notin P$ holds is called the *minimal order m_A of A*. If all terms $a_\alpha D^\alpha$ of A with $a_\alpha \neq 0$ have the same order the operator A is called *homogenous of order m_A*. Order and minimal order are not defined for elements of P.

LEMMA 2. *No multidifferential operator $A = \sum_\alpha a_\alpha D^\alpha$ of minimal order $m_A > 1$ defines a derivation.*

Proof. We have obviously $A(x_i) = 0$ ($i = 1, 2, \ldots, n$). Suppose now that A is a derivation, then it must follow that also $A(x^\mu) = 0$ where

$$x^\mu = \prod_{i=1}^n x_i^{\mu_i}, \qquad \mu = (\mu_1, \mu_2, \ldots, \mu_n)$$

can be chosen such that

$$\sum_{i=1}^n \mu_i = m_A,$$

all $\mu_i < p$ and $a_\mu \neq 0$. But it can be seen easily that

$$D^\alpha(x^\mu)\begin{cases} \neq 0 \text{ if } & \alpha = \mu \\ = 0 \text{ if } & \sum \alpha_i = m_A, \text{ but } \alpha \neq \mu \\ = 0 \text{ if } & \sum \alpha_i > m_A. \end{cases}$$

Hence we have $A(x^\mu) = \sum_\alpha a_\alpha D^\alpha(x^\mu) \neq 0$, a contradiction.

COROLLARY 1. *Let H be the set of all multidifferential operators of minimal orders greater than* 1. *Then $I = H \cup P$ is a left-ideal of Ω.*

COROLLARY 2. *A multidifferential operator A is a zero operator if and only if $A \in P$.*

Since two multidifferential operators A and B define the same mapping of F into itself if and only if $A \equiv B \pmod{P}$, it is sufficient to consider the operator ring Ω/P. A multidifferential operator $A = \sum_\alpha a_\alpha D^\alpha$ is called *reduced with respect to P* if none of its non-zero terms $a_\alpha D^\alpha$ lies in P; the reduced operators can be taken as representatives of the residue classes of Ω/P. Furthermore we can identify the additive group, consisting of the operator 0 and all homogenous multidifferential operators of order 1, with the F-module $M = M(D, F)$ of all derivations of F over K.

COROLLARY 3. *If $A^q \equiv A_1 \pmod{I}$ holds with $q = p^t$ for some positive integer t and $A_1 \in M$, then $A^q \equiv A_1 \pmod{P}$.*

Proof. A^q and A_1 and hence, $A^q - A_1$ define derivations. Since all derivations induced by elements of I are trivial it follows that $A^q - A_1 \in P$.

THEOREM. *There exists a differentiation d of F over K such that for each multidifferential operator A of $\Omega(D, F)$ there exists a differential operator $\bar{A}$ of $\Omega(d, F)$ which defines the same mapping of F into itself as A.*

Proof. It is sufficient to show the existence of a basis $d = (d_1, d_2, \ldots, d_n)$ of the F-module M such that $d^q \equiv d_{t+1} \pmod{P}$ holds for $q = p^t, t = 1, 2, \ldots, n-1$.

Example. Take $d = \sum x_i^{p+1} D_i$, then it follows that

$$d^k \equiv \sum_{i=1}^{n} x_i^{kp+1} D_i \qquad \pmod{I}$$

(k any positive integer), and, by using Corollary 3, we have especially

$$d^q \equiv \sum_{i=1}^{n} x_i^{qp+1} D_i \qquad \pmod{P}.$$

But the elements

$$d_k = \sum_{i=1}^{n} x_i^{p^k+1} D_i \qquad k = 1, 2, \ldots, n$$

form a basis of the F-module M since

$$\det(x_i^{p^k+1})_{i,k=1,2,\ldots,n} \neq 0.$$

COROLLARY. *Every partial differential equation in F over K can be written as an ordinary differential equation in F over K.*

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University of Cincinnati

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