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A GENERALIZATION OF COMMUTATIVE AND ALTERNATIVE RINGS II

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We shall call a linear function on the elements of a ring R skew-symmetric if it vanishes whenever at least two of the variables are equal. Here we shall study rings R of characteristic not 2 which satisfy the following two identities:

$$(x, x, x) = 0, \tag{1}$$

$$F(w, x, y, z) \equiv ((w, x), y, z) + (w, x, (y, z)) \text{ is skew-symmetric.} \tag{2}$$

Both of these identities hold in alternative rings. The fact that $F(w, x, y, z)$ is skew-symmetric in alternative rings is an important tool in the study of such rings. It is also obvious that both identities hold in commutative rings. But unlike other recent generalizations of commutative and alternative rings it turns out that there exist simple, finite dimensional algebras of degree two which are neither alternative nor commutative and satisfy (1) and (2). Nevertheless we are able to determine all those simple rings R which possess an idempotent e such that $(e, R) \not\equiv 0$, while $(e, e, R) = 0 = (R, e, e)$, for it turns out that they must be alternative.

Because of our assumptions about e, we can use (1) to prove that $(e, R, e) = 0$. But then we have the usual Peirce decomposition $R = R_{11} \oplus R_{10} \oplus R_{01} \oplus R_{00}$, where $x_{ij} \in R_{ij}$ if and only if $e(x_{ij}) = ix_{ij}$, $(x_{ij})e = jx_{ij}$, and $i, j = 0, 1$. We shall now concern ourselves with the multiplication table of the Peirce decomposition. We have $0 = F(e, y_{10}, e, x_{11}) = ((e, y_{10}), e, x_{11}) + (e, y_{10}, (e, x_{11})) = (y_{10}, e, x_{11}) = -y_{10}x_{11}$. Thus

$$R_{10}R_{11} = 0. \tag{3}$$

Similarly $0 = F(x_{11}, e, e, y_{01}) = ((x_{11}, e), e, y_{01}) + (x_{11}, e, (e, y_{01})) = -(x_{11}, e, y_{01}) = -x_{11}y_{01}$. Thus

$$R_{11}R_{01} = 0. \tag{4}$$

Also $0 = F(e, y_{01}, e, x_{00}) = ((e, y_{01}), e, x_{00}) + (e, y_{01}, (e, x_{00})) = -(y_{01}, e, x_{00}) = -y_{01}x_{00}$. This shows

$$R_{01}R_{00} = 0. \tag{5}$$

Finally $0 = F(x_{00}, e, e, y_{10}) = ((x_{00}, e), e, y_{10}) + (x_{00}, e, (e, y_{10})) = (x_{00}, e, y_{10}) = -x_{00}y_{10}$, so that

$$R_{00}R_{10} = 0. \tag{6}$$

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Because of (3), it follows that $0 = (y_{10}, x_{11}, e) = (e, y_{10}, x_{11}) = (y_{10}, e, x_{11})$. Also by expansion $(x_{11}, e, y_{10}) = 0$. But then $0 = F(e, x_{11}, e, y_{10}) = ((e, x_{11}), e, y_{10}) + (e, x_{11}, (e, y_{10})) = (e, x_{11}, y_{10})$. A linearization of (1) implies that $(y_{10}, x_{11}, e) + (x_{11}, e, y_{10}) + (e, y_{10}, x_{11}) + (y_{10}, e, x_{11}) + (x_{11}, y_{10}, e) + (e, x_{11}, y_{10}) = 0$. But five of these associators have already been shown to be zero. Hence the sixth must be zero. From $0 = (e, x_{11}, y_{10}) = (x_{11}, y_{10}, e)$, it follows that $e(x_{11}y_{10}) = x_{11}y_{10}$, and $(x_{11}y_{10})e = 0$. Thus $x_{11}y_{10} \in R_{10}$. We have shown

(7) $$R_{11}R_{10} \subset R_{10}.$$

By switching subscripts in the proof of (7) and by going to the anti-isomorphic copy of R, as well as by a combination of the two we are quickly led to

(8) $$R_{00}R_{01} \subset R_{01},$$

(9) $$R_{01}R_{11} \subset R_{01},$$

(10) $$R_{10}R_{00} \subset R_{10}.$$

Then $0 = F(e, x_{10}, e, y_{10}) = ((e, x_{10}), e, y_{10}) + (e, x_{10}, (e, y_{10})) = (x_{10}, e, y_{10}) + (e, x_{10}, y_{10}) = -x_{10}y_{10} + x_{10}y_{10} - e(x_{10}y_{10}) = -e(x_{10}y_{10})$. Also $0 = F(e, x_{10}, y_{10}, e) = ((e, x_{10}), y_{10}, e) + (e, x_{10}, (y_{10}, e)) = (x_{10}, y_{10}, e) - (e, x_{10}, y_{10}) = (x_{10}y_{10})e - x_{10}y_{10} + e(x_{10}y_{10}) = (x_{10}y_{10})e - x_{10}y_{10}$. Clearly $x_{10}y_{10} \in R_{01}$. Thus

(11) $$R_{10}R_{10} \subset R_{01}.$$

By reversing subscripts in the proof of (11), we obtain

(12) $$R_{01}R_{01} \subset R_{10}.$$

But then $0 = F(e, x_{10}, y_{01}, e) = ((e, x_{10}), y_{01}, e) + (e, x_{10}, (y_{01}, e)) = (x_{10}, y_{01}, e) + (e, x_{10}, y_{01})$. Also $0 = F(x_{10}, e, y_{01}, e) = ((x_{10}, e), y_{01}, e) + (x_{10}, e, (y_{01}, e)) = -(x_{10}, y_{01}, e) + (x_{10}, e, y_{01}) = -(x_{10}, y_{01}, e)$. But then $(x_{10}, y_{01}, e) = 0$, so that $(e, x_{10}, y_{01}) = 0$. Expanding one gets $(x_{10}y_{01})e = x_{10}y_{01} = e(x_{10}y_{01})$. Thence $x_{10}y_{01} \in R_{11}$, proving

(13) $$R_{10}R_{01} \subset R_{11}.$$

Reversing subscripts in the proof of (13) one obtains

(14) $$R_{01}R_{10} \subset R_{00}.$$

A linearization of (1) implies that $0 = (e, x_{10}, x_{10}) + (x_{10}, e, x_{10}) + (x_{10}, x_{10}, e)$. Expanding these associators and using (11), we see that $0 = x_{10}^2 - x_{10}^2 + x_{10}^2 = x_{10}^2$. Thus

(15) $$x_{10}^2 = 0 = x_{10}y_{10} + y_{10}x_{10}.$$

By reversing subscripts in the proof of (15) it follows that

(16) $$x_{01}^2 = 0 = x_{01}y_{01} + y_{01}x_{01}.$$

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Then $F(a_{10}, e, c_{11}, b_{01}) = ((a_{10}, e), c_{11}, b_{01}) + (a_{10}, e, (c_{11}, b_{01})) = -(a_{10}, c_{11}, b_{01}) - (a_{10}, e, b_{01}c_{11}) = 0$, using (3), (4) and (9). Hence we may use (2) to obtain $0 = F(a_{10}, e, b_{01}, c_{11}) = ((a_{10}, e), b_{01}, c_{11}) + (a_{10}, e, (b_{01}, c_{11})) = -(a_{10}, b_{01}, c_{11}) + (a_{10}, e, b_{01}c_{11}) = -(a_{10}, b_{01}, c_{11})$, using (4) and (9). Therefore

(17) $$(R_{10}, R_{01}, R_{11}) = 0 \quad \text{and} \quad (R_{10}R_{01})R_{11} \subset R_{10}R_{01}.$$

Again

$$\begin{aligned} 0 &= F(c_{11}, b_{01}, e, a_{10}) = ((c_{11}, b_{01}), e, a_{10}) + (c_{11}, b_{01}, (e, a_{10})) \\ &= -(b_{01}c_{11}, e, a_{10}) + (c_{11}, b_{01}, a_{10}) \\ &= (c_{11}, b_{01}, a_{10}) \\ &= -c_{11}(b_{01}a_{10}), \end{aligned}$$

using (4), so that

(18) $$(R_{11}, R_{01}, R_{10}) = 0 \quad \text{and} \quad R_{11}(R_{01}R_{10}) = 0.$$

Next

$$\begin{aligned} 0 &= F(e, b_{01}, a_{10}, c_{11}) = ((e, b_{01}), a_{10}, c_{11}) + (e, b_{01}, (a_{10}, c_{11})) \\ &= -(b_{01}, a_{10}, c_{11}) - (e, b_{01}, c_{11}a_{10}) \\ &= -(b_{01}, a_{10}, c_{11}), \end{aligned}$$

using (4) and (14). Hence

(19) $$(R_{01}, R_{10}, R_{11}) = 0 \quad \text{and} \quad (R_{01}R_{10})R_{11} = 0.$$

Again

$$\begin{aligned} 0 &= F(c_{11}, a_{10}, e, b_{01}) = ((c_{11}, a_{10}), e, b_{01}) + (c_{11}, a_{10}, (e, b_{01})) \\ &= (c_{11}a_{10}, e, b_{01}) - (c_{11}, a_{10}, b_{01}) \\ &= -(c_{11}, a_{10}, b_{01}), \end{aligned}$$

using (4) and (7). Hence

(20) $$(R_{11}, R_{10}, R_{01}) = 0 \quad \text{and} \quad R_{11}(R_{10}R_{01}) \subset R_{10}R_{01}.$$

Similarly, reversing subscripts in the proofs of (17)–(20) we get

(21) $$(R_{01}, R_{10}, R_{00}) = 0 \quad \text{and} \quad (R_{01}R_{10})R_{00} \subset R_{01}R_{10};$$

(22) $$(R_{00}, R_{10}, R_{01}) = 0 \quad \text{and} \quad R_{00}(R_{10}R_{01}) = 0;$$

(23) $$(R_{10}, R_{01}, R_{00}) = 0 \quad \text{and} \quad (R_{10}R_{01})R_{00} = 0;$$

(24) $$(R_{00}, R_{01}, R_{10}) = 0 \quad \text{and} \quad R_{00}(R_{01}R_{10}) \subset R_{01}R_{10}.$$

We are now ready to construct an ideal.

THEOREM 1. *$A = R_{10}R_{01} + R_{10} + R_{01} + R_{01}R_{10}$ is an ideal of R.*

Proof. $(R_{10}R_{01})R_{11} \subset R_{10}R_{01} \subset A$, using (17). $(R_{10}R_{01})R_{10} \subset R_{11}R_{10} \subset R_{10} \subset A$, using (13) and (7). $(R_{10}R_{01})R_{01} \subset R_{11}R_{01} = 0$, using (13) and (4). $(R_{10}R_{01})R_{00} = 0$, using (23). $R_{11}(R_{10}R_{01}) \subset R_{10}R_{01} \subset A$, using (20). $R_{10}(R_{10}R_{01}) \subset R_{10}R_{11} = 0$,

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using (13) and (3). $R_{01}(R_{10}R_{01}) \subset R_{01}R_{11} \subset R_{01} \subset A$, using (13) and (9). $R_{00}(R_{10}R_{01}) = 0$, using (22). $R_{10}R_{11} = 0$, using (3). $R_{10}R_{10} \subset R_{01} \subset A$, using (11). $R_{10}R_{01} \subset A$. Then $R_{10}R_{00} \subset R_{10} \subset A$, using (10). $R_{11}R_{10} \subset R_{10} \subset A$, using (7). $R_{10}R_{10} \subset R_{01} \subset A$, using (11). $R_{01}R_{10} \subset A$, while $R_{00}R_{10} = 0$, using (6). Then $R_{01}R_{11} \subset R_{01} \subset A$, using (9). $R_{01}R_{10} \subset A$, while $R_{01}R_{01} \subset R_{10} \subset A$, using (12). Then $R_{01}R_{00} = 0$, using (5). Similarly $R_{11}R_{01} = 0$, using (4). $R_{10}R_{01} \subset A$, while $R_{01}R_{01} \subset R_{10} \subset A$, using (12). $R_{00}R_{01} \subset R_{01} \subset A$, using (8). $(R_{01}R_{10})R_{11} = 0$, using (19). $(R_{01}R_{10})R_{10} \subset R_{00}R_{10} = 0$, using (14) and (6). $(R_{01}R_{10})R_{01} \subset R_{00}R_{01} \subset R_{01} \subset A$, using (14) and (8). $(R_{01}R_{10})R_{00} \subset R_{01}R_{10}$, using (21). $R_{11}(R_{01}R_{10}) = 0$, using (18). $R_{10}(R_{01}R_{10}) \subset R_{10}R_{00} \subset R_{10} \subset A$, using (14) and (10). $R_{01}(R_{01}R_{10}) \subset R_{01}R_{00} = 0$, using (14) and (5). $R_{00}(R_{01}R_{10}) \subset R_{01}R_{10}$, using (24). This completes the proof of the theorem.

COROLLARY. $A = R, R_{01}R_{10} = R_{00}, R_{10}R_{01} = R_{11}, R_{11}R_{11} \subset R_{11}, R_{00}R_{00} \subset R_{00}$, $R_{11}R_{00} = 0$, $R_{00}R_{11} = 0$.

Proof. Since R is simple, either $A = 0$ or $A = R$. But $A = 0$ implies $R_{10} = R_{01} = 0$, so that $R_{11} + R_{00} = R$ and hence $(e, R) = 0$, contrary to assumption. Thus $A = R$. But then it follows from the directness of the Peirce decomposition that $R_{10}R_{01} = R_{11}$, and that $R_{01}R_{10} = R_{00}$. The rest of the corollary follows from substituting the latest two equalities in (17), (21), (22) and (23).

THEOREM 2. *R is alternative, hence is either associative or a Cayley vector matrix algebra over its center.*

Proof. As in the case of an alternative ring it is possible to prove that all associators with components from R_{ij} vanish except possibly those which have all three components in the same R_{ii}, or those which have at least two components in the same R_{ij}, where $i \neq j$. Most of the calculations are simple applications of the appropriate equation chosen from (3)–(14) and (17)–(24). We present now a sample of those which require additional calculations. Thus it is a direct consequence of (3) and the corollary that $(R_{10}, R_{11}, R_{11}) = 0$, and $(R_{11}, R_{10}, R_{11}) = 0$. But then

$$\begin{aligned} F(e, x_{10}, y_{11}, z_{11}) &= ((e, x_{10}), y_{11}, z_{11}) + (e, x_{10}, (y_{11}, z_{11})) \\ &= (x_{10}, y_{11}, z_{11}) + (e, x_{10}, w_{11}) = 0. \end{aligned}$$

Hence

$$\begin{aligned} 0 = F(y_{11}, z_{11}, e, x_{10}) &= ((y_{11}, z_{11}), e, x_{10}) + (y_{11}, z_{11}, (e, x_{10})) \\ &= (w_{11}, e, x_{10}) + (y_{11}, z_{11}, x_{10}) = (y_{11}, z_{11}, x_{10}). \end{aligned}$$

This proves $(R_{11}, R_{11}, R_{10}) = 0$. In a like manner we can establish $(R_{01}, R_{11}, R_{11}) = 0$, $(R_{10}, R_{00}, R_{00}) = 0$, and $(R_{00}, R_{00}, R_{01}) = 0$. The remaining associators involving either two elements from R_{11} or two elements from R_{00} are easy to dispose of. Suppose we need to prove $(x_{11}, y_{10}, z_{00}) = 0$. It is clear that $(x_{11}, z_{00}, y_{10}) = 0$, using (6) and the corollary. Then $(z_{00}, y_{10}, x_{11}) = 0$, using (6) and (3). Also $(z_{00}, x_{11}, y_{10}) = 0$, using the corollary, (7) and (6). But $(y_{10}, z_{00}, x_{11}) = 0$, using

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(10), (3) and the corollary, while $(y_{10}, x_{11}, z_{00}) = 0$, using (3) and the corollary. But a linearization of (1) implies that $(x_{11}, y_{10}, z_{00}) + (y_{10}, z_{00}, x_{11}) + (z_{00}, x_{11}, y_{10}) + (x_{11}, z_{00}, y_{10}) + (y_{10}, x_{11}, z_{00}) + (z_{00}, y_{10}, x_{11}) = 0$. Since five of the six terms are known to be zero then $(x_{11}, y_{10}, z_{00}) = 0$. We can establish $(y_{10}, x_{11}, z_{01}) = 0$ using (3) and (4). Also $(x_{11}, z_{01}, y_{10}) = 0$, as a result of (18), while $(z_{01}, y_{10}, x_{11}) = 0$, because of (19). Now (17) implies $(y_{10}, z_{01}, x_{11}) = 0$, while (20) implies $(x_{11}, y_{10}, z_{01}) = 0$. By using the linearization of (1) we obtain $(z_{01}, x_{11}, y_{10}) = 0$. The remaining calculations can be obtained by reversing subscripts in the previous calculations as well as by going to the anti-isomorphic copy of R. For those associators which do not vanish we must prove the alternative law. First we will establish that R_{11} and R_{00} are associative. In an arbitrary ring we have the Teichmüller identity $(wx, y, z) - (w, xy, z) + (w, x, yz) = w(x, y, z) + (w, x, y)z$. Therefore $(w_{11}x_{11}, y_{11}, z_{10}) - (w_{11}, x_{11}y_{11}, z_{10}) + (w_{11}, x_{11}, y_{11}z_{10}) = w_{11}(x_{11}, y_{11}, z_{10}) + (w_{11}, x_{11}, y_{11})z_{10}$. Then using the corollary, (7), and the fact that

$$0 = (R_{11}, R_{11}, R_{10}),$$

which we established earlier in the proof, we see that $(w_{11}, x_{11}, y_{11})z_{10} = 0$. Consequently

$$(R_{11}, R_{11}, R_{11})R_{10} = 0. \tag{25}$$

Let B be the associator ideal of R_{11}. This ideal can be described as the additive subgroup generated by all associators and right multiples of associators involving elements of R_{11}. The same ideal results if we substitute left multiples for right multiples. Since $(R_{11}, R_{11}, R_{11}) \subset R_{11}$, and $(R_{11}, R_{11}, R_{10}) = 0$, then use of the corollary leads to $[R_{11}(R_{11}, R_{11}, R_{11})]R_{10} = R_{11}[(R_{11}, R_{11}, R_{11})R_{10}] = 0$, using (25). Thus

$$BR_{10} = 0. \tag{26}$$

Since we showed earlier that $(R_{11}, R_{10}, R_{01}) = 0$, then $0 = (B, R_{10}, R_{01}) = B(R_{10}R_{01})$. Now we have all the information necessary to prove that $B(R_{10}R_{01} + R_{10} + R_{01} + R_{01}R_{10}) = 0$, or $BA = 0$, where $A = R_{10}R_{01} + R_{10} + R_{01} + R_{01}R_{10}$. But the corollary established that $A = R$, so we must have $B = 0$. In similar fashion if C is the associator ideal of R_{00}, then $CA = 0$, so that $C = 0$. Consequently R_{11} and R_{00} are associative subrings of R. Expanding,

$$\begin{aligned} f &= (x_{10}, y_{10}, z_{11}) = (x_{10}y_{10})z_{11} = -(y_{10}x_{10})z_{11} \\ &= -(y_{10}, x_{10}, z_{11}), \end{aligned}$$

using (3) and (15). Then

$$\begin{aligned} g &= (x_{10}, z_{11}, y_{10}) = -x_{10}(z_{11}y_{10}) \\ &= (z_{11}y_{10})x_{10} = (z_{11}, y_{10}, x_{10}), \end{aligned}$$

using (3), (15), (7) and (11). Interchanging x_{10} and y_{10} in the last equation we obtain $h = (y_{10}, z_{11}, x_{10}) = (z_{11}, x_{10}, y_{10})$. Using a linearization of (1) we get

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$f - f + 2g + 2h = 0$, implying $g = -h$. Also $F(x_{10}, y_{10}, e, z_{11}) = ((x_{10}, y_{10}), e, z_{11}) + (x_{10}, y_{10}, (e, z_{11})) = ((x_{10}, y_{10}), e, z_{11}) = (w_{01}, e, z_{11}) = 0$. Thus (2) implies $0 = F(x_{10}, e, y_{10}, z_{11}) = ((x_{10}, e), y_{10}, z_{11}) + (x_{10}, e, (y_{10}, z_{11})) = -(x_{10}, y_{10}, z_{11}) - (x_{10}, e, z_{11}y_{10}) = -f + x_{10}(z_{11}y_{10}) = -f - (x_{10}, z_{11}, y_{10}) = -f - g$. But then $f = -g = h$. This proves the alternative law for the triple x_{10}, y_{10}, z_{11}. Interchanging subscripts in the proof and going to the anti-isomorphic copy of R yields the same result for x_{01}, y_{01}, z_{00}, then for x_{10}, y_{10}, z_{00} and finally for x_{01}, y_{01}, z_{11}. Note that $F(x_{10}, e, y_{10}, z_{10}) = ((x_{10}, e), y_{10}, z_{10}) + (x_{10}, e, (y_{10}, z_{10})) = -(x_{10}, y_{10}, z_{10}) + (x_{10}, e, w_{01}) = -(x_{10}, y_{10}, z_{10})$, using (11). Now it follows from (2) that we have the alternative law for the triple x_{10}, y_{10}, z_{10}. Then interchanging subscripts in the proof we are led to the same result for x_{01}, y_{01}, z_{01}. Let $a = F(x_{10}, e, y_{10}, z_{01}) = ((x_{10}, e), y_{10}, z_{01}) + (x_{10}, e, (y_{10}, z_{01})) = -(x_{10}, y_{10}, z_{01}) - x_{10}(y_{10}z_{01}) = -(x_{10}, y_{10}, z_{01}) = (y_{10}, x_{10}, z_{01})$, using (2). Since $-(x_{10}, y_{10}, z_{01}) = -(x_{10}y_{10})z_{01} = z_{01}(x_{10}y_{10}) = -(z_{01}, x_{10}, y_{10})$, using (16), we see that $a = (y_{10}, x_{10}, z_{01}) = -(x_{10}, y_{10}, z_{01}) = -(z_{01}, x_{10}, y_{10}) = (z_{01}, y_{10}, x_{10})$. By using a linearization of (1) we get $(y_{10}, z_{01}, x_{10}) + (x_{10}, z_{01}, y_{10}) = 0$. Let $b = (y_{10}, z_{01}, x_{10}) = -(x_{10}, z_{01}, y_{10})$. Then $-a = F(x_{10}, e, z_{01}, y_{10}) = ((x_{10}, e), z_{01}, y_{10}) + (x_{10}, e, (z_{01}, y_{10})) = -(x_{10}, z_{01}, y_{10}) + (x_{10}, e, w_{00} - w_{11}) = -(x_{10}, z_{01}, y_{10}) = b$. This proves the alternative law for x_{10}, y_{10}, z_{01}. By interchanging subscripts in the proof we get the same result for x_{01}, y_{01}, z_{10}. Thus we have proved that R is alternative. Then by a theorem of A. A. Albert [**1**] R must be either associative or a Cayley vector matrix algebra of dimension 8 over its center. This completes the proof of the theorem.

In [**2**] we gave an example of a three dimensional algebra with basis $1, x, y$ such that $xy = 1$, while $yx = x^2 = y^2 = 0$. This algebra turns out to be simple, power-associative, quadratic and every commutator lies in the center. Hence it certainly satisfies (1) and (2). Moreover this algebra has an idempotent $e = \frac{1}{2}(1 + x + y)$, with $ey - ye = \frac{1}{2} \neq 0$. Clearly this algebra is neither commutative nor alternative. This shows the necessity of assuming the existence of a Peirce decomposition relative to e before one can hope to prove alternativity.

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