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THE *N*-SOLITON SOLUTION OF A GENERALISED VAKHNENKO EQUATION

A. J. MORRISON and E. J. PARKES

Department of Mathematics, University of Strathclyde, Glasgow G1 1XH, UK
e-mail: ta.amor@maths.strath.ac.uk and ejp@maths.strath.ac.uk

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Abstract. The N-soliton solution of a generalised Vakhnenko equation is found, where N is an arbitrary positive integer. The solution, which is obtained by using a blend of transformations of the independent variables and Hirota's method, is expressed in terms of a Moloney & Hodnett (1989) type decomposition. Different types of soliton are possible, namely loops, humps or cusps. Details of the different types of interactions between solitons, including resonant soliton interactions, are discussed in detail for the case $N = 2$. A proof of the 'N-soliton condition' is given in the Appendix.

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1. Introduction. In [**1**] and [**2**] loop soliton solutions of the nonlinear evolution equation

$$\frac{\partial}{\partial x}\mathcal{D}u + u = 0, \tag{1.1}$$

where

$$\mathcal{D} := \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \tag{1.2}$$

hereafter referred to as the Vakhnenko equation (VE), were discussed. The key step in finding these solutions was to transform the independent variables in (1.1). This led to an equation that can be expressed in bilinear form in terms of the Hirota D operator [**3**]. This equation is a very basic version of Ito's equation [**4**, equation (A.1) with (B.10)]. It was straightforward to find the exact explicit N-soliton solution to this equation by use of Hirota's method for a general positive integer $N \geq 2$. The solution was expressed in terms of a decomposition first proposed by Moloney & Hodnett [**5**] in the context of the Korteweg-de Vries equation. The exact N loop soliton solution to the VE was then found in implicit form by means of a transformation back to the original independent variables.

The aim of the present paper is to consider a more general version of Ito's equation, again in terms of the transformed variables. This equation can be transformed back to the original variables to give what we shall call the Generalised Vakhnenko Equation (GVE). The exact N-soliton solution to the GVE can then be found in implicit form.

In §2 we summarise the transformation of the VE into an equation in bilinear form. In §3 we consider the more general version of Ito's equation and find the GVE. In §4 we discuss the 1-soliton solution of the GVE and we find that different types of soliton are possible, namely loops, humps and cusps. In §5 we find the N-soliton solution. In §6 we interpret this solution in terms of the dynamics of individual solitons and we calculate the shift of each soliton due to its interaction with the other solitons. In §7 we illustrate our results by considering in detail the case $N=2$. A proof of the 'N-soliton condition' is given in the Appendix.

2. Transformation of the Vakhnenko equation. Here we summarise the transformation of the VE into an equation in bilinear form as described in [**1**].

We introduce new variables X, T defined by

$$x=\theta(X,T):=T+\int_{-\infty}^{X}U(X',T)dX'+x_0,\qquad t=X, \tag{2.1}$$

where $u(x,t)=U(X,T)$, and x_0 is a constant. We also introduce W defined by

$$W_X=U \tag{2.2}$$

and assume that, as $|X|\to\infty$, the derivatives of W vanish and W tends to a constant.

From (2.1), it follows that

$$\frac{\partial}{\partial X}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x},\qquad \frac{\partial}{\partial T}=(1+W_T)\frac{\partial}{\partial x}. \tag{2.3}$$

By using (2.2) and (2.3), we can express (1.1) in terms of our new variables X and T as

$$U_{XT}+U\int_{-\infty}^{X}U_T(X',T)dX'+U=0 \tag{2.4}$$

or equivalently,

$$W_{XXT}+W_XW_T+W_X=0. \tag{2.5}$$

By taking

$$W=6(\ln f)_X, \tag{2.6}$$

(2.5) may be written in Hirota bilinear form as

$$(D_TD_X^3+D_X^2)(f\cdot f)=0. \tag{2.7}$$

3. The generalised Vakhnenko equation. We can see that (2.7) is a very basic version of Ito's equation [**4**, equation (A.1) with (B.10)]. Now we shall consider a more general form of Ito's equation, namely

$$F(D_X,D_T)(f\cdot f)=0, \tag{3.1}$$

with

$$F(D_X, D_T) := D_T D_X^3 + D_X^2 + \beta D_X D_T, \tag{3.2}$$

where β is a free parameter. We shall now transform this equation into the GVE by reversing the process outlined in §2.

Introducing W as before, and observing that

$$W_X = \frac{3D_X^2 f \cdot f}{f^2}, \quad W_T = \frac{3D_X D_T f \cdot f}{f^2}, \quad \text{and} \quad W_{XXT} + W_X W_T = \frac{3D_T D_X^3 f \cdot f}{f^2}, \tag{3.3}$$

(3.1) with (3.2) may be written as

$$W_{XXT} + W_X W_T + W_X + \beta W_T = 0 \tag{3.4}$$

or equivalently in terms of $U(X, T)$ as

$$U_{XXT} + UU_T + U_X \int_{-\infty}^{X} U_T(X', T)dX' + U_X + \beta U_T = 0. \tag{3.5}$$

Using (2.3) we can transform (3.5) into the GVE, namely

$$\frac{\partial}{\partial x}\left(\mathcal{D}^2 u + \frac{1}{2}u^2 + \beta u\right) + \mathcal{D}u = 0 \tag{3.6}$$

or equivalently

$$\left(\frac{\partial u}{\partial x} + \mathcal{D}\right)\left(\frac{\partial}{\partial x}\mathcal{D}u + u + \beta\right) = 0. \tag{3.7}$$

Note that if $\beta = 0$, (3.7) can be reduced to the VE given by (1.1) as expected. For $\beta \neq 0$ we note that the GVE is not simply

$$\frac{\partial}{\partial x}\mathcal{D}u + u + \beta = 0. \tag{3.8}$$

This may be explained as follows. Using (2.3), (3.8) becomes

$$W_{XXT} + (1 + W_T)(W_X + \beta) = 0. \tag{3.9}$$

As noted earlier, we assume that, as $|X| \to \infty$, the derivatives of W vanish. However this means that (3.8) can only be satisfied for $\beta = 0$. Because of this we must take (3.7) as the GVE to allow $\beta \neq 0$.

The solution procedure for the GVE is as follows. We solve (3.1) with (3.2) for f by use of Hirota's method [**3**] and hence find $W(X, T)$ and $U(X, T)$ by using (2.6) and (2.2) respectively. The solution to the GVE is then given in parametric form by

$$u(x, t) = U(t, T), \qquad x = \theta(t, T), \tag{3.10}$$

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where

$$\theta(X, T) = T + W(X, T) + x_0. \tag{3.11}$$

4. The one-soliton solution of the generalised Vakhnenko equation. The solution to (3.1) corresponding to one soliton is given by

$$f = 1 + e^{2\eta}, \quad \text{where } \eta = kX - \omega T + \alpha, \tag{4.1}$$

and k, ω and α are constants. The dispersion relation is $F(2k, -2\omega) = 0$ from which we find that $\omega = k/(4k^2 + \beta)$ and then

$$\eta = k(X - cT) + \alpha \quad \text{with } c = 1/(4k^2 + \beta). \tag{4.2}$$

Note that c can be positive or negative depending on the value of β.

Substitution of (4.1) into (2.6) gives

$$W(X, T) = 6k(1 + \tanh \eta) \tag{4.3}$$

so that

$$U(X, T) = 6k^2 \operatorname{sech}^2 \eta. \tag{4.4}$$

The one-soliton solution to the GVE is given by (3.10) with (4.3) and (4.4). From (3.11) with $v = 1/c$ we have

$$x - vt = -v(X - cT) + 6k(1 + \tanh[k(X - cT) + \alpha]) + x_0. \tag{4.5}$$

Clearly, from (4.4) and (4.5), $U(X, T)$ and $x - vt$ are related by the parameter $X - cT$ so that $u(x, t)$ is a soliton that travels with speed $|v|$ in the positive x-direction if $\beta > -4k^2$ and in the negative x-direction if $\beta < -4k^2$.

From (2.3), (4.3) and (4.4) we can show that

$$u_x = -\frac{U_X}{v - U}. \tag{4.6}$$

As $X - cT$ goes from $-\infty$ to $+\infty$, U_X changes sign once and remains finite. Furthermore

- if $\beta/k^2 < -4$ or $\beta/k^2 > 2$, $v - U$ is never zero. Therefore we deduce that u is hump shaped;
- if $-4 < \beta/k^2 < 2$, $v - U = 0$ twice and so u_x changes sign 3 times and goes infinite twice. Therefore we deduce that u is loop shaped. Note that $\beta = 0$ lies in this region and corresponds to the loop soliton solutions of the VE;
- if $\beta/k^2 = 2$, $|u_x| \to \infty$ as $\eta \to 0$. Therefore we deduce that u is cusp shaped.

Finally, if we require symmetry in X,T-space i.e. $U(X, T) = U(-X, -T)$ we take $\alpha = 0$ in (4.2) and then, for symmetry in x,t-space, we take $x_0 = -6k$ in (3.11).

Now let us look in more detail at the shape of the loop soliton. Let w be the maximum width of the loop and H be the height at which this occurs. Note that this will be when $v - U = 0$. Furthermore, let h be the height at which the crossover point occurs. This is all summarised in Figure 1.

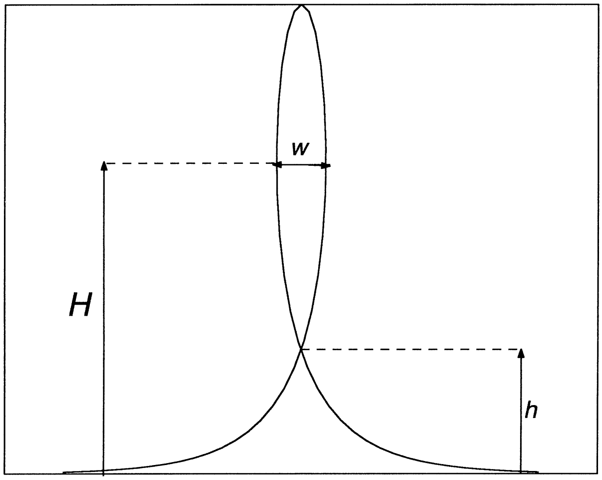

Figure 1. w, H, and h.

It is convenient to introduce q defined by $q := \beta/k^2$. Hence loops occur for $-4 < q < 2$. We shall begin by considering h/u_{MAX} for $-4 < q < 2$. Since $u = 6k^2\text{sech}^2\eta$, $u_{MAX} = 6k^2$. For simplicity, and without loss of generality, let us consider the symmetric case, i.e. $\alpha = 0$ and $x_0 = -6k$, and consider what happens at $t = 0$. Hence the crossover point will occur at $x = 0$. From (4.2), (4.4) and (4.5) the solution $u(x, 0)$ can be expressed in parametric form, with parameter η, as

$$u(\eta) = 6k^2\text{sech}^2\eta \tag{4.7}$$

$$x(\eta) = -\left(\frac{4k^2 + \beta}{k}\right)\eta + 6k\tanh\eta. \tag{4.8}$$

Suppose $x = 0$ when $\eta = \eta_1$ so that

$$\tanh\eta_1 = \left(\frac{2}{3} + \frac{q}{6}\right)\eta_1. \tag{4.9}$$

Hence, from (4.7), when $\eta = \eta_1$,

$$\frac{h}{6k^2} = 1 - \left(\frac{2}{3} + \frac{q}{6}\right)^2\eta_1^2. \tag{4.10}$$

We solve (4.9) numerically and then plot (4.10) for $-4 < q < 2$. This is shown in Figure 2.

We shall now consider H/u_{MAX} and w/k. Once again for simplicity we shall consider the symmetric case. As already mentioned, to obtain w and H we have to consider $v - U = 0$ so that

$$\cosh \eta = \sqrt{\frac{6k^2}{4k^2 + \beta}}. \tag{4.11}$$

When η satisfies (4.11), $u = H = 4k^2 + \beta$. Therefore

$$\frac{H}{6k^2} = \frac{2}{3} + \frac{q}{6}. \tag{4.12}$$

Clearly as $q \to -4$, $H/6k^2 \to 0$ and as $q \to 2$, $H/6k^2 \to 1$. A plot of $H/6k^2$ is shown alongside $h/6k^2$ in Figure 2.

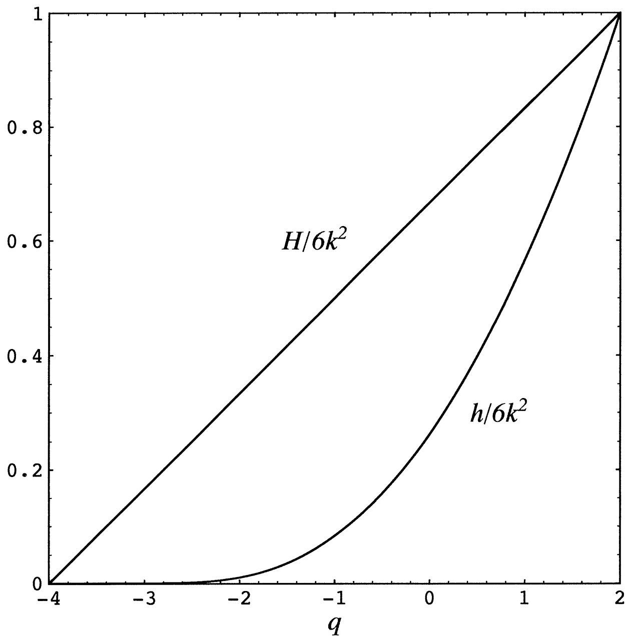

Figure 2. $h/6k^2$ and $H/6k^2$ for $-4 < q < 2$.

Finally, using (4.8) and (4.11), it can be shown that

$$\frac{w}{k} = -2(4+q)\ln\left(\sqrt{\frac{6}{4+q}} + \sqrt{\frac{2-q}{4+q}}\right) + 12\sqrt{\frac{2-q}{6}}. \tag{4.13}$$

A plot of w/k as given by (4.13) is shown in Figure 3.

We can see from Figures 2 and 3 that for q near -4, h and H are near zero but the maximum width is at its largest. As q increases both h and H increase and the maximum width decreases until, at $q = 2$, $h = H = 6k^2$ and $w = 0$ as we would expect since the soliton is no longer a loop but is instead a cusp. All of the above properties are observed in Figure 4 as we look at the solution for $q = -5, -3, 0, 1, 2, 4$.

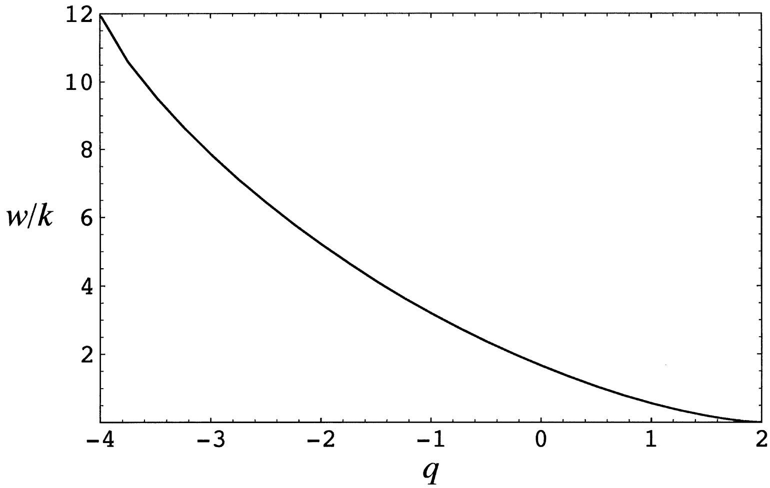

Figure 3. w/k for $-4 < q < 2$.

5. The N-soliton solution of the generalised Vakhnenko equation. The solution to (3.1) with (3.2) corresponding to N solitons is given by

$$f = \sum_{\mu=0,1} \exp\left[2\left(\sum_{i=1}^{N} \mu_i \eta_i + \sum_{i<j}^{(N)} \mu_i \mu_j \ln b_{ij}\right)\right], \quad \text{where} \quad \eta_i = k_i X - \omega_i T + \alpha_i, \tag{5.1}$$

$$b_{ij}^2 = -\frac{F[2(k_i - k_j), -2(\omega_i - \omega_j)]}{F[2(k_i + k_j), -2(\omega_i + \omega_j)]}, \tag{5.2}$$

and k_i, ω_i and α_i are constants. In (5.1) $\sum_{\mu=0,1}$ means the summation over all possible combinations of $\mu_1 = 0$ or 1, $\mu_2 = 0$ or 1, $\ldots$, $\mu_N = 0$ or 1, and $\sum_{i<j}^{(N)}$ means the summation over all possible combinations of N elements under the condition $i < j$.

(5.1) is a solution to (3.1) provided the 'N-soliton condition' holds [**3**]. In the Appendix we discuss this condition with F given by (3.2).

With F given by (3.2) the dispersion relations $F(2k_i, -2\omega_i) = 0$ $(i = 1, \ldots, N)$ give $\omega_i = k_i/(4k_i^2 + \beta)$ and then

$$\eta_i = k_i(X - c_i T) + \alpha_i \quad \text{with} \quad c_i = 1/(4k_i^2 + \beta). \tag{5.3}$$

Also, without loss of generality, we may take $k_1 < \ldots < k_N$ and then

$$b_{ij} = \frac{k_j - k_i}{k_j + k_i}\sqrt{\frac{4k_i^2 + 4k_j^2 - 4k_i k_j + 3\beta}{4k_i^2 + 4k_j^2 + 4k_i k_j + 3\beta}}, \quad \text{where} \quad i < j. \tag{5.4}$$

In principle, substitution of (5.1) into (2.6) gives $W(X, T)$. However, following Moloney & Hodnett [**5**], it is more convenient to express f in the form

$$f = h_i + \hat{h}_i e^{2\eta_i} \tag{5.5}$$

for a given i with $1 \leq i \leq N$, where

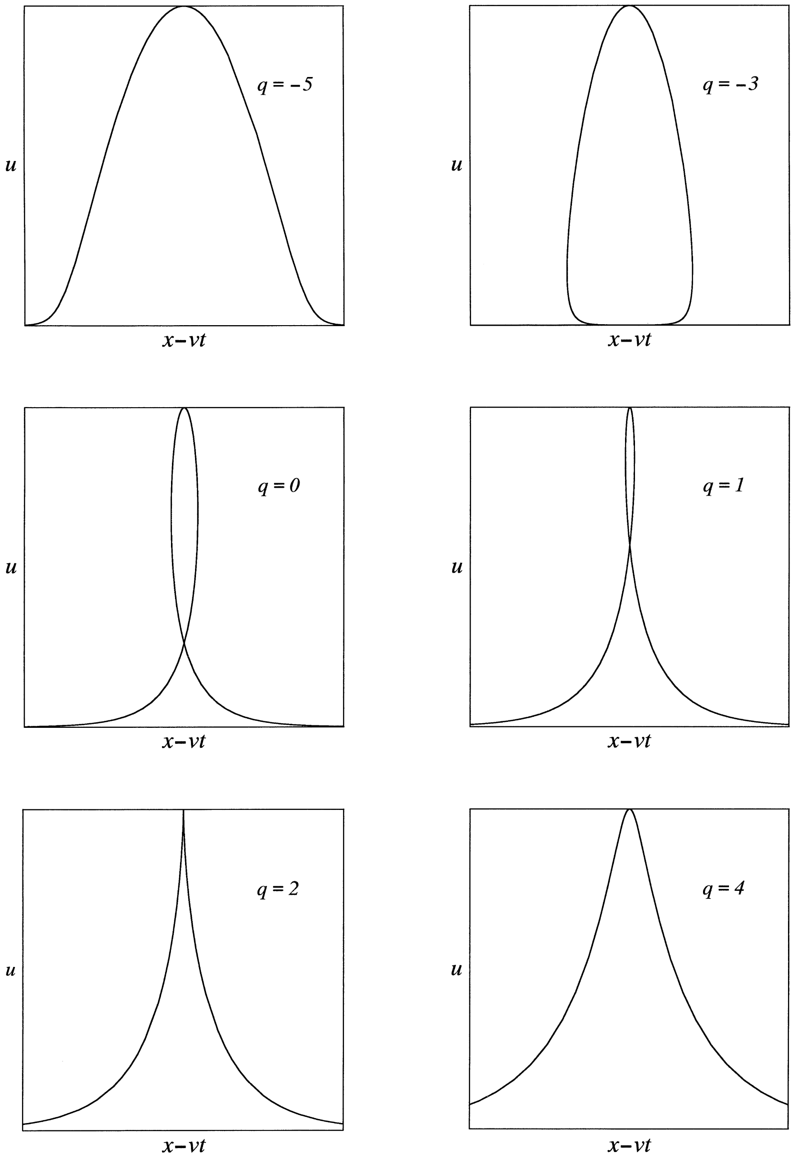

Figure 4. $u(x, t)$ for $q = -5, -3, 0, 1, 2, 4$.

$$h_i = \sum_{\mu=0,1} \exp\left[2\left(\sum_{\substack{r=1\\(r\neq i)}}^{N} \mu_r \eta_r + \sum_{\substack{r<s\\(r\neq i, s\neq i)}}^{(N)} \mu_r \mu_s \ln b_{rs}\right)\right], \tag{5.6}$$

$$\hat{h}_i = \sum_{\mu=0,1} \exp\left[2\left(\sum_{\substack{r=1\\(r\neq i)}}^{N} \mu_r \eta_r + \sum_{\substack{r<s\\(r\neq i, s\neq i)}}^{(N)} \mu_r \mu_s \ln b_{rs} + \sum_{r=1}^{i-1} \mu_r \ln b_{ri} + \sum_{r=i+1}^{N} \mu_r \ln b_{ir}\right)\right], \tag{5.7}$$

and then we may write $W(X, T)$ in the form

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$$W = \sum_{i=1}^{N} W_i, \quad \text{where} \quad W_i = 6k_i(1 + \tanh g_i) \quad \text{and} \quad g_i(X, T) = \eta_i + \tfrac{1}{2}\ln\left[\frac{\hat{h}_i}{h_i}\right]. \tag{5.8}$$

It follows that U may be written

$$U = \sum_{i=1}^{N} U_i, \quad \text{where} \quad U_i = 6k_i \frac{\partial g_i}{\partial X} \operatorname{sech}^2 g_i. \tag{5.9}$$

The N-soliton solution to the GVE is given by (3.10) and (3.11) with (5.8) and (5.9).

6. Discussion of the N-soliton solution. We now interpret the N-soliton solution found in §5 in terms of individual solitons.

First it is instructive to consider what happens in X,T-space. Introduce $c' = c_i - c_j$. We must consider when $c' > 0$ and $c' < 0$. Now

$$X - c_jT = X - c_iT + c'T. \tag{6.1}$$

Hence for η_i fixed (i.e. $X - c_iT$ fixed),

$$\left.\begin{array}{llll} \eta_j \to \pm\infty & \text{as} & T \to \pm\infty & \text{for} \quad c' > 0, \\ \eta_j \to \mp\infty & \text{as} & T \to \pm\infty & \text{for} \quad c' < 0. \end{array}\right\} \tag{6.2}$$

Suppose that $v_l < 0$ but $v_{l+1} > 0$. Then since $v_1 < v_2 < \cdots < v_N$,

$$v_i < 0 \quad \text{for} \quad 1 \le i \le l \tag{6.3}$$

$$v_i > 0 \quad \text{for} \quad l+1 \le i \le N \tag{6.4}$$

i.e.

$$v_1 < v_2 < \cdots < v_l < 0 < v_{l+1} < \cdots < v_N$$

so

$$c_l < c_{l-1} < \cdots < c_2 < c_1 < 0 < c_N < c_{N-1} < \cdots < c_{l+1}. \tag{6.5}$$

Note that if $l = 0$ then (6.5) yields $0 < c_N < \cdots < c_1$ and if we take $l = N$ then (6.5) yields $c_N < c_{N-1} < \cdots < c_2 < c_1 < 0$.

To decide whether c' is positive or negative in (6.1) we must consider the cases $c_i > 0$ and $c_i < 0$ separately. If $c_i > 0$ then we have

$$\left.\begin{array}{lll} c' > 0 & \Leftrightarrow & i+1 \le j \le N \text{ or } 1 \le j \le l \\ c' < 0 & \Leftrightarrow & l+1 \le j \le i-1. \end{array}\right\} \tag{6.6}$$

and if $c_i < 0$ we have

$$\left.\begin{array}{lll} c' > 0 & \Leftrightarrow & i+1 \le j \le l \\ c' < 0 & \Leftrightarrow & 1 \le j \le i-1 \text{ or } l+1 \le j \le N. \end{array}\right\} \tag{6.7}$$

Hence from (6.2), (6.6) and (6.7) we obtain for $c_i > 0$,

$$\left.\begin{array}{llll} \eta_j \to \pm\infty & \text{as} & T \to \pm\infty & \text{for} \quad i+1 \le j \le N \text{ or } 1 \le j \le l, \\ \eta_j \to \mp\infty & \text{as} & T \to \pm\infty & \text{for} \quad l+1 \le j \le i-1, \end{array}\right\} \tag{6.8}$$

and for $c_i < 0$,

$$\left.\begin{array}{llll} \eta_j \to \pm\infty & \text{as} & T \to \pm\infty & \text{for} \quad i+1 \le j \le l \\ \eta_j \to \mp\infty & \text{as} & T \to \pm\infty & \text{for} \quad 1 \le j \le i-1 \text{ or } l+1 \le j \le N. \end{array}\right\} \tag{6.9}$$

We now investigate the asymptotic form of each U_i. First we consider the case where $c_i > 0$. From (5.6), (5.9) and (6.8) we deduce that with $X - c_i T$ fixed and $T \to -\infty$

$$U_i \sim \begin{cases} 6k_1^2 \operatorname{sech}^2 \eta_1 & \text{if } i = 1, \\ 6k_i^2 \operatorname{sech}^2 \left(\eta_i + \displaystyle\sum_{r=l+1}^{i-1} \ln b_{ri} \right) & \text{if } 2 \le i \le N-1, \\ 6k_N^2 \operatorname{sech}^2 \left(\eta_N + \displaystyle\sum_{r=l+1}^{N-1} \ln b_{rN} \right) & \text{if } i = N, \end{cases} \tag{6.10}$$

and as $T \to +\infty$,

$$U_i \sim \begin{cases} 6k_1^2 \operatorname{sech}^2 \left(\eta_1 + \displaystyle\sum_{r=2}^{N} \ln b_{1r} \right) & \text{if } i = 1, \\ 6k_i^2 \operatorname{sech}^2 \left(\eta_i + \displaystyle\sum_{r=1}^{l} \ln b_{ri} + \displaystyle\sum_{r=i+1}^{N} \ln b_{ir} \right) & \text{if } 2 \le i \le N-1, \\ 6k_N^2 \operatorname{sech}^2 \left(\eta_N + \displaystyle\sum_{r=1}^{l} \ln b_{rN} \right) & \text{if } i = N. \end{cases} \tag{6.11}$$

Hence it is apparent that, in the limits $T \to \pm\infty$, each U_i may be identified as an individual soliton moving with speed c_i in the positive X-direction (since $c_i > 0$).

Similar calculations for $c_i < 0$ give as $T \to -\infty$,

$$U_i \sim \begin{cases} 6k_1^2 \operatorname{sech}^2 \left(\eta_1 + \displaystyle\sum_{r=l+1}^{N} \ln b_{1r} \right) & \text{if } i = 1, \\ 6k_i^2 \operatorname{sech}^2 \left(\eta_i + \displaystyle\sum_{r=1}^{i-1} \ln b_{ri} + \displaystyle\sum_{r=l+1}^{N} \ln b_{ir} \right) & \text{if } 2 \le i \le N-1, \\ 6k_N^2 \operatorname{sech}^2 \left(\eta_N + \displaystyle\sum_{r=1}^{N-1} \ln b_{rN} \right) & \text{if } i = N, \end{cases} \tag{6.12}$$

and as $T \to +\infty$,

$$U_i \sim \begin{cases} 6k_1^2 \operatorname{sech}^2\left(\eta_1 + \sum_{r=2}^{l} \ln b_{1r}\right) & \text{if } \ i = 1, \\ 6k_i^2 \operatorname{sech}^2\left(\eta_i + \sum_{r=i+1}^{l} \ln b_{ir}\right) & \text{if } \ 2 \le i \le N-1, \\ 6k_N^2 \operatorname{sech}^2(\eta_N) & \text{if } \ i = N. \end{cases} \tag{6.13}$$

Hence this time it is apparent that, in the limits $T \to \pm\infty$, each U_i may be identified as an individual soliton moving with speed $|c_i|$ in the negative X-direction (since $c_i < 0$). Recall that in (6.10), (6.11), (6.12) and (6.13) l is found from $c_l < 0$ and $c_{l+1} > 0$.

The shifts, Δ_i, of the solitons in the positive X-direction due to the interactions between the N solitons can be found from

$$\Delta_i = \Big[X - c_i T\Big]_{T\to-\infty}^{T\to+\infty}. \tag{6.14}$$

Hence, for $c_i > 0$ (and so $l < i$), we have

$$\Delta_i \sim \begin{cases} -\dfrac{1}{k_1}\sum_{r=2}^{N} \ln b_{1r} & \text{if } \ i = 1, \\ -\dfrac{1}{k_i}\left(\sum_{r=1}^{l} \ln b_{ri} - \sum_{r=l+1}^{i-1} \ln b_{ri} + \sum_{r=i+1}^{N} \ln b_{ir}\right) & \text{if } \ 2 \le i \le N-1, \\ -\dfrac{1}{k_N}\left(\sum_{r=1}^{l} \ln b_{rN} - \sum_{r=l+1}^{N-1} \ln b_{rN}\right) & \text{if } \ i = N, \end{cases} \tag{6.15}$$

and, for $c_i < 0$ (so that $l + 1 > i$) we have

$$\Delta_i \sim \begin{cases} -\dfrac{1}{k_1}\left(\sum_{r=2}^{l} \ln b_{1r} - \sum_{r=l+1}^{N} \ln b_{1r}\right) & \text{if } \ i = 1, \\ \dfrac{1}{k_i}\left(\sum_{r=1}^{i-1} \ln b_{ri} - \sum_{r=i+1}^{l} \ln b_{ir} + \sum_{r=l+1}^{N} \ln b_{ir}\right) & \text{if } \ 2 \le i \le N-1, \\ \dfrac{1}{k_N}\sum_{r=1}^{N-1} \ln b_{rN} & \text{if } \ i = N. \end{cases} \tag{6.16}$$

Now let us consider what happens in x,t-space. From (3.11) with $v_i = 1/c_i$ we have

$$x - v_i t = -v_i(X - c_i T) + W(X, T) + x_0. \tag{6.17}$$

Note that in (6.10) and (6.11), taking the limits $T \to \pm\infty$ with $X - c_i T$ fixed is equivalent to taking the limits $X \to \pm\infty$ with $X - c_i T$ fixed and from (6.12) and

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(6.13), taking the limits $T \to \pm\infty$ with $X - c_i T$ fixed is equivalent to taking the limits $X \to \mp\infty$ with $X - c_i T$ fixed. Also note that $X = t$ from (2.1). Accordingly from (6.10)–(6.13) and (6.17), with a given i, we see that in the limits $t \to \pm\infty$ with $X - c_i T$ fixed, $U_i(X, T)$ and $x - v_i t$ are related by the parameter $X - c_i T$. It follows that in the limits $t \to \pm\infty$, u_i may be identified as an individual soliton moving with speed $|v_i|$ in the positive x-direction if $\beta > -4k_i^2$ and in the negative x-direction if $\beta < -4k_i^2$, where $u_i(x, t) = U_i(X, T)$.

The shifts, δ_i, of the solitons u_i in the positive x-direction due to the interaction between the N solitons are defined by

$$\delta_i = \Big[x - v_i t\Big]_{t\to-\infty}^{t\to+\infty}, \tag{6.18}$$

so that

$$\delta_i = \operatorname{sign}(v_i)\Big[x - v_i t\Big]_{T\to-\infty}^{T\to+\infty}. \tag{6.19}$$

In order to calculate the shifts δ_i, we have to consider the cases $v_i > 0$ and $v_i < 0$ separately. First let us consider the case where $v_i > 0$. From (6.10), as $T \to -\infty$, $U_i \to U_{i\max} = 6k_i^2$ where

$$X - c_i T = \begin{cases} -\dfrac{\alpha_1}{k_1} & \text{if } \ i = 1, \\[2ex] -\dfrac{\alpha_i}{k_i} - \dfrac{1}{k_i}\displaystyle\sum_{r=l+1}^{i-1} \ln b_{ri} & \text{if } \ 2 \le i \le N-1, \\[2ex] -\dfrac{\alpha_N}{k_N} - \dfrac{1}{k_N}\displaystyle\sum_{r=l+1}^{N-1} \ln b_{rN} & \text{if } \ i = N. \end{cases} \tag{6.20}$$

Hence from (5.8) and (6.17) we obtain,

$$\Big[x - v_i t\Big]_{T\to-\infty} = \begin{cases} \left(4k_1^2 + \beta\right)\dfrac{\alpha_1}{k_1} + 6k_1 + x_0 & \text{if } \ i = 1, \\[2ex] \left(4k_i^2 + \beta\right)\left(\dfrac{\alpha_i}{k_i} + \dfrac{1}{k_i}\displaystyle\sum_{r=l+1}^{i-1} \ln b_{ri}\right) & \\ \qquad + 6k_i + 12\displaystyle\sum_{r=l+1}^{i-1} k_r + x_0 & \text{if } \ 2 \le i \le N-1, \\[2ex] \left(4k_N^2 + \beta\right)\left(\dfrac{\alpha_N}{k_N} + \dfrac{1}{k_N}\displaystyle\sum_{r=l+1}^{N-1} \ln b_{rN}\right) & \\ \qquad + 6k_N + 12\displaystyle\sum_{r=l+1}^{N-1} k_r + x_0 & \text{if } \ i = N. \end{cases} \tag{6.21}$$

From (6.11) as $T \to +\infty$, $U_i \to U_{i\max} = 6k_i^2$ where

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$$X - c_iT = \begin{cases} -\dfrac{\alpha_1}{k_1} - \dfrac{1}{k_1}\displaystyle\sum_{r=2}^{N} \ln b_{1r} & \text{if } \ i = 1, \\ -\dfrac{\alpha_i}{k_i} - \dfrac{1}{k_i}\displaystyle\sum_{r=1}^{l} \ln b_{ri} - \dfrac{1}{k_i}\sum_{r=i+1}^{N} \ln b_{ir} & \text{if } \ 2 \le i \le N-1, \\ -\dfrac{\alpha_N}{k_N} - \dfrac{1}{k_N}\displaystyle\sum_{r=1}^{l} \ln b_{rN} & \text{if } \ i = N. \end{cases} \tag{6.22}$$

From (5.8) and (6.17) we obtain

$$\left[x - v_it\right]_{T\to+\infty} = \begin{cases} (4k_1^2 + \beta)\left(\dfrac{\alpha_1}{k_1} + \dfrac{1}{k_1}\displaystyle\sum_{r=2}^{N} \ln b_{1r}\right) & \\ \qquad +6k_1 + 12\displaystyle\sum_{r=2}^{N} k_r + x_0 & \text{if } \ i = 1, \\ (4k_i^2 + \beta)\left(\dfrac{\alpha_i}{k_i} + \dfrac{1}{k_i}\displaystyle\sum_{r=1}^{l} \ln b_{ri} + \dfrac{1}{k_i}\sum_{r=i+1}^{N} \ln b_{ir}\right) & \\ \qquad +6k_i + 12\displaystyle\sum_{r=1}^{l} k_r + 12\sum_{r=i+1}^{N} k_r + x_0 & \text{if } \ 2 \le i \le N-1, \\ (4k_N^2 + \beta)\left(\dfrac{\alpha_N}{k_N} + \dfrac{1}{k_N}\displaystyle\sum_{r=1}^{l} \ln b_{rN}\right) & \\ \qquad +6k_N + 12\displaystyle\sum_{r=1}^{l} k_r + x_0 & \text{if } \ i = N. \end{cases} \tag{6.23}$$

Therefore, from (6.19) we can calculate the shifts in the positive x-direction for $v_i > 0$. For $i = 1$ we obtain

$$\delta_1 = \frac{1}{k_1}(4k_1^2 + \beta)\sum_{r=2}^{N} \ln b_{1r} + 12\sum_{r=2}^{N} k_r; \tag{6.24}$$

for $2 \le i \le N-1$ we obtain

$$\delta_i = \frac{1}{k_i}(4k_i^2 + \beta)\left(\sum_{r=1}^{l} \ln b_{ri} - \sum_{r=l+1}^{i-1} \ln b_{ri} + \sum_{r=i+1}^{N} \ln b_{ir}\right) + 12\left(\sum_{r=1}^{l} k_r - \sum_{r=l+1}^{i-1} k_r + \sum_{r=i+1}^{N} k_r\right), \tag{6.25}$$

and for $i = N$ we obtain

$$\delta_N = \frac{1}{k_N}(4k_N^2 + \beta)\left(\sum_{r=1}^{l} \ln b_{rN} - \sum_{r=l+1}^{N-1} \ln b_{rN}\right) + 12\left(\sum_{r=1}^{l} k_r - \sum_{r=l+1}^{N-1} k_r\right). \tag{6.26}$$

In (6.24)–(6.26) $0 \le l \le i-1$.

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Note that if we put $\beta = 0$ and $l = 0$ in (6.24)–(6.26) we get the results obtained for the ordinary Vakhnenko equation [**2**, equations (4.8), (4.9) and (4.10)].

By performing similar calculations we can calulate the shifts in the positive x-direction for $v_i < 0$. For $i = 1$ we obtain

$$\delta_1 = \frac{1}{k_1}\left(4k_1^2 + \beta\right)\left(\sum_{r=l+1}^{N} \ln b_{1r} - \sum_{r=2}^{l} \ln b_{1r}\right) + 12\left(\sum_{r=l+1}^{N} k_r - \sum_{r=2}^{l} k_r\right), \tag{6.27}$$

for $2 \leq i \leq N-1$ we obtain

$$\delta_i = \frac{1}{k_i}\left(4k_i^2 + \beta\right)\left(\sum_{r=1}^{i-1} \ln b_{ri} - \sum_{r=i+1}^{l} \ln b_{ir} + \sum_{r=l+1}^{N} \ln b_{ir}\right) + 12\left(\sum_{r=1}^{i-1} k_r - \sum_{r=i+1}^{l} k_r + \sum_{r=l+1}^{N} k_r\right), \tag{6.28}$$

and for $i = N$ we obtain

$$\delta_N = \frac{1}{k_N}\left(4k_N^2 + \beta\right)\sum_{r=1}^{N-1} \ln b_{rN} + 12\sum_{r=1}^{N-1} k_r. \tag{6.29}$$

In (6.27)–(6.29) $i \leq l \leq N$.

Finally we note that, for the interactions to be centred at $X = 0$ and $T = 0$ in X,T-space, we require

$$\alpha_1 = -\frac{1}{2}\sum_{r=2}^{N} \ln b_{1r}, \tag{6.30}$$

$$\alpha_i = -\frac{1}{2}\left(\sum_{r=1}^{i-1} \ln b_{ri} + \sum_{r=i+1}^{N} \ln b_{ir}\right) \quad 2 \leq i \leq N-1, \tag{6.31}$$

$$\alpha_N = -\frac{1}{2}\sum_{r=1}^{N-1} \ln b_{rN}, \tag{6.32}$$

and then, for the interactions to be centred at $x = 0$ and $t = 0$ in x,t-space, we require

$$x_0 = -6\sum_{r=1}^{N} k_r. \tag{6.33}$$

7. Example: $N = 2$. We shall now consider in detail the case $N = 2$.

For $N = 2$, (5.1) gives

$$f = 1 + e^{2\eta_1} + e^{2\eta_2} + b^2 e^{2(\eta_1+\eta_2)}, \tag{7.1}$$

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where

$$\eta_i = k_i(X - c_i T) + \alpha_i, \quad \text{with} \quad c_i = 1/(4k_i^2 + \beta), \quad (i = 1, 2) \tag{7.2}$$

and

$$b^2 := b_{12}^2 = -\frac{F[2(k_2 - k_1), -2(\omega_2 - \omega_1)]}{F[2(k_2 + k_1), -2(\omega_2 + \omega_1)]}. \tag{7.3}$$

Without loss of generality we choose $k_1 < k_2$ and obtain

$$b := b_{12} = \frac{k_2 - k_1}{k_2 + k_1}\sqrt{\frac{4k_1^2 + 4k_2^2 - 4k_1k_2 + 3\beta}{4k_1^2 + 4k_2^2 + 4k_1k_2 + 3\beta}}. \tag{7.4}$$

Next, from (5.8)

$$W = W_1 + W_2, \quad \text{where} \quad W_i = 6k_i(1 + \tanh g_i) \tag{7.5}$$

and

$$g_1(X, T) = \eta_1 + \frac{1}{2}\ln\left[\frac{1 + b^2 e^{2\eta_2}}{1 + e^{2\eta_2}}\right], \quad g_2(X, T) = \eta_2 + \frac{1}{2}\ln\left[\frac{1 + b^2 e^{2\eta_1}}{1 + e^{2\eta_1}}\right]. \tag{7.6}$$

Hence

$$U = U_1 + U_2, \quad \text{where} \quad U_i = 6k_i \frac{\partial g_i}{\partial X}\operatorname{sech}^2 g_i. \tag{7.7}$$

The shifts δ_1 and δ_2 of u_1 and u_2, respectively, in the positive x-direction due to the interaction between the two solitons can be calculated from (6.24), (6.26), (6.27) and (6.29) with $N = 2$. However for the case $N = 2$ we can express the shifts in the more convenient form

$$\delta_1 = \operatorname{sign}(v_2)\left[\frac{1}{k_1}\left(4k_1^2 + \beta\right)\ln b + 12k_2\right] \tag{7.8}$$

and

$$\delta_2 = \operatorname{sign}(v_1)\left[-\frac{1}{k_2}\left(4k_2^2 + \beta\right)\ln b - 12k_1\right]. \tag{7.9}$$

7.1. Types of solitons. For convenience, we shall introduce the ratios

$$s = \frac{\beta}{k_2^2} \quad \text{and} \quad r = \frac{k_1}{k_2}. \tag{7.10}$$

Note that since $0 < k_1 < k_2$, $0 < r < 1$. We cannot have $s = -4r^2$ or $s = -4$, as this would result in c_1 or c_2 being infinite respectively. Also we must have $b^2 > 0$.

From §4 we expect u_1 to be a loop if $-4r^2 < s < 2r^2$, a cusp if $s = 2r^2$ and a hump shape otherwise. Similarly u_2 will be a loop if $-4 < s < 2$, a cusp if $s = 2$ and a hump shape otherwise. Also $v_1 < 0$ if $s < -4r^2$ and $v_1 > 0$ if $s > -4r^2$, and $v_2 < 0$ if $s < -4$ and $v_2 > 0$ if $s > -4$.

Different types of soliton solution are possible; these are summarised below.

1. If $s < -4$ both u_1 and u_2 are hump shaped with $v_1 < 0$ and $v_2 < 0$.
2. If $-4 < s < -4r^2$ then u_1 is hump shaped with $v_1 < 0$ and u_2 is a loop with $v_2 > 0$. An example of this is shown in Figure 5 where we have $r = 0.45$ and $s = -1$. Here $v_1 = -0.19$ and $v_2 = 3$ and it can be observed that $\delta_1 > 0$ and $\delta_2 < 0$. In fact, from (7.8) and (7.9) we obtain $\delta_1 = 13.65$ and $\delta_2 = -6.34$.

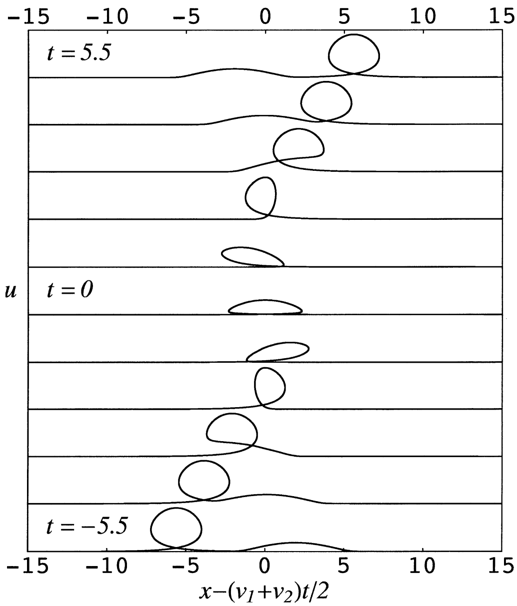

Figure 5. The interaction process for $r = 0.45$ and $s = -1$.

3. If $-4r^2 < s < 2r^2$ then both u_1 and u_2 are loops with $v_1 > 0$ and $v_2 > 0$. An example of this is shown in Figure 6 where we have $r = 0.5$ and $s = -0.75$. Here $v_1 = 0.25$ and $v_2 = 3.25$ and it can be observed that $\delta_1 > 0$ and $\delta_2 > 0$. In fact, from (7.8) and (7.9) we obtain $\delta_1 = 10.99$ and $\delta_2 = 0.57$.
4. If $s = 2r^2$ then u_1 is a cusp with $v_1 > 0$ and u_2 is a loop with $v_2 > 0$.
5. If $2r^2 < s < 2$ then u_1 is hump shaped with $v_1 > 0$ and u_2 is a loop with $v_2 > 0$. An example of this is shown in Figure 7 where we have $r = 0.6$ and $s = 1$. Here $v_1 = 2.44$ and $v_2 = 5$ and it can be observed that $\delta_1 > 0$ and $\delta_2 > 0$. In fact, from (7.8) and (7.9) we obtain $\delta_1 = 5.17$ and $\delta_2 = 1.19$.
6. If $s = 2$, u_1 is a hump with $v_1 > 0$ and u_2 is a cusp with $v_2 > 0$.
7. If $s > 2$ both u_1 and u_2 are hump shaped with $v_1 > 0$ and $v_2 > 0$.

7.2. Resonant soliton interactions. Here we shall closely follow the work of Mussette, Lambert and Decuyper [**6**], in the context of the second modified regularised long wave equation, by investigating the resonant solutions on the boundary curves of the segment of s,r-space in which $b^2 < 0$.

If we label $D(0, -\frac{4}{3})$, $A(\frac{1}{2}, -1)$, $B(1, -\frac{4}{3})$ and $P(1, -4)$ then the upper curve, where $b^2 = 0$, is given by $\widehat{\mathrm{DAB}}$ and the lower curve, where $1/b^2 = 0$, is given by $\widehat{\mathrm{DP}}$, as

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shown in Figure 8. From (7.4), on curve $\widehat{\mathrm{DAB}}$ $s = -\frac{4}{3}\left(r^2 - r + 1\right)$ and on curve $\widehat{\mathrm{DP}}$ $s = -\frac{4}{3}\left(r^2 + r + 1\right)$.

We shall begin by considering what happens in X,T-space. On $\widehat{\mathrm{DAB}}$ $b^2 = 0$ and so, from (7.3),

$$F[2(k_2 - k_1), -2(\omega_2 - \omega_1)] = 0 \tag{7.11}$$

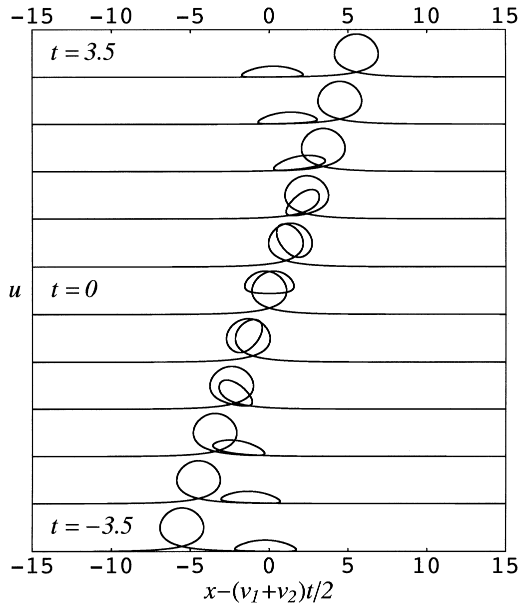

Figure 6. The interaction process for $r = 0.5$ and $s = -0.75$.

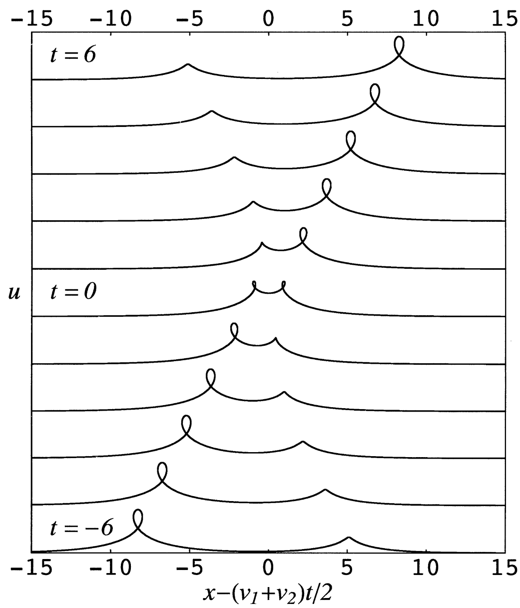

Figure 7. The interaction process for $r = 0.6$ and $s = 1$.

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and on $\widehat{\mathrm{DP}}$ $1/b^2 = 0$ and so, once again from (7.3),

$$F[2(k_2 + k_1), -2(\omega_2 + \omega_1)] = 0. \tag{7.12}$$

We introduce

$$k_R^{\pm} := k_2 \pm k_1 \tag{7.13}$$

and

$$\omega_R^{\pm} := \omega_2 \pm \omega_1. \tag{7.14}$$

so that (7.11) and (7.12) become

$$F[2k_R^{\pm}, -2\omega_R^{\pm}] = 0 \tag{7.15}$$

and so

$$\omega_R^{\pm} = \frac{k_R^{\pm}}{4(k_R^{\pm})^2 + \beta}. \tag{7.16}$$

We also define

$$\eta_R^{\pm} := k_R^{\pm} X - \omega_R^{\pm} T + \alpha_R^{\pm}, \tag{7.17}$$

where $\alpha_R^{\pm} := \alpha_2 \pm \alpha_1$ and so

$$\eta_R^{\pm} = \eta_2 \pm \eta_1. \tag{7.18}$$

Furthermore

$$c_R^{\pm} := \frac{\omega_R^{\pm}}{k_R^{\pm}} = \frac{1}{4(k_R^{\pm})^2 + \beta} \tag{7.19}$$

but $c_R^{\pm} \neq c_2 \pm c_1$.

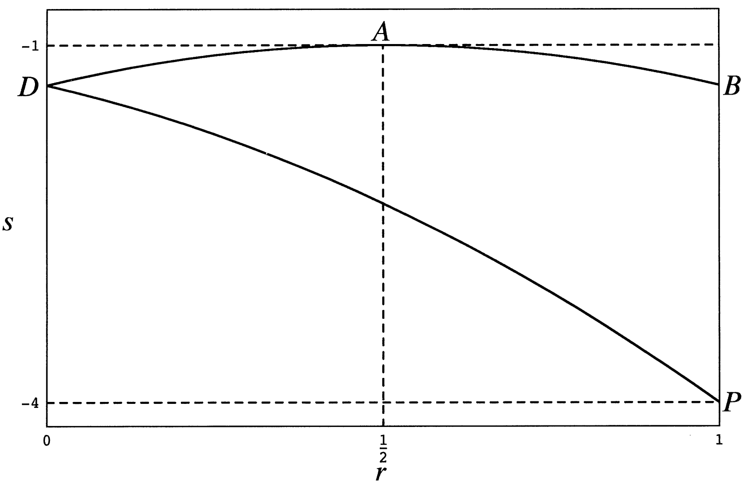

Figure 8. The segment of s,r-space where $b^2 < 0$.

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We can now look at the solutions on the two curves $\widehat{\mathrm{DAB}}$ and $\widehat{\mathrm{DP}}$ separately.

First, let us investigate the solution on $\widehat{\mathrm{DAB}}$. On $\widehat{\mathrm{DAB}}$ $b^2 = 0$ and so, from (7.1), the solution to (3.1) is

$$f = 1 + e^{2\eta_1} + e^{2\eta_2}. \tag{7.20}$$

Hence, from (2.6) and (7.20),

$$W = \frac{12\left(k_1 e^{2\eta_1} + k_2 e^{2\eta_2}\right)}{1 + e^{2\eta_1} + e^{2\eta_2}}. \tag{7.21}$$

It is also useful to express W in terms of k_1, k_R^-, η_1 and η_R^- as

$$W = \frac{12\left(k_1 e^{2\eta_1}(1 + e^{2\eta_R^-}) + k_R^-(e^{2\eta_R^-} e^{2\eta_1})\right)}{1 + e^{2\eta_1}(1 + e^{2\eta_R^-})}, \tag{7.22}$$

or in terms of k_2, k_R^-, η_2 and η_R^- as

$$W = \frac{12\left(k_2 e^{2\eta_2}(1 + e^{-2\eta_R^-}) - k_R^-(e^{2\eta_2} e^{-2\eta_R^-})\right)}{1 + e^{2\eta_2}(1 + e^{-2\eta_R^-})}. \tag{7.23}$$

We want to find a solution for U as $T \to \pm\infty$. To do this we consider the behaviour of W as $T \to \pm\infty$ with each of η_1, η_2 and η_R^- fixed in turn (i.e. fix $X - c_1 T$, $X - c_2 T$ and $X - c_R^- T$ respectively). In order to do this we must order the speeds c_1, c_2 and c_R^- and it turns out that we have to break $\widehat{\mathrm{DAB}}$ into $\widehat{\mathrm{DA}}$ and $\widehat{\mathrm{AB}}$ and consider these two cases separately.

On $\widehat{\mathrm{DA}}$, $0 < r < \frac{1}{2}$ and $k_1 < k_R^- < k_2$. Consequently, we can show that

$$c_1 < 0 < c_2 < c_R^- < |c_1|. \tag{7.24}$$

We can now describe the behaviour of η_j with η_i fixed as $T \to \pm\infty$. This is summarised in Table 1. From the results in Table 1 together with (7.21), (7.22) and (7.23) we can describe the behaviour of W as $T \to \pm\infty$ with η_i fixed. This is summarised in Table 2. Hence we can deduce that as $T \to -\infty$,

$$W \sim \frac{12 k_2 e^{2\eta_2}}{1 + e^{2\eta_2}} \tag{7.25}$$

Table 1: The behaviour of η_j with η_i fixed as $T \to \pm\infty$ on $\widehat{\mathrm{DA}}$

	$T \to -\infty$	$T \to +\infty$
η_1 fixed	$\eta_2 \to +\infty$ $\eta_R^- \to +\infty$	$\eta_2 \to -\infty$ $\eta_R^- \to -\infty$
η_2 fixed	$\eta_1 \to -\infty$ $\eta_R^- \to +\infty$	$\eta_1 \to +\infty$ $\eta_R^- \to -\infty$
η_R^- fixed	$\eta_1 \to -\infty$ $\eta_2 \to -\infty$	$\eta_1 \to +\infty$ $\eta_2 \to +\infty$

Table 2: The behaviour of W with η_i fixed as $T \to \pm\infty$ on $\widehat{\mathrm{DA}}$

	$T \to -\infty$	$T \to +\infty$
η_1 fixed	$W \to 12k_2$	$W \to \dfrac{12k_1 e^{2\eta_1}}{1+e^{2\eta_1}}$
η_2 fixed	$W \to \dfrac{12k_2 e^{2\eta_2}}{1+e^{2\eta_2}}$	$W \to 12k_1$
η_R^- fixed	$W \to 0$	$W \to 12\left(k_1 + \dfrac{k_R^- e^{2\eta_R^-}}{1+e^{2\eta_R^-}}\right)$

and, as $U = W_X$, we obtain, as $T \to -\infty$,

$$U \sim 6k_2^2 \operatorname{sech}^2 \eta_2. \tag{7.26}$$

Next, as $T \to +\infty$,

$$W \sim \frac{12k_1 e^{2\eta_1}}{1+e^{2\eta_1}} + \frac{12k_R^- e^{2\eta_R^-}}{1+e^{2\eta_R^-}} \tag{7.27}$$

and so

$$U \sim 6k_1^2 \operatorname{sech}^2 \eta_1 + 6(k_R^-)^2 \operatorname{sech}^2 \eta_R^-. \tag{7.28}$$

This solution describes the decay of one soliton travelling with speed c_2 in the positive x-direction into two solitons, one moving with speed c_R^- in the positive x-direction and the other moving with speed $|c_1|$ in the negative x-direction.

At this point we note that (3.1) with (3.2) is invariant under the transformation

$$X \to -X, \ T \to -T. \tag{7.29}$$

As a consequence of this there are two solutions to (3.1) with (3.2) and hence two solutions of the GVE. Off the resonance curves both solutions are the same. However on the resonance curves the two solutions are different. On $\widehat{\mathrm{DA}}$, the second solution is, as $T \to -\infty$,

$$U \sim 6k_1^2 \operatorname{sech}^2 \eta_1 + 6(k_R^-)^2 \operatorname{sech}^2 \eta_R^- \tag{7.30}$$

and as $T \to +\infty$

$$U \sim 6k_2^2 \operatorname{sech}^2 \eta_2. \tag{7.31}$$

This solution describes the fusion of two solitons, one moving with speed $|c_1|$ in the negative x-direction and the other moving with speed c_R^- in the positive x-direction, into one soliton moving with speed c_2 in the positive x-direction. Clearly, the two solutions obtained on $\widehat{\mathrm{DA}}$ are the reverse of each other.

We now investigate the solution on $\widehat{\mathrm{AB}}$. On $\widehat{\mathrm{AB}}$ $\frac{1}{2} < r < 1$ and $k_R^- < k_1 < k_2$ so we can show that

$$c_R^- < 0 < c_2 < c_1 < |c_R^-|. \tag{7.32}$$

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Using (7.32) we can perform similar calculations to those on $\widehat{\mathrm{DA}}$ to obtain the two solutions on $\widehat{\mathrm{AB}}$. One solution is

$$\left.\begin{aligned} T\to-\infty &\quad:\quad U\sim 6k_1^2\mathrm{sech}^2\eta_1+6(k_R^-)^2\mathrm{sech}^2\eta_R^-,\\ T\to+\infty &\quad:\quad U\sim 6k_2^2\mathrm{sech}^2\eta_2;\end{aligned}\right\} \tag{7.33}$$

and the other, obtained by the transformation (7.29), is

$$\left.\begin{aligned} T\to-\infty &\quad:\quad U\sim 6k_2^2\mathrm{sech}^2\eta_2,\\ T\to+\infty &\quad:\quad U\sim 6k_1^2\mathrm{sech}^2\eta_1+6(k_R^-)^2\mathrm{sech}^2\eta_R^-.\end{aligned}\right\} \tag{7.34}$$

We now investigate the solution on $\widehat{\mathrm{DP}}$. On $\widehat{\mathrm{DP}}$ $0<r<1$ and $k_1<k_2<k_R^+$. Consequently,

$$c_1<0<c_R^+<c_2<|c_1|. \tag{7.35}$$

If we introduce the transformations $\alpha_1=\alpha_1'$ and $\alpha_2=\alpha_2'-\ln b$ then

$$\eta_1=k_1(X-c_1T)+\alpha_1'=:\eta_1' \tag{7.36}$$

and

$$\eta_2=k_1(X-c_2T)+\alpha_2'-\ln b=:\eta_2'-\ln b. \tag{7.37}$$

Therefore from (7.1),

$$\begin{aligned} f&=1+e^{2\eta_1'}+e^{2(\eta_2'-\ln b)}+b^2e^{2(\eta_1'+\eta_2'-\ln b)}\\ &=1+e^{2\eta_1'}+e^{2\eta_R^{+\prime}},\end{aligned} \tag{7.38}$$

where $1/b^2=0$, $\eta_R^{+\prime}=k_R^+(X-c_R^+T)+\alpha_R^{+\prime}$ and $\alpha_R^{+\prime}=\alpha_2'+\alpha_1'$. If we compare (7.38) and (7.35) with (7.20) and (7.24) we can obtain the solution on $\widehat{\mathrm{DP}}$ by the same analysis without repeating all the details. As a result, we conclude that a solution on $\widehat{\mathrm{DP}}$ is given by

$$\left.\begin{aligned} T\to-\infty &\quad:\quad U\sim 6(k_R^+)^2\mathrm{sech}^2\eta_R^{+\prime},\\ T\to+\infty &\quad:\quad U\sim 6k_1^2\mathrm{sech}^2\eta_1'+6k_2^2\mathrm{sech}^2\eta_2'.\end{aligned}\right\} \tag{7.39}$$

Similarly, if we repeat the above with $\alpha_1=\alpha_1'-\ln b$ and $\alpha_2=\alpha_2'$ and compare to (7.20) and (7.32) we obtain the second solution on $\widehat{\mathrm{DP}}$, namely

$$\left.\begin{aligned} T\to-\infty &\quad:\quad U\sim 6k_1^2\mathrm{sech}^2\eta_1'+6k_2^2\mathrm{sech}^2\eta_2',\\ T\to+\infty &\quad:\quad U\sim 6(k_R^+)^2\mathrm{sech}^2\eta_R^{+\prime}.\end{aligned}\right\} \tag{7.40}$$

Note that if we used the transformation $X\to-X$, $T\to-T$ in (7.39) we would obtain the second solution (7.40).

We can now discuss the solutions obtained on the resonance curves in x,t-space. This is best represented pictorially. On $\widehat{DA}$, the two solutions in the x,t-space are represented pictorially by Figure 9. On $\widehat{AB}$, the two solutions in x,t-space are shown in Figure 10 and on $\widehat{DP}$ the two solutions are shown in Figure 11. As can be seen from Figure 9, Figure 10 and Figure 11 the two solutions on each of the three curves all have the same form. One solution consists of a large fast loop moving in the positive x-direction and a small slow hump moving in the negative x-direction fusing together to form a medium sized loop travelling with intermediate speed in the positive x-direction. The other solution is the reverse of this, namely a medium sized loop travelling with intermediate speed in the positive x-direction splitting into a

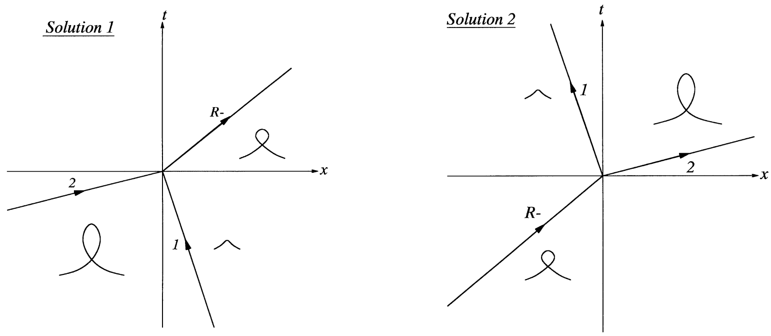

Figure 9. $u(x, t)$ on $\widehat{DA}$.

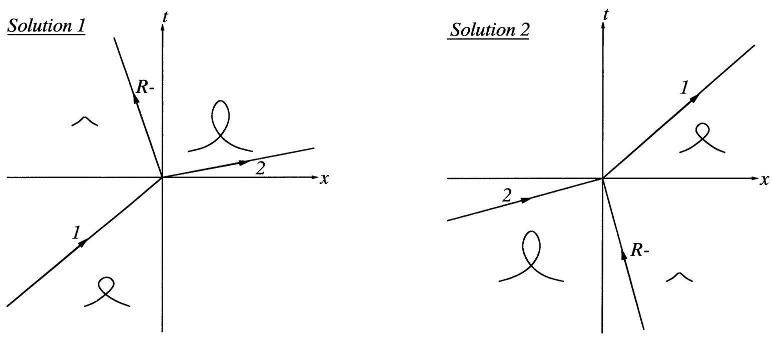

Figure 10. $u(x, t)$ on $\widehat{AB}$.

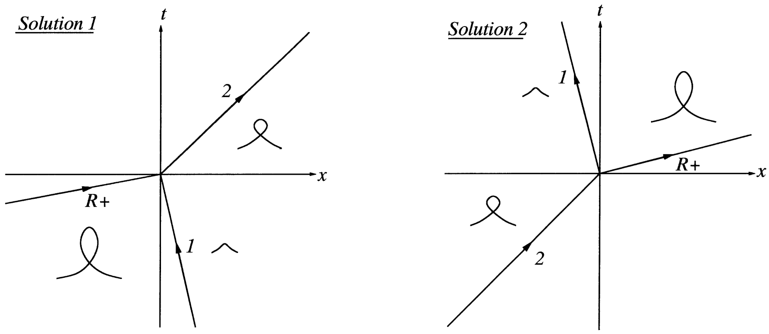

Figure 11. $u(x, t)$ on $\widehat{DP}$.

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small slow hump travelling in the negative x-direction and a large fast loop travelling in the positive x-direction.

An example of solution 1 on $\widehat{\mathrm{DA}}$ is shown in Figure 12. In this example $r = 0.4$, $s \simeq -1.0133$, $v_1 = -0.37$, $v_2 = 2.99$ and $v_R^- = 0.43$. An example of solution 1 on $\widehat{\mathrm{AB}}$ is shown in Figure 13. In this example $r = 0.55$, $s \simeq -1.0033$, $v_1 = 0.21$, $v_2 = 3$ and $v_R^- = -0.19$. These two examples illustrate the two different types of interactions observed on the resonance curves.

8. Conclusion. We have found the N-soliton solution to the GVE by using a blend of transformations and Hirota's method.

We are currently investigating the N-soliton solution of the following nonlinear evolution equation

$$\frac{\partial}{\partial x}\left(\mathcal{D}^2 u + \frac{1}{2}pu^2 + \beta u\right) + q\mathcal{D}u = 0, \tag{8.1}$$

where p, q and β are free parameters. This equation, when transformed into X,T-space, can be expressed as the following version of the well known shallow water wave equation

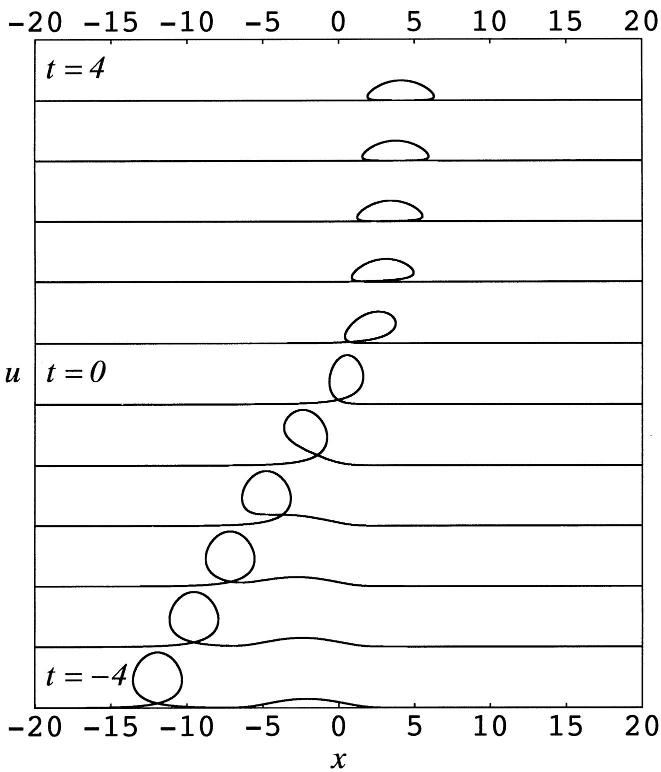

Figure 12. The interaction process on $\widehat{\mathrm{DA}}$ for $r = 0.4$.

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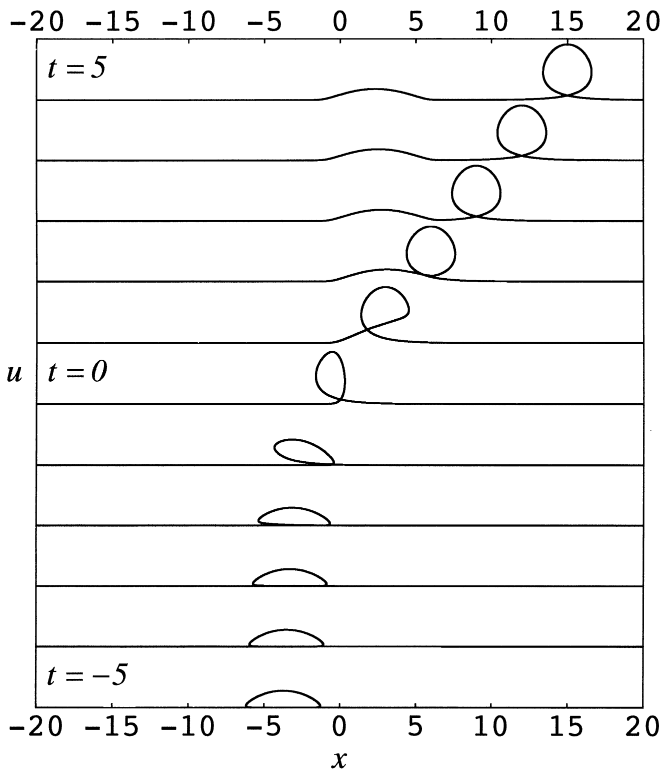

Figure 13. The interaction process on $\widehat{\mathrm{AB}}$ for $r = 0.55$.

$$U_{XXT} + pUU_T - qU_X \int_X^\infty U_T dX' + \beta U_T + qU_X = 0. \tag{8.2}$$

Hirota and Satsuma [7] have shown that this equation is integrable when $p = q$ or $p = 2q$. The case $p = q = 1$ gives the GVE, however the case $p = 2q$ is new and it is this case which is currently under investigation.

Appendix. The N-soliton condition. For there to be an N-soliton solution (NSS) to (3.1) with $N(\geq 1)$ arbitrary, $F(D_X, D_T)$ must satisfy the 'N-soliton condition' (NSC) [3], namely

$$G^{(n)}(p_1, \ldots, p_n) = 0, \qquad n = 1, 2, \ldots, N, \tag{A.1}$$

where

$$G^{(1)}(p_1) := 0 \tag{A.2}$$

and, for $n \geq 2$,

$$G^{(n)}(p_1, \ldots, p_n) := C \sum_{\sigma=\pm 1} \left\{ F\left(\sum_{i=1}^{n} \sigma_i p_i, \sum_{i=1}^{n} \sigma_i \Omega_i \right) \prod_{i>j}^{(n)} F(\sigma_i p_i - \sigma_j p_j, \sigma_i \Omega_i - \sigma_j \Omega_j) \sigma_i \sigma_j \right\}. \tag{A.3}$$

In (A.3) the Ω_i are given in terms of the p_i by the dispersion relations $F(p_i, \Omega_i) = 0$ ($i = 1, \ldots, N$), $\sum_{\sigma=\pm 1}$ means the summation over all possible combinations of $\sigma_1 = \pm 1, \sigma_2 = \pm 1, \ldots, \sigma_n = \pm 1$, and C is a function of the p_i that is independent of the summation indices $\sigma_1, \ldots, \sigma_n$.

From (A.2) it follows that (A.1) is satisfied for $n = 1$. If $F(p, \Omega) = F(-p, -\Omega)$, which is true of (3.2), then (A.1) is satisfied for $n = 2$. Hence there is a 2SS. However, whether or not (A.1) is satisfied for $n \geq 3$ depends on the particular form of $F(p, \Omega)$.

With F given by (3.2), the dispersion relations give $\Omega_i = -p_i/((p_i^2 + \beta)$ and (A.3) may be written

$$G^{(n)}(p_1, \ldots, p_n) := \sum_{\sigma=\pm 1} \left\{ \left(\sum_{i=1}^{n} \sigma_i p_i \right) \left[\sum_{i=1}^{n} \sigma_i p_i - \left(\sum_{i=1}^{n} \frac{\sigma_i p_i}{p_i^2 + \beta} \right) \left(\beta + \left(\sum_{i=1}^{n} \sigma_i p_i \right)^2 \right) \right] \right.$$
$$\left. \times \prod_{i>j}^{(n)} (\sigma_i p_i - \sigma_j p_j)^2 (p_i^2 + p_j^2 - \sigma_i \sigma_j p_i p_j + 3\beta) \right\}. \tag{A.4}$$

In order to prove that the NSC is satisfied, we closely followed the work by Musette et al in [**6**]. In [**6**], the expression for $G^{(n)}$ is equivalent to (A.4) with $\beta = -1$. When $\beta < 0$, (3.2) may be rescaled to correspond to the case $\beta = -1$. However for $\beta > 0$, (3.2) cannot be rescaled to correspond to the case $\beta = -1$. Consequently we need to prove the NSC for general β.

We need the following properties of $G^{(n)}$ (as given by (A.4)) for $n \geq 3$:

(i) $G^{(n)}(p_1, \ldots, p_n)|_{p_1=0} = 2\prod_{i=2}^{n} p_i^2(p_i^2 + 3\beta) G^{(n-1)}(p_2, \ldots, p_n)$,

(ii) $G^{(n)}(p_1, \ldots, p_n)|_{p_1=\pm p_2} = 24 p_1^2(p_1^2 + \beta) \prod_{i=3}^{n} (p_i^2 - p_1^2)^2 [(p_1^2 + p_i^2 + 3\beta)^2 - p_1^2 p_i^2]$ $G^{(n-2)}(p_3, \ldots, p_n)$,

(iii) $G^{(n)}(p_1, \ldots, p_n)|_{p_1^2+p_2^2 \pm p_1 p_2 + 3\beta = 0} = (p_1 \mp p_2)^2 (p_1^2 + p_2^2 \mp p_1 p_2 + 3\beta)$ $\times \prod_{i=3}^{n} \{[p_i^2 + (p_1 \pm p_2)^2 + 3\beta]^2 - p_i^2 (p_1 \pm p_2)^2\} G^{(n-1)}(p_1 \pm p_2, p_3, \ldots, p_n)$.

Furthermore, because of the σ summation in (A.4), $G^{(n)}$ is an even, symmetric function of the p_i.

Now consider the polynomial $P^{(n)}$ defined by

$$P^{(n)}(p_1, \ldots, p_n) := \prod_{i=1}^{n} (p_i^2 + \beta) G^{(n)}(p_1, \ldots, p_n). \tag{A.5}$$

As already noted, the condition (A.1) is satisfied for $n = 1$ and $n = 2$. We now assume that the condition is satisfied for all $n \leq m - 1$, where $m \geq 3$; then the properties of $G^{(n)}$ imply that the polynomial $P^{(m)}$ may be factorised as follows:

$$P^{(m)}(p_1, \ldots, p_m) = \left[\prod_{i=1}^{m} p_i^2 \right] \left[\prod_{i>j}^{(m)} (p_i^2 - p_j^2)^2 (p_i^2 + p_j^2 + p_i p_j + 3\beta)(p_i^2 + p_j^2 - p_i p_j + 3\beta) \right]$$
$$\times \tilde{P}^{(m)}(p_1, \ldots, p_m) \tag{A.6}$$

where $\tilde{P}^{(m)}$ is some polynomial.

From (A.6) the degree of $P^{(m)}$ is at least $4m^2 - 2m$. On the other hand, from (A.4) and (A.5), the degree of $P^{(m)}$ is at most $2m^2 + 2$. As $4m^2 - 2m > 2m^2 + 2$ for

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$m \geq 3$, it follows that $P^{(m)} \equiv 0$ and hence $G^{(m)} \equiv 0$. It now follows by induction that the NSC is satisfied.

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