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A presentation for the Eisenstein-Picard modular group in three complex dimensions

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Abstract
A. Mark and J. Paupert [Presentations for cusped arithmetic hyperbolic lattices, 2018, arXiv:1709.06691.] presented a method to compute a presentation for any cusped complex hyperbolic lattice. In this note, we will use their method to give a presentation for the Eisenstein-Picard modular group in three complex dimensions.

1. Introduction

To study discrete subgroups of Lie groups, it is a challenging work to determine a presentation of a discrete subgroup. The purpose of this paper is to give a presentation of an arithmetic lattice in PU(3, 1), which is the holomorphic isometry group of complex hyperbolic space $\mathbf{H}^3_{\mathbb{C}}$. It is well known that complex hyperbolic space is one of the non-compact rank one symmetric spaces.

As is widely known, the classical modular group $\mathrm{PSL}(2, \mathbb{Z})$ is a free product of a cyclic group of order 2 and a cyclic group of order 3. To see this, there are many different ways. One of them is an algebraic method that uses a continued fraction algorithm to compute it directly. The other one is a geometric method that uses the action of $\mathrm{PSL}(2, \mathbb{Z})$ on the hyperbolic plane. One can consider $\mathrm{PSL}(2, \mathbb{Z})$ as a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ that acts isometrically on the hyperbolic plane $\mathbf{H}^2$. There will be a fundamental domain for this action. The presentation for $\mathrm{PSL}(2, \mathbb{Z})$ can be obtained by applying Poincaré's polygon theorem to the fundamental domain.

There are two kinds of generalizations of $\mathrm{PSL}(2, \mathbb{Z})$ in higher dimensional hyperbolic space, (real) hyperbolic space $\mathbf{H}^n_{\mathbb{R}}$, and complex hyperbolic space $\mathbf{H}^n_{\mathbb{C}}$. Before we introduce these generalizations in detail, we would like to explain some notations. Let $\mathcal{O}_d$ be the ring of integers in $\mathbb{Q}(\sqrt{-d})$ with d being a positive square-free integer. $\mathcal{O}_d$ is a Euclidean domain when $d = 1, 2, 3, 7, 11$.

In the real case, Bianchi groups $\mathrm{PSL}(2; \mathcal{O}_d)$ are subgroups of $\mathrm{PSL}(2, \mathbb{C})$ with entries in $\mathcal{O}_d$. Here, $\mathrm{PSL}(2, \mathbb{C})$ is the orientation-preserving isometry group of $\mathbf{H}^3_{\mathbb{R}}$. These groups are arithmetic lattices that are finitely presentable. Presentations for Bianchi groups can be obtained geometrically. Considering the actions of the Bianchi groups on hyperbolic space, Swan [13] developed a method to find the Ford domains. Based on these domains and using the generalization theorem of Macbeath [7], Swan derived a method to obtain presentations for the Bianchi groups. In particular, finite presentations for the Euclidean Bianchi groups can be derived from a result of Cohn, see [1] and [2].

In the complex case, Picard modular groups $\mathrm{PU}(n, 1; \mathcal{O}_d)$ are subgroups of $\mathrm{PU}(n, 1)$ with entries in $\mathcal{O}_d$. Here $\mathrm{PU}(n, 1)$ is the holomorphic isometry group of $\mathbf{H}^n_{\mathbb{C}}$. These groups are arithmetic lattices that are finitely presentable. To the best of our knowledge, the presentations for some of them are known. A presentation for the Eisenstein-Picard modular group $\mathrm{PU}(2, 1; \mathcal{O}_3)$ was found by Falbel and Parker in [5], in which they studied the geometry of the group. A few years later, a presentation for the

Gauss-Picard modular group PU(2, 1; $\mathcal{O}_1$) was given in [4]. In his paper [20], Zhao gave a finite system of generators for each one of the Euclidean-Picard modular groups PU(2, 1; $\mathcal{O}_d$). We remark that the sister of the Eisenstein-Picard modular group was studied in [10], and a presentation was given in [19]. In particular, Stover [12] showed that the quotient spaces of $\mathbf{H}^2_{\mathbb{C}}$ by the Eisenstein-Picard modular group and its sister group are the only two arithmetic cusped complex hyperbolic 2-orbifolds with minimal volume. The sister of the other Euclidean-Picard modular groups was studied in [16], in which a finite system of generators of them was constructed. The methods in these papers are geometric, that is, they construct fundamental domains for the Euclidean-Picard modular groups PU(2, 1; $\mathcal{O}_d$) acting on complex hyperbolic space. On the other hand, one can give a finite system of generators for some of the Picard modular groups using an algebraic method. In [3], the authors obtained a system of generators for the Gauss-Picard modular group PU(2, 1; $\mathcal{O}_1$) by using the continued fraction algorithm. One can also use this method to derive a system of generators for the Eisenstein-Picard modular group PU(2, 1; $\mathcal{O}_3$), see [14].

For the Picard modular groups in three complex dimensions PU(3, 1; $\mathcal{O}_d$), we know very little. Since it is very difficult to construct a fundamental domain for PU(3, 1; $\mathcal{O}_d$) acing on $\mathbf{H}^3_{\mathbb{C}}$, we do not know a presentation of PU(3, 1; $\mathcal{O}_d$). In [17], the authors gave a system of finite generators for PU(3, 1; $\mathcal{O}_3$) by using the continued fraction algorithm. In [18], a finitely generating set of PU(3, 1; $\mathcal{O}_1$) was given by using a combined geometric and algebraic method. Unfortunately, it seems that it is difficult to give a presentation for these two groups. In a recent work, Mark and Paupert [8] present a method to obtain a presentation for Picard modular group PU(2, 1; $\mathcal{O}_d$) with $d = 1, 3, 7$, by applying a result of Macbeath [7]. Polletta [11] discussed the application of their method to the Picard modular groups, PU(2, 1; $\mathcal{O}_d$), when $d = 2$, 11, and obtained presentations for these groups. Inspired by their work, in this paper, we obtain a presentation for PU(3, 1; $\mathcal{O}_3$) (see Theorem 4.4).

To do this, we use the method of Mark and Paupert [8]. Let $\Gamma = \mathrm{PU}(3, 1; \mathcal{O}_3)$. Consider the action of Γ on $\mathbf{H}^3_{\mathbb{C}}$. One can construct a suitable horoball V based on the cusp of Γ, such that the Γ-orbits of V form a covering of $\mathbf{H}^3_{\mathbb{C}}$. Let $E(V) = \{g \in \Gamma | g(V) \cap V \neq \emptyset\}$. Then, by Macbeath's theorem [7], Γ will be generated by $E(V)$ together with relations $g \cdot h = gh$ for each pair $g, h \in E(V)$ with $V \cap g(V) \cap gh(V) \neq \emptyset$, where $gh \in E(V)$.

In practice, our process is as follows.

Step 1: give a presentation for the stabilizer subgroup Γ_∞ of Γ, which fixing the point at infinity q_∞, see Corollary 2.6.
Step 2: give a suitable coarse fundamental domain D_∞ of Γ_∞, see Definition 3.1.
Step 3: determine the covering depth of Γ, which is at most 4, see Section 3.2.
Step 4: find all the $\mathcal{O}_3$-rational points of depth at most 4 in D_∞ and a system of representatives $\{p_\alpha : \alpha \in \{0, 31, 32, 41, 42, 43\}\}$ of Γ_∞-orbits of these $\mathcal{O}_3$-rational points, see Section 3.3.
Step 5: give a map $A_\alpha \in \Gamma$ sending q_∞ to p_α for each $\alpha \in \{0, 31, 32, 41, 42, 43\}$, see Section 4.1.
Step 6: determine $\gamma, W, W' \in \Gamma_\infty$ such that $A_c^{-1}W^{-1}A_a\gamma A_b = W'$ for each triple (A_a, p_b, p_c), where $A_a \in \{I_0, A_{31}, A_{32}, A_{41}, A_{42}, A_{43}\}$, p_b, p_c be two $\mathcal{O}_3$-rational points of depth at most 4 in D_∞ and A_b, A_c be two maps sending q_∞ to p_b, p_c, respectively. See Section 4.2.
Step 7: obtain a presentation $\Gamma = \langle S|R\rangle$, where S consists of the generators of Γ_∞ and A_α with $\alpha \in \{0, 31, 32, 41, 42, 43\}$, and R consists of relations of Γ_∞ and the forms $A_c^{-1}W^{-1}A_a\gamma A_b = W'$ given in Step 6.

2. Background

2.1. Complex hyperbolic space and its isometry group

We will introduce some material on complex hyperbolic geometry in this subsection. More details can be found in Goldman's book [6] and Parker's notes [9].

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Let H be a Hermitian matrix given by the following

$$H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & I_{n-1} & 0 \\ 1 & 0 & 0 \end{pmatrix},$$

where I_{n-1} is the identity matrix. H induces a Hermitian form on the complex vector space $\mathbb{C}^{n+1}$ given by

$$\langle z, w\rangle = \bar{w}^t H z = z_1\overline{w_{n+1}} + z_{n+1}\overline{w_1} + z_2\overline{w_2} + \cdots + z_n\overline{w_n},$$

where $z = (z_1, z_2, \cdots, z_n, z_{n+1})^t$ and $w = (w_1, w_2, \cdots, w_n, w_{n+1})^t$ are vectors in $\mathbb{C}^{n+1}$. Equipped with this Hermitian form, we write $\mathbb{C}^{n,1}$ instead of $\mathbb{C}^{n+1}$.

Let V_+ (resp. V_0 or V_-) be the set of positive (resp. null or negative) vectors in $\mathbb{C}^{n,1}$, that is the vectors with $\langle z, z\rangle > 0$ (resp. $\langle z, z\rangle = 0$ or $\langle z, z\rangle < 0$). Let $P : \mathbb{C}^{n,1} \to \mathbb{C}P^n$ be the canonical projection. The complex hyperbolic space is defined to be $\mathbf{H}^n_{\mathbb{C}} = P(V_-)$ associated with the Bergman metric, which is the Siegel domain in $\mathbb{C}P^n$. The boundary of complex hyperbolic space is defined to be $\partial\mathbf{H}^n_{\mathbb{C}} = P(V_0)$, which can be viewed as the one-point compactification of the Heisenberg group.

Let $\mathrm{U}(n, 1)$ be the unitary group preserving the Hermitian form. The holomorphic isometry group of complex hyperbolic space is $\mathrm{PU}(n, 1) = \mathrm{U}(n, 1)/\mathrm{U}(1)$.

Now, let us focus on the boundary of three-dimensional complex hyperbolic space $\partial\mathbf{H}^3_{\mathbb{C}}$. Let $\mathcal{N} = \mathbb{C}^2 \times \mathbb{R}$ be the five-dimensional Heisenberg group with the group law

$$(z_1, z_2, t)(\zeta_1, \zeta_2, v) = (z_1 + \zeta_1, z_2 + \zeta_2, t + v + 2\mathrm{Im}(\overline{\zeta_1}z_1 + \overline{\zeta_2}z_2)).$$

Thus, $\partial\mathbf{H}^3_{\mathbb{C}}$ identifies with $\mathcal{N} \cup \{q_\infty\}$ and the Siegel domain can be identified with $\mathcal{N} \times \mathbb{R}_+$. This identification gives the horospherical coordinates of points in $\mathbf{H}^3_{\mathbb{C}}$. In details, the standard lift of $(\zeta_1, \zeta_2, v, u) \in \mathcal{N} \times \mathbb{R}_+$ is $\left(\frac{-|\zeta_1|^2-|\zeta_2|^2-u+iv}{2}, \zeta_1, \zeta_2, 1\right)^t$. For each $u > 0$, the horosphere of height u is the subset of the Siegel domain given by $H_u = \mathcal{N} \times \{u\}$ and horoball of height u is $B_u = \mathcal{N} \times (u, +\infty)$.

The Cygan metric on $\mathcal{N}$ is defined to be

$$d_{\mathrm{Cyg}}\left((z_1, z_2, t), (\zeta_1, \zeta_2, v)\right) = \left||\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2 - it + iv - 2i\mathrm{Im}\left(\overline{\zeta_1}z_1 + \overline{\zeta_2}z_2\right)\right|^{1/2}. \tag{2.1}$$

This metric can be extended to a metric on $\overline{\mathbf{H}^3_{\mathbb{C}}} - \{q_\infty\}$ as the following

$$d_{\mathrm{Cyg}}\left((z_1, z_2, t, u_1), (\zeta_1, \zeta_2, v, u_2)\right) = \left||\zeta_1 - z_1|^2 + |\zeta_2 - z_2|^2 + |u_2 - u_1| - it + iv - 2i\mathrm{Im}\left(\overline{\zeta_1}z_1 + \overline{\zeta_2}z_2\right)\right|^{1/2}. \tag{2.2}$$

The Cygan sphere centered at $p_0 = (z_1, z_2, t) \in \mathcal{N}$ with radius r is defined by

$$S_r(p_0) = \{p = (\zeta_1, \zeta_2, v, u) \in \mathbf{H}^3_{\mathbb{C}} \quad | \quad d_{\mathrm{Cyg}}(p, p_0) = r\}.$$

Let $g = (g_{ij})^4_{i,j=1} \in \mathrm{PU}(3, 1)$ such that $g(q_\infty) \neq q_\infty$, i.e., $g_{41} \neq 0$. The isometric sphere of g is defined to be

$$\{z \in \mathbf{H}^3_{\mathbb{C}} : |\langle z, q_\infty\rangle| = |\langle z, g^{-1}(q_\infty)\rangle|\}.$$

The isometric sphere of g is a Cygan sphere centered at $g^{-1}(q_\infty)$ with radius $r = \sqrt{2/|g_{41}|}$.

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For any given point $(z_1, z_2, t) \in \mathcal{N}$, let

$$N_{(z_1,z_2,t)} = \begin{bmatrix} 1 & -\bar{z}_1 & -\bar{z}_2 & \frac{-|z_1|^2-|z_2|^2+it}{2} \\ 0 & 1 & 0 & z_1 \\ 0 & 0 & 1 & z_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then, $N_{(z_1,z_2,t)} \in \mathrm{PU}(3, 1)$ fixes q_∞ and translates the origin $(0, 0, 0) \in \mathcal{N}$ to the given point.

2.2. Picard modular groups

Let $\mathcal{O}_d$ be the ring of integers in the imaginary quadratic number field $\mathbb{Q}(\sqrt{-d})$, where d is a positive square-free integer. When $d \equiv 1, 2 \pmod 4$, $\mathcal{O}_d = \mathbb{Z}[\sqrt{-d}]$, and $\mathcal{O}_d = \mathbb{Z}\left[\frac{1+\sqrt{-d}}{2}\right]$ when $d \equiv 3 \pmod 4$. The Picard modular group $\mathrm{PU}(n, 1; \mathcal{O}_d)$ is the subgroup of $\mathrm{PU}(n, 1)$ with matrix entries in $\mathcal{O}_d$.

In this paper, we will study the Picard modular group in three complex dimensions with $d = 3$. It is well known that $\mathcal{O}_3 = \mathbb{Z}[\omega]$, where $\omega = \frac{-1+\sqrt{3}i}{2}$ is a cubic root of unit.

Theorem 2.1 ([17]). *Let* $\Gamma = \mathrm{PU}(3, 1; \mathbb{Z}[\omega])$. *Then,* Γ *can be generated by four elements* I_0, M_1, M_2 *and* $N_{(1,0,\sqrt{3})}$, *where*

$$I_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad N_{(1,0,\sqrt{3})} = \begin{bmatrix} 1 & -1 & 0 & \omega \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$M_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Moreover, the stabilizer subgroup Γ_∞ *that fixes the point at infinity* q_∞ *is generated by* M_1, M_2 *and* $N_{(1,0,\sqrt{3})}$.

Lemma 2.2. *The subgroup generated by* M_1 *and* M_2 *has the following presentation*

$$\langle M_1, M_2 | M_1^2, M_2^6, (M_1M_2)^2 = (M_2M_1)^2 \rangle.$$

Remark 2.3. *The subgroup generated by* M_1 *and* M_2 *is a finite group with order* 72, *which is isomorphic to the semidirect product of* $\mathbb{Z}_6 \times \mathbb{Z}_6$ *with* $\mathbb{Z}_2$.

Remark 2.4. *For any* M *in the subgroup generated by* M_1 *and* M_2, M *can be written as*

$$M = M_1^p M_2^j M_1 M_2^k,$$

where $0 \le j \le 5$, $0 \le k \le 5$ *and* $p = 0, 1$.

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To obtain a presentation for Γ_∞, we define the following notations:

$$T_0 = N_{(0,0,2\sqrt{3})},$$
$$T_1 = N_{(0,1,\sqrt{3})},$$
$$T_2 = N_{(1,0,\sqrt{3})},$$
$$T_3 = N_{(0,\omega,\sqrt{3})},$$
$$T_4 = N_{(\omega,0,\sqrt{3})}.$$

Besides, for simplicity, we define

$$T_5 = N_{\left(-\bar{\omega},-\bar{\omega},-2\sqrt{3}\right)} = T_0^{-2}T_1T_2T_3T_4,$$
$$T_6 = N_{\left(-\omega,-\omega,2\sqrt{3}\right)} = T_0^2T_3^{-1}T_4^{-1},$$

which will be used to describe some relations of PU(3, 1; $\mathbb{Z}[\omega]$) in Section 4.

Lemma 2.5. *Let T be the group generated by T_0, T_1, T_2, T_3, and T_4. Then, T has the presentation*

$$T = \langle T_0, T_1, T_2, T_3, T_4 \mid [T_1, T_2], [T_1, T_4], [T_2, T_3], [T_3, T_4], [T_1, T_3]T_0, [T_2, T_4]T_0 \rangle .$$

Furthermore, T is a normal subgroup of Γ_∞.

Proof. Let $\Pi : \mathcal{N} \to \mathbb{C}^2$ be defined by $\Pi(z_1, z_2, t) = (z_1, z_2)$, for all $(z_1, z_2, t) \in \mathcal{N}$. Then, we have the sequence

$$0 \to \mathbb{R} \to \mathcal{N} \xrightarrow{\Pi} \mathbb{C}^2 \to 0.$$

This induces the following exact sequence

$$0 \to \mathbb{R} \to Isom(\mathcal{N}) \xrightarrow{\Pi_*} Isom(\mathbb{C}^2) \to 0.$$

Since T_0, T_1, T_2, T_3, and T_4 are Heisenberg translations, the group T is a subgroup of $Isom(\mathcal{N})$. When we restrict to the subgroup T, we obtain the exact sequence

$$0 \to 2\sqrt{3}\mathbb{Z} \to T \xrightarrow{\Pi_*} \mathbb{Z}^4 \to 0.$$

Observe that the kernel of Π_* is generated by T_0. The generators of $\Pi_*(T) \cong \mathbb{Z}^4$ are $\Pi_*(T_1)$, $\Pi_*(T_2)$, $\Pi_*(T_3)$, and $\Pi_*(T_4)$, of which each pair is commutative. Hence, the relations of T are generated by the commutators

$$[T_1, T_2], [T_1, T_4], [T_2, T_3], [T_3, T_4], [T_1, T_3] = T_0^{-1}, [T_2, T_4] = T_0^{-1}.$$

It is obvious that T_0 commutes with T_1, T_2, T_3, T_4.

Note that Γ_∞ is generated by T_2, M_1 and M_2. Since

$$M_1T_1M_1^{-1} = T_2, M_1T_2M_1^{-1} = T_1, M_1T_3M_1^{-1} = T_4, M_1T_4M_1^{-1} = T_3,$$

and

$$M_2T_1M_2^{-1} = T_1, M_2T_2M_2^{-1} = T_0T_4^{-1}, M_2T_3M_2^{-1} = T_3, M_2T_4M_2^{-1} = T_2T_4,$$

we have

$$M_1T_0M_1^{-1} = T_0, M_2T_0M_2^{-1} = T_0.$$

Thus, the group T is normal in Γ_∞. □

We can obtain the presentation of the stabilizer subgroup Γ_∞ by using the procedure introduced in Lemma 15 of [8]. In short, we know that Γ_∞ has two subgroups, T and $\langle M_1, M_2\rangle$, where T is normal in Γ_∞. Then, Γ_∞ admits a presentation $\langle \mathbf{gen}|\mathbf{rel}\rangle$, where the set **gen** consists of M_1, M_2 and the generators

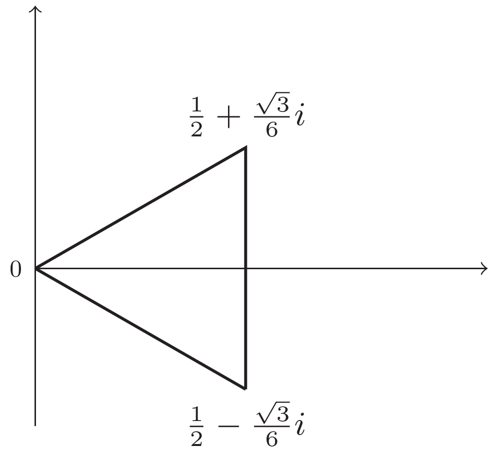

Figure 1. *A fundamental domain $\triangle$ for the group of orientation-preserving symmetries of $\mathbb{Z}[\omega] \subset \mathbb{C}$.*

of T, the set **rel** consists of relations of those two subgroups and of the form $M_iT_jM_i^{-1} = G \in T$ with $i = 1, 2$ and $j = 0, 1, 2, 3, 4$.

Corollary 2.6. *The stabilizer subgroup Γ_∞ has a presentation as the following:*

$$\Gamma_\infty = \left\langle \begin{matrix} T_0, T_1, T_2, T_3, T_4, \\ M_1, M_2 \end{matrix} \;\middle|\; \begin{matrix} [T_1, T_2], [T_1, T_4], [T_2, T_3], [T_3, T_4], [T_1, T_3]T_0, [T_2, T_4]T_0, \\ M_1T_1M_1^{-1} = T_2, M_2T_1M_2^{-1} = T_1, M_2T_2M_2^{-1} = T_0T_4^{-1}, \\ M_1T_3M_1^{-1} = T_4, M_2T_3M_2^{-1} = T_3, M_2T_4M_2^{-1} = T_2T_4, \\ M_1^2, M_2^6, (M_1M_2)^2 = (M_2M_1)^2 \end{matrix} \right\rangle.$$

3. The action of PU(3, 1; $\mathbb{Z}[\omega]$)

Let $\Gamma = \mathrm{PU}(3, 1; \mathbb{Z}[\omega])$, and Γ_∞ be the stabilizer subgroup of $\mathrm{PU}(3, 1; \mathbb{Z}[\omega])$ fixing the point at infinity.

3.1. A coarse fundamental domain for Γ_∞

Definition 3.1. *The coarse fundamental domain $D_\infty \subset \partial\mathbf{H}^3_{\mathbb{C}}$ for the cusp stabilizer Γ_∞ is defined to be*

$$D_\infty = \{(z_1, z_2, t) \in \partial\mathbf{H}^3_{\mathbb{C}} | z_1 \in \Delta, z_2 \in \Delta, 0 \le t \le 2\sqrt{3}\},$$

where Δ is the isosceles triangle in $\mathbb{C}$ with vertices $0, \frac{1}{2} - \frac{i\sqrt{3}}{6}, \frac{1}{2} + \frac{i\sqrt{3}}{6}$, see Figure 1.

Lemma 3.2. *The Γ_∞-translates of D_∞ cover $\partial\mathbf{H}^3_{\mathbb{C}}$.*

Proof. The restriction of Γ_∞ on the subsets $0 \times \mathbb{C} \times \{0\}$ or $\mathbb{C} \times 0 \times \{0\}$ is the group of orientation-preserving symmetries of $\mathbb{Z}[\omega] \subset \mathbb{C}$, which is the (2, 3, 6)-triangle group. A fundamental domain for this group acting on $\mathbb{C}$ is the isosceles triangle Δ. By the same argument of [5, 18], we conclude that the fundamental domain for Γ_∞ on $\partial\mathbf{H}^3_{\mathbb{C}}$ is contained in D_∞. □

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We review some terminology from [8]. A vector $v \in \mathcal{O}_3^4$ is called primitive if for every $0 \neq \lambda \in \mathcal{O}_3$, $\frac{1}{\lambda} v \in \mathcal{O}_3^4$ implies that λ is a unit. The units in $\mathcal{O}_3$ are $1, \omega, \omega^2$. Moreover, every $\mathbb{Z}[\omega]$-rational point in $\mathbb{C}P^3$ has a primitive representation, which is unique up to multiplication by a unit in $\mathcal{O}_3$.

Definition 3.3. *The depth of a $\mathbb{Z}[\omega]$-rational point x is given by $|\langle v, \mathbf{q}_\infty \rangle|^2 = |v_4|^2$, where $v = (v_1, v_2, v_3, v_4)^t$ is any primitive integral lift of x.*

3.2. Covering depth of Γ

Recall that $u(n) = \frac{2}{\sqrt{n}}$ is the height at which balls of depth n appear, in the sense of Corollary 1 of [8]. Let $B\left((z_1, z_2, t), r\right)$ denote the open Cygan ball centered at $p = (z_1, z_2, t) \in \partial \mathbf{H}^3_{\mathbb{C}}$ with radius r. It is easy to see that the isometric sphere of $g \in \mathrm{PU}(3, 1; \mathbb{Z}[\omega])$ has radius $r \leq \sqrt{2}$. In particular, the isometric sphere of I_0 is a Cygan sphere centered at $(0, 0, 0)$ with radius $r = \sqrt{2}$. We will show that four Cygan balls of radius $\sqrt{2}$ will cover the set $D_\infty \times \{u(5)\}$.

Proposition 3.4. *Let $u_0 = 2/\sqrt{5}$ and H_{u_0} be the horosphere of height u_0 based at ∞. Then, the prism $D_\infty \times \{u_0\}$ is covered by the intersections with H_{u_0} of the isometric spheres*

$$B\left((0,0,0), \sqrt{2}\right), B\left((0,0,2\sqrt{3}), \sqrt{2}\right), B\left((0,1,\sqrt{3}), \sqrt{2}\right), B\left((1,0,\sqrt{3}), \sqrt{2}\right).$$

Proof. The method of the proof is similar to the proof of the main result in [18]. We proceed with the following steps.

- If $z_1, z_2 \in \Delta$ and $t \in [0, 2/\sqrt{3}]$, then

  $$(|z_1|^2 + |z_2|^2 + u_0)^2 + t^2 \leq (2/3 + u_0)^2 + 4/3 = 3.77\ldots < 4.$$

  So the point is in $B\left((0,0,0), \sqrt{2}\right)$.
- If $z_1, z_2 \in \Delta$ and $t \in [4/\sqrt{3}, 2\sqrt{3}]$, then

  $$(|z_1|^2 + |z_2|^2 + u_0)^2 + (2\sqrt{3} - t)^2 \leq (2/3 + u_0)^2 + 4/3 = 3.77\ldots < 4.$$

  So the point is in $B\left((0,0,2\sqrt{3}), \sqrt{2}\right)$.
- If $z_1, z_2 \in \Delta$ with $\mathrm{Re}(z_2) \leq \mathrm{Re}(z_1)$, and $t \in [2/\sqrt{3}, 4/\sqrt{3}]$ then consider

  $$(|z_1|^2 + |z_2|^2 + u_0)^2 + \left(t - \sqrt{3} + 2\mathrm{Im}(z_1)\right)^2.$$

  Note that $-\mathrm{Re}(z_1)/\sqrt{3} \leq \mathrm{Im}(z_1) \leq \mathrm{Re}(z_1)/\sqrt{3}$. For a given $\mathrm{Re}(z_1) = x$, this expression is bounded above by

  $$f(x) = (1 - 2x + 4x^2/3 + 4x^2/3 + u_0)^2 + (1/\sqrt{3} + 2x/\sqrt{3})^2.$$

  Elementary calculus shows that this function attains its maximum values at one of the endpoints. (To see this, note that $f'(0) < 0 < f'(1/2)$ and $f''(x) > 0$.) We have

  $$f(0) = (1 + u_0)^2 + 1/3 = 3.922\ldots < 4,$$
  $$f(1/2) = (2/3 + u_0)^2 + 4/3 = 3.77\ldots < 4.$$

  Thus, the point is in $B\left((1,0,\sqrt{3}), \sqrt{2}\right)$.

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- If $z_1, z_2 \in \Delta$ with $\mathrm{Re}(z_1) \le \mathrm{Re}(z_2)$, and $t \in [2/\sqrt{3}, 4/\sqrt{3}]$ then consider
$$(|z_1|^2 + |z_2|^2 + u_0)^2 + \left(t - \sqrt{3} + 2\mathrm{Im}(z_2)\right)^2.$$
Note that $-\mathrm{Re}(z_2)/\sqrt{3} \le \mathrm{Im}(z_2) \le \mathrm{Re}(z_2)/\sqrt{3}$. For a given $\mathrm{Re}(z_2) = x$, this expression is bounded above by the same function $f(x)$. As above, this function is less than 4 and so the point is in $B\left((0, 1, \sqrt{3}), \sqrt{2}\right)$. □

Remark 3.5. *This proof of Proposition 3.4 was given by the referee. We would like to thank him for allowing us to write it here.*

Corollary 3.6. *The covering depth of* Γ *is at most* 4.

3.3. $\mathbb{Z}[\omega]$-rational points in D_∞

We will give the points of depth at most 4 in D_∞. The standard lift of a $\mathbb{Z}[\omega]$-rational point $q = (z_1, z_2, t)$ in D_∞ and the primitive lift of q_∞ are given by

$$\mathbf{q}_\infty = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \qquad \mathbf{q} = \begin{bmatrix} \dfrac{-|z_1|^2 - |z_2|^2 + it}{2} \\ z_1 \\ z_2 \\ 1 \end{bmatrix},$$

where $z_1, z_2 \in \mathbb{C}$ and $t \in \mathbb{R}$. Note that the standard lift of q may not be integral. But we can choose some $\lambda \in \mathbb{Z}[\omega]$ such that $\lambda\mathbf{q}$ is a primitive integral lift. The depth of q is

$$|\langle \mathbf{q}_\infty, \lambda\mathbf{q}\rangle|^2 = |\lambda|^2.$$

In order to find the $\mathbb{Z}[\omega]$-rational points of depth h for $1 \le h \le 4$, we need to find $\lambda \in \mathbb{Z}[\omega]$ such that $|\lambda|^2 = h$ up to multiplication by a unit. Let $\lambda = a + b\omega$ with $a, b \in \mathbb{Z}$. Then, $|\lambda|^2 = a^2 - ab + b^2$, and so we need to find $a, b \in \mathbb{Z}$ such that $a^2 - ab + b^2 = h$ for $1 \le h \le 4$. Note that there are no $\mathbb{Z}[\omega]$-rational points of depth 2. By a simple calculation, we have the following:

Depth h	λ
1	1
3	$-1+\omega$, $2+\omega$, $2+\omega^2$
4	2, $3+\omega$, $3+\omega^2$

Now we can find the point $q = (z_1, z_2, t) \in D_\infty$ such that $\lambda\mathbf{q}$ is a primitive integral lift for a fixed λ, that is, $\lambda z_1 \in \mathbb{Z}[\omega]$, $\lambda z_2 \in \mathbb{Z}[\omega]$, and $\lambda\frac{-|z_1|^2 - |z_2|^2 + it}{2} \in \mathbb{Z}[\omega]$. We will list depths that contain $\mathbb{Z}[\omega]$-rational points in D_∞. Furthermore, we need only consider one representative from each Γ_∞-orbit. We list the set of Γ_∞-orbit up to depth 4 in the following:

- Depth 1: $(0, 0, 0)$ and $(0, 0, 2\sqrt{3})$, both of them are in the same Γ_∞-orbit;
- Depth 3: $\left(0, 0, \frac{2}{3}\sqrt{3}\right)$ in one Γ_∞-orbit, and $\left(0, 0, \frac{4}{3}\sqrt{3}\right)$ in the other Γ_∞-orbit;
- Depth 4: $(0, 0, \sqrt{3})$ in one Γ_∞-orbit; $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\sqrt{3}\right)$ and $\left(\frac{1}{2}, \frac{1}{2}, \frac{3}{2}\sqrt{3}\right)$ in the other two different orbits.

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Integral lifts of the representatives of Γ_∞-orbits of these points are as follows:

$$p_0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad p_{31} = \begin{bmatrix} -1 \\ 0 \\ 0 \\ i\sqrt{3} \end{bmatrix}, \quad p_{32} = \begin{bmatrix} -2 \\ 0 \\ 0 \\ i\sqrt{3} \end{bmatrix},$$

$$p_{41} = \begin{bmatrix} i\sqrt{3} \\ 0 \\ 0 \\ 2 \end{bmatrix}, \quad p_{42} = \begin{bmatrix} \frac{-1+\sqrt{3}i}{2} \\ 1 \\ 1 \\ 2 \end{bmatrix}, \quad p_{43} = \begin{bmatrix} \frac{-1+3\sqrt{3}i}{2} \\ 1 \\ 1 \\ 2 \end{bmatrix}.$$

4. Generators and relations of PU (3, 1; $\mathbb{Z}[\omega]$)

4.1. The generators

Let $A_\alpha \in \Gamma$ be a map sending q_∞ to p_α, where α is in the set $\{0, 31, 32, 41, 42, 43\}$. The matrix form of A_α is given by

$$A_0 = I_0 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, A_{31} = \begin{bmatrix} -1 & 0 & 0 & i\sqrt{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ i\sqrt{3} & 0 & 0 & 2 \end{bmatrix}, A_{32} = A_{31}^{-1},$$

$$A_{41} = \begin{bmatrix} -i\sqrt{3} & 0 & 0 & -2 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -2 & 0 & 0 & i\sqrt{3} \end{bmatrix}, A_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \dfrac{-1-\sqrt{3}i}{2} & 1 & 0 & 0 \\ \dfrac{-1-\sqrt{3}i}{2} & 0 & 1 & 0 \\ -1-\sqrt{3}i & \dfrac{1-\sqrt{3}i}{2} & \dfrac{1-\sqrt{3}i}{2} & 1 \end{bmatrix},$$

$$A_{43} = \begin{bmatrix} -2-i\sqrt{3} & \dfrac{-3+\sqrt{3}i}{2} & \dfrac{-3+\sqrt{3}i}{2} & i\sqrt{3} \\ \dfrac{-1+\sqrt{3}i}{2} & 1 & 0 & 0 \\ \dfrac{-1+\sqrt{3}i}{2} & 0 & 1 & 0 \\ -1+\sqrt{3}i & \dfrac{1+\sqrt{3}i}{2} & \dfrac{1+\sqrt{3}i}{2} & 1 \end{bmatrix}.$$

4.2. The relations

By applying generators to points of depth at most 4, we obtain the cycles. The general method to get these can be found in [8]. Here, we just describe it briefly. Let A_a be a map in the set of generators $\{I_0, A_{31}, A_{32}, A_{41}, A_{42}, A_{43}\}$. Let p_b, p_c be two $\mathbb{Z}[\omega]$-rational points of depth at most 4 which are listed in

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Section 3.3. Let $\gamma \in \Gamma_\infty$. If $A_a(\gamma p_b)$ and p_c have the same depth, and there exists a map $W \in \Gamma_\infty$ such that $A_a(\gamma p_b) = W(p_c)$, then one obtains $A_c^{-1}W^{-1}A_a\gamma A_b = W' \in \Gamma_\infty$, here A_b and A_c be two maps sending q_∞ to p_b and p_c, respectively. Therefore, one obtains a relation $A_c^{-1}W^{-1}A_a\gamma A_b W'^{-1} = Id$.

In practice, by using the following observations, we can simplify the process of solving the equations $A_a(\gamma p_b) = W(p_c)$.

Lemma 4.1. *M_1 commutes with $I_0, A_{31}, A_{32}, A_{41}, A_{42}, A_{43}$, and M_2 commutes with I_0, A_{31}, A_{32}, A_{41}.*

Proof. All of these can be computed directly. □

Lemma 4.2. *Suppose that $A_a(\gamma p_b) = W(p_c)$ and $A_c^{-1}W^{-1}A_a\gamma A_b = W'$. Let $M \in \langle M_1, M_2\rangle$.*

1. *If $A_aM = MA_a$, then $A_a(M\gamma p_b) = MW(p_c)$ and $A_c^{-1}W^{-1}M^{-1}A_aM\gamma A_b = W'$.*
2. *If $A_bM = MA_b$, then $A_a(\gamma M p_b) = W(p_c)$ and $A_c^{-1}W^{-1}A_a\gamma MA_b = W'M$.*

Proof. The first item is obvious. To see the second item, it is suffice to show that $M(p_b) = p_b$. Since $p_b = A_b(q_\infty)$ and $M \in \Gamma_\infty$,

$$M(p_b) = MA_b(q_\infty) = A_bM(q_\infty) = A_b(q_\infty) = p_b.$$

□

Lemma 4.3. *Suppose that $A_a(\gamma_i p_b) = W_i(p_c)$ and $A_c^{-1}W_i^{-1}A_a\gamma_i A_b = W_i'$ with $i = 1, 2$. Let $\gamma_2 = M\gamma_1M^{-1}$ with $M \in \langle M_1, M_2\rangle$.*

1. *If M commutes with A_a, A_b and A_c, then $W_2 = MW_1M^{-1}$ and $W_2' = MW_1'M^{-1}$.*
2. *If M commutes with A_a, A_b, but not A_c, then $W_2 = MW_1$ and $W_2' = W_1'M^{-1}$.*

Moreover, in both of the two cases, the relation coming from γ_2 can be derived from the relation coming from γ_1.

Proof. (1). If M commutes with A_a, A_b and A_c, then $M(p_b) = p_b$ and $M(p_c) = p_c$. Therefore,

$$A_a\gamma_2 p_b = A_aM\gamma_1M^{-1}p_b = MA_a\gamma_1 p_b = MW_1M^{-1}p_c,$$

which means that $W_2 = MW_1M^{-1}$. It implies that

$$A_c^{-1}W_2^{-1}A_a\gamma_2A_b = A_c^{-1}MW_1^{-1}M^{-1}A_aM\gamma_1M^{-1}A_b = MA_c^{-1}W_1^{-1}A_a\gamma_1A_bM^{-1} = MW_1'M^{-1}$$

and so $W_2' = MW_1'M^{-1}$. Thus,

$$A_c^{-1}W_2^{-1}A_a\gamma_2A_bW_2'^{-1} = M\cdot A_c^{-1}W_1^{-1}A_a\gamma_1A_bW_1'^{-1}\cdot M^{-1} = M\cdot Id\cdot M^{-1} = Id.$$

(2). If M commutes with A_a, A_b, but not A_c, then

$$A_a\gamma_2p_b = A_aM\gamma_1M^{-1}p_b = MA_a\gamma_1p_b = MW_1p_c,$$

and so $W_2 = MW_1$. Thus,

$$A_c^{-1}W_2^{-1}A_a\gamma_2A_b = A_c^{-1}W_1^{-1}M^{-1}A_aM\gamma_1M^{-1}A_b = A_c^{-1}W_1^{-1}A_a\gamma_1A_bM^{-1} = W_1'M^{-1},$$

and so $W_2' = W_1'M^{-1}$. Thus,

$$A_c^{-1}W_2^{-1}A_a\gamma_2A_bW_2'^{-1} = A_c^{-1}W_1^{-1}A_a\gamma_1A_bW_1'^{-1} = Id.$$

□

Our main result is the following.

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Table 1. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation. If $A_a(\gamma p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma A_b = W'$. The case $A_a = I_0$, $p_b = p_0, p_{31}, p_{32}, p_{41}$*

p_b	γ	$W(p_c)$	W'
p_0	Id	q_∞	Id
p_0	$T_0^{-1}T_1T_2$	$T_0^{-1}T_2T_1(p_0)$	$T_2^{-1}T_1^{-1}T_0M_2^3M_1M_2^3$
p_0	T_2	$M_1T_0^{-1}T_1T_3(p_0)$	$M_2^{-1}M_1T_4^{-1}$
p_0	$T_0^{-1}T_2$	$T_0T_4^{-1}(p_0)$	$M_1M_2M_1T_2T_4$
p_0	T_0	$T_0^{-1}(p_{32})$	$M_1M_2^3M_1M_2^3$
p_0	T_0^{-1}	p_{31}	$T_0M_1M_2^3M_1M_2^3$
p_0	$T_1T_3^{-1}$	$T_0^{-1}T_1(p_{31})$	$T_3M_2M_1M_2^3M_1$
p_0	$T_0^{-1}T_1T_3^{-1}$	$T_0^{-1}T_1T_3^{-1}T_1^{-1}(p_{32})$	$T_0T_1^{-1}M_2^{-1}M_1M_2^3M_1$
p_0	$T_0^{-1}T_3^2$	$T_0^{-1}T_3(p_{41})$	$T_3M_2^3$
p_0	$T_1T_2T_0^{-1}T_3^2$	$T_0^{-1}T_3(p_{42})$	$M_2^{-2}M_1M_2T_0^{-1}T_3T_2$
p_0	T_1T_2	$M_2^{-1}M_1M_2^{-1}T_0^{-1}(p_{43})$	$M_2M_1M_2$
p_0	$T_0^{-2}T_1T_2$	$M_2M_1M_2(p_{42})$	$M_2^{-1}M_1M_2^{-1}$
p_{31}	$T_0^{-1}T_2$	$T_0T_4^{-1}T_2(p_0)$	$M_1M_2^{-1}M_1M_2^3T_4^{-1}$
p_{31}	Id	$T_0^{-1}(p_0)$	$T_0^{-1}M_1M_2^3M_1M_2^3$
p_{31}	T_0^{-1}	p_{41}	$M_1M_2^3M_1M_2^3$
p_{31}	$T_0^{-1}T_1T_2$	$M_2M_1M_2T_6^{-1}(p_{42})$	$T_0^{-2}T_6M_1M_2^{-1}M_1M_2^{-1}$
p_{32}	$T_0^{-1}T_1$	$T_0^{-1}T_1^2T_3(p_0)$	$T_0T_1^{-1}T_3^{-1}M_2M_1M_2^3M_1$
p_{32}	T_0^{-1}	$T_0(p_0)$	$M_1M_2^3M_1M_2^3$
p_{32}	Id	$T_0^{-1}(p_{41})$	$M_1M_2^3M_1M_2^3$
p_{32}	$T_1^{-1}T_2^{-1}$	$T_2^{-1}T_1^{-1}(M_1M_2^{-1})^2(p_{43})$	$M_2M_1M_2$
p_{41}	T_1^{-1}	$T_0T_1^{-2}(p_0)$	$T_0^{-1}T_1M_2^3$
p_{41}	Id	$T_0^{-1}(p_{31})$	$M_1M_2^3M_1M_2^3$
p_{41}	T_0^{-1}	p_{32}	$M_1M_2^3M_1M_2^3$

Theorem 4.4. *Let $\Gamma = \mathrm{PU}(3,1;\mathbb{Z}[\omega])$. Then, Γ has a presentation $\langle S|R\rangle$, where S consists of $I_0, A_{31}, A_{32}, A_{41}, A_{42}, A_{43}$ and the generators of Γ_∞, R consist of the relations of Γ_∞, the relations given in Lemma 4.1 and the relations from Tables 1 to 16.*

Proof. Let $\gamma = N_{(z_1,z_2,t)}M \in \Gamma_\infty$, where $N_{(z_1,z_2,t)}$ be a Heisenberg translation and $M \in \langle M_1, M_2\rangle$. We need to solve $A_a(\gamma p_b) = A_aN_{(z_1,z_2,t)}M(p_b) = W(p_c)$ for every pair (A_a, p_b).

If $MA_a = A_aM$, then by Lemma 4.2,

$$A_aN_{(z_1,z_2,t)}M(p_b) = A_aM\cdot M^{-1}N_{(z_1,z_2,t)}M(p_b) = MA_aN_{(\tilde{z}_1,\tilde{z}_2,\tilde{t})}(p_b) = M\widetilde{W}(p_c),$$

where $N_{(\tilde{z}_1,\tilde{z}_2,\tilde{t})} = M^{-1}N_{(z_1,z_2,t)}M$. Moreover, suppose that $A_c^{-1}\widetilde{W}^{-1}A_aN_{(\tilde{z}_1,\tilde{z}_2,\tilde{t})}A_b = \widetilde{W}' \in \Gamma_\infty$, then

$$A_c^{-1}\widetilde{W}^{-1}M^{-1}A_aN_{(z_1,z_2,t)}MA_b = A_c^{-1}\widetilde{W}^{-1}A_aN_{(\tilde{z}_1,\tilde{z}_2,\tilde{t})}A_b = \widetilde{W}'.$$

This means that the relation coming from $N_{(z_1,z_2,t)}M$ is the same as the relation from $N_{(\tilde{z}_1,\tilde{z}_2,\tilde{t})}$. Thus, it is equivalent to solve $A_aN_{(\tilde{z}_1,\tilde{z}_2,\tilde{t})}(p_b) = \widetilde{W}(p_c)$. If $MA_b = A_bM$, then by Lemma 4.2,

$$A_aN_{(z_1,z_2,t)}M(p_b) = A_aN_{(z_1,z_2,t)}(p_b) = W(p_c).$$

Similarly, the relation coming from $N_{(z_1,z_2,t)}M$ is the same as relation from $N_{(z_1,z_2,t)}$. Therefore, in both of the two cases, it suffices to solve $A_aN_{(z_1,z_2,t)}(p_b) = W(p_c)$.

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***Table* 2.** *Let* $\gamma = N_{(z_1,z_2,t)}$ *be a Heisenberg translation. If* $A_a(\gamma p_b) = W(p_c)$, *then* $A_c^{-1}W^{-1}A_a\gamma A_b = W'$. *The case* $A_a = I_0$, $p_b = p_{42}$

p_b	γ	$W(p_c)$	W'
p_{42}	Id	$T_2T_1T_6^{-1}(p_0)$	Id
p_{42}	$T_0T_1^{-1}T_2^{-1}$	$T_0^{-2}T_5^{-1}(p_0)$	$T_0^{-1}T_3T_4$
p_{42}	$T_0^{-1}T_4T_5^{-1}$	$T_0^3T_1^{-1}T_2^{-1}T_3^{-2}(p_0)$	$T_1T_3M_2^{-1}M_1M_2^2$
p_{42}	T_1^{-1}	$T_0T_3T_4^{-1}(p_0)$	$M_1M_2M_1T_2T_4M_2$
p_{42}	$T_0^{-1}T_3$	$T_0^{-1}T_1T_2T_3^2(p_0)$	$T_0T_2^{-1}T_3^{-1}M_2^{-1}M_1M_2^2$
p_{42}	$T_0^{-1}T_5^{-1}$	$T_0^{-1}T_5^{-1}(p_{31})$	T_0M_1
p_{42}	$T_0T_1^{-1}T_3^{-1}T_4$	$T_0^{-1}T_3^{-1}T_4(p_{32})$	$T_0^2T_1^{-1}T_3^{-1}M_2^{-2}M_1M_2^{-2}$
p_{42}	T_6^{-1}	$T_0T_5(p_{31})$	$T_0^2T_6^{-1}M_1$
p_{42}	$T_0^{-1}T_2^{-1}T_3$	$M_2M_1M_2T_1^{-1}T_4(p_{42})$	$T_0^2T_2^{-1}T_4^{-1}M_2^{-1}M_1$
p_{42}	$T_0^{-1}T_1T_3$	$T_0^{-1}T_1T_3(p_{42})$	$T_2^{-1}T_6M_1M_2^2M_1M_2^{-2}M_1$
p_{42}	$T_0T_1^{-1}T_3^{-1}$	$M_2M_1M_2^{-2}T_0^{-1}T_4(p_{43})$	$T_1T_3M_2^3M_1M_2$
p_{42}	$T_0^{-1}T_1^{-1}T_3$	$T_0^{-1}T_1^{-1}T_3(p_{42})$	$T_4^{-1}T_2^{-1}T_1M_2^{-1}M_1M_2$
p_{42}	$T_0T_2^{-1}T_3^{-1}$	$T_0T_2^{-1}T_3^{-1}(p_{42})$	$M_1M_2M_1M_2^{-1}M_1$
p_{42}	$T_2^{-1}T_3$	$M_2^2M_1M_2^{-1}T_0^{-2}T_4(p_{43})$	$T_2M_2^2M_1M_2^{-2}$
p_{42}	$T_0^3T_1^{-2}T_2^{-1}T_3^{-1}$	$T_0^3T_1^{-2}T_2^{-1}T_3^{-1}(p_{42})$	$T_1T_4^{-1}M_1M_2^2M_1M_2^{-2}M_1$
p_{42}	$T_0^2T_1^{-1}T_3^{-1}$	$M_2^{-1}M_1M_2^2T_0^{-2}T_4(p_{43})$	$T_1T_3M_2^{-1}M_1M_2$

***Table* 3.** *Let* $\gamma = N_{(z_1,z_2,t)}$ *be a Heisenberg translation. If* $A_a(\gamma p_b) = W(p_c)$, *then* $A_c^{-1}W^{-1}A_a\gamma A_b = W'$. *The case* $A_a = I_0$, $p_b = p_{43}$

p_b	γ	$W(p_c)$	W'
p_{43}	T_0^{-1}	$T_6(p_0)$	Id
p_{43}	$T_1^{-1}T_2^{-1}$	$T_3T_4(p_0)$	$T_0^{-1}T_5^{-1}$
p_{43}	$T_0^{-1}T_1^{-1}T_4$	$T_1^{-1}T_2T_4^2(p_0)$	$T_4^{-1}M_2^{-2}M_1M_2$
p_{43}	T_1^{-1}	$T_0^4T_1^{-2}T_3^{-2}T_5(p_0)$	$T_4^{-1}M_1M_2^{-1}M_1M_2^{-1}$
p_{43}	$T_0T_1^{-1}T_3^{-1}$	$T_0^2T_1^{-1}T_2T_3^{-2}(p_0)$	$T_1T_2^{-1}T_3M_2M_1M_2^{-2}$
p_{43}	$T_0^{-1}T_5^{-1}$	$T_0^{-2}T_6(p_{32})$	$T_0^{-1}T_5^{-1}M_1$
p_{43}	T_6^{-1}	$T_6^{-1}(p_{32})$	M_1
p_{43}	$T_1^{-1}T_3^{-1}T_4$	$T_0T_1^{-1}T_2T_3^{-1}T_4(p_{31})$	$T_4M_2^2M_1M_2^2$
p_{43}	$T_0^{-2}T_3$	$M_2M_1M_2T_0^{-1}T_4(p_{43})$	$M_2^{-1}M_1$
p_{43}	$T_0^{-1}T_1T_2^{-1}T_3$	$T_0^{-1}T_1T_2^{-1}T_3(p_{43})$	$M_2M_1M_2^{-1}$
p_{43}	$T_0^{-2}T_4T_5^{-1}$	$M_2^{-2}M_1M_2^{-2}T_0^{-1}T_4(p_{43})$	$T_0^{-1}T_1T_3T_4M_2^2M_1M_2^3$
p_{43}	$T_0^{-1}T_1^{-1}T_2^{-1}T_3$	$T_0^{-1}T_1^{-1}T_2^{-1}T_3(p_{43})$	$T_0^{-1}T_1T_2T_4M_2^2M_1M_2^{-2}$
p_{43}	T_3^{-1}	$T_3^{-1}(p_{43})$	$T_0^{-1}T_1T_3T_4M_2^2M_1M_2^{-2}$
p_{43}	$T_0^{-1}T_3$	$M_2^{-1}M_1M_2^{-1}T_0^{-2}T_4(p_{43})$	M_2M_1
p_{43}	$T_0^2T_1^{-2}T_3^{-1}$	$T_0^2T_1^{-2}T_3^{-1}(p_{43})$	$T_0^{-1}T_1T_4M_2M_1M_2^{-1}$
p_{43}	$T_0^{-1}T_4T_5^{-1}$	$M_2^2M_1M_2^2T_0^{-2}T_4(p_{43})$	$T_0^2T_4^{-1}T_5M_2^{-2}M_1M_2^3$

Table 4. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation. If $A_a(\gamma p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma A_b = W'$. The case $A_a = A_{31}$*

p_b	γ	$W(p_c)$	W'
p_0	T_1	$T_1T_3(p_0)$	$T_0^3T_1^{-2}T_3^{-1}M_1M_2^3M_1M_2^{-1}$
p_0	T_0	p_0	$T_0^{-1}M_1M_2^3M_1M_2^3$
p_0	Id	p_{41}	$T_0M_1M_2^3M_1M_2^3$
p_0	T_1T_2	$M_2M_1M_2T_0T_1^{-1}T_2^{-1}(p_{42})$	$T_0^{-2}T_1T_2M_1M_2^{-1}M_1M_2^{-1}$
p_{31}	Id	p_{32}	$M_1M_2^3M_1M_2^3$
p_{31}	T_1	$T_0^{-1}T_3(p_{32})$	$T_0^{-1}T_1T_3M_1M_2^3M_1M_2$
p_{32}	Id	q_∞	Id
p_{41}	Id	$T_0(p_0)$	$M_1M_2^3M_1M_2^3$
p_{42}	$T_0T_1^{-1}$	$T_3^{-1}T_4(p_0)$	$T_0T_1^{-1}T_4^{-1}M_2^{-2}M_1M_2^{-2}$
p_{43}	Id	$T_0T_6^{-1}(p_0)$	$T_0^{-1}T_6M_1$
p_{43}	$T_0T_1^{-1}T_2^{-1}$	$T_0^{-1}T_6(p_0)$	$T_1^{-1}T_2^{-1}M_1$

Table 5. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation. If $A_a(\gamma p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma A_b = W'$. The case $A_a = A_{32}$*

p_b	γ	$W(p_c)$	W'
p_0	Id	$T_0(p_0)$	$T_0M_1M_2^3M_1M_2^3$
p_0	T_1	$T_0T_3^{-1}(p_0)$	$T_0^{-1}T_1^{-1}T_3M_1M_2^3M_1M_2$
p_0	$T_1^{-1}T_2$	$M_1M_2^{-1}M_1M_2^{-1}T_0T_2^{-1}(p_{42})$	$T_0T_1^{-1}T_2T_3^{-1}M_2^3M_1M_2^3$
p_0	T_0	p_{41}	$M_1M_2^3M_1M_2^3$
p_{31}	Id	q_∞	Id
p_{32}	Id	p_{31}	$M_1M_2^3M_1M_2^3$
p_{32}	T_1^{-1}	$T_1T_3(p_{31})$	$T_3M_1M_2^3M_1M_2^{-1}$
p_{41}	Id	p_0	$T_0^{-1}M_1M_2^3M_1M_2^3$
p_{42}	Id	$T_5^{-1}(p_0)$	$T_0T_5M_1$
p_{42}	$T_0T_1^{-1}T_2^{-1}$	$T_0^2T_5(p_0)$	$T_0^2T_1^{-1}T_2^{-1}M_1$
p_{43}	T_1^{-1}	$T_1^2T_3^2T_5^{-1}(p_0)$	$T_1^{-1}T_2T_4M_2^2M_1M_2^2$

By Lemma 4.1, M commutes with I_0, A_{31}, A_{32} and A_{41}. Thus, we have to solve $A_aN_{(z_1,z_2,t)}(p_b) = W(p_c)$ for every pair (A_a, p_b).

Besides, we should solve $A_aN_{(z_1,z_2,t)}M(p_b) = W(p_c)$ for the pairs

$$(A_a, p_b) \in \{(A_{42}, p_{42}), (A_{42}, p_{43}), (A_{43}, p_{42}), (A_{43}, p_{43})\},$$

with $M \neq Id$, since M does not commute with A_{42} and A_{43}. For these four cases, it suffices to solve $A_aN_{(z_1,z_2,t)}Mp_b = Wp_c$ with $M = M_2^jM_1M_2^k$ and $M = M_2^j$ $(j, k = 1, 2, 3, 4, 5)$ by Lemma 4.2, since $M = M_1^pM_2^jM_1M_2^k$ (see remark 2.4) and M_1 commutes with A_{42} and A_{43}.

In practice, there should be several solutions (γ, W) of $A_a(\gamma p_b) = W(p_c)$ for fixed triple (A_a, p_b, p_c). However, we do not need to consider all of the solutions. Suppose that (γ_1, W_1) and (γ_2, W_2) be two different solutions. If $\gamma_2 = M\gamma_1M^{-1}$ for some $M \in \langle M_1, M_2\rangle$ commuting with A_a and A_b, then by Lemma 4.3, the relation coming from γ_2 can be derived from the relation coming from γ_1. Thus, the relation coming from γ_2 can be omitted.

The solutions of $A_a(\gamma p_b) = W(p_c)$ and $A_c^{-1}W^{-1}A_a\gamma A_b = W'$ are given in Tables 1–16.

Table 6. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation. If $A_a(\gamma p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma A_b = W'$. The case $A_a = A_{41}$*

p_b	γ	$W(p_c)$	W'
p_0	T_1	$T_0T_1^{-1}(p_0)$	$T_0^{-1}T_1^2M_2^3$
p_0	Id	p_{32}	$T_0M_1M_2^3M_1M_2^3$
p_0	T_0	p_{31}	$M_1M_2^3M_1M_2^3$
p_{31}	Id	$T_0(p_0)$	$M_1M_2^3M_1M_2^3$
p_{32}	Id	p_0	$T_0^{-1}M_1M_2^3M_1M_2^3$
p_{41}	Id	q_∞	Id
p_{42}	Id	$(M_1M_2)^2T_0T_1^{-1}T_2^{-1}(p_{43})$	$T_1T_2M_2^2M_1M_2^2$
p_{42}	$T_2^{-1}T_3$	$T_0^2T_1^{-1}T_3^{-1}(p_{42})$	$T_3T_4^{-1}M_2^3$
p_{42}	$T_0T_1^{-1}T_2^{-1}$	$(M_1M_2)^2(p_{43})$	$T_0^2T_5M_2^2M_1M_2^2$
p_{42}	$T_0^2T_1^{-1}T_3^{-1}$	$T_2^{-1}T_3(p_{42})$	$T_0T_3^{-1}T_4^{-1}M_2^3$
p_{42}	$T_0T_1^{-1}$	$(M_1M_2^{-1})^2T_2^{-1}(p_{43})$	$T_2M_2^3M_1M_2^3$
p_{43}	Id	$M_2^2M_1M_2^2(p_{42})$	$T_0^{-1}T_5^{-1}M_1M_2M_1M_2$
p_{43}	$T_0^{-1}T_4T_5^{-1}$	$T_0^{-1}T_3(p_{43})$	$T_0T_1^{-1}T_2T_3^{-1}T_4M_2^3$
p_{43}	$T_0T_1^{-1}T_2^{-1}$	$M_2^{-1}M_1M_2^{-1}(p_{42})$	$T_0^{-2}T_6M_1M_2^{-2}M_1M_2^{-2}$
p_{43}	$T_0^{-1}T_3$	$T_0^2T_1^{-1}T_2^{-1}T_3^{-1}(p_{43})$	$T_0T_5M_2^3$
p_{43}	T_1^{-1}	$(M_1M_2)^2T_0T_2^{-1}(p_{42})$	$T_0^{-1}T_2M_2^3M_1M_2^3$

As follows, we shall explain the computations for the case $A_a = I_0$ and $p_b = p_0$ in Table 1. The others are similar.

Let $\gamma = N_{(z_1,z_2,t)} \in \Gamma_\infty$ be a Heisenberg translation. Recall that $z_1, z_2 \in \mathbb{Z}[\omega]$, $t/\sqrt{3} \in \mathbb{Z}$, and the integers $|z_1|^2 + |z_2|^2$ and $t/\sqrt{3}$ have the same parity. One can compute that the depth of $I_0(\gamma p_0)$ is

$$\mathrm{dep} = \frac{(|z_1|^2 + |z_2|^2)^2 + t^2}{4}.$$

Obviously, we have the following.

- If $\mathrm{dep} = 0$, then $z_1 = z_2 = t = 0$.
- If $\mathrm{dep} = 1$, then $|z_1| = |z_2| = 1, t = 0$ or $|z_1|^2 + |z_2|^2 = 1, t = \pm\sqrt{3}$.
- If $\mathrm{dep} = 3$, then $z_1 = z_2 = 0, t = \pm 2\sqrt{3}$ or $|z_1|^2 + |z_2|^2 = 3, t = \pm\sqrt{3}$.
- If $\mathrm{dep} = 4$, then $|z_1|^2 + |z_2|^2 = 4, t = 0$ or $|z_1| = |z_2| = 1, t = \pm 2\sqrt{3}$.

It is obvious that there are many solutions of $N_{(z_1,z_2,t)}$. Note that $M_1N_{(z_1,z_2,t)}M_1^{-1} = N_{(z_2,z_1,t)}$ and $M_2N_{(z_1,z_2,t)}M_2^{-1} = N_{(-\omega z_1,z_2,t)}$. Thus according to Lemma 4.1 and Lemma 4.3, it suffices to consider the following 12 cases.

- $\mathrm{dep} = 0 : N_{(0,0,0)} = Id$.
- $\mathrm{dep} = 1 : N_{(1,1,0)} = T_0^{-1}T_1T_2$, $N_{(1,0,\sqrt{3})} = T_2$, $N_{(1,0,-\sqrt{3})} = T_0^{-1}T_2$.
- $\mathrm{dep} = 3 : N_{(0,0,2\sqrt{3})} = T_0$, $N_{(0,0,-2\sqrt{3})} = T_0^{-1}$, $N_{(0,1-\omega,\sqrt{3})} = T_1T_3^{-1}$, $N_{(0,1-\omega,-\sqrt{3})} = T_0^{-1}T_1T_3^{-1}$.
- $\mathrm{dep} = 4 : N_{(0,2\omega,0)} = T_0^{-1}T_3^2$, $N_{(1,1+2\omega,0)} = T_1T_2T_0^{-1}T_3^2$, $N_{(1,1,2\sqrt{3})} = T_1T_2$, $N_{(1,1,-2\sqrt{3})} = T_0^{-2}T_1T_2$.

Finally, for each one of the above cases, we find $W(p_c)$ and W' such that $A_c^{-1}W^{-1}I_0\gamma I_0 = W'$, where $W, W' \in \Gamma_\infty$. □

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Table 7. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation. If $A_a(\gamma p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma A_b = W'$. The case $A_a = A_{42}$*

p_b	γ	$W(p_c)$	W'
p_0	Id	p_0	T_5
p_0	$T_0^{-1}T_6$	$T_0^{-1}T_1T_2(p_0)$	$T_0^{-2}T_5^{-1}$
p_0	$T_1^{-1}T_6$	$T_0T_4^{-1}(p_0)$	$T_0^{-1}T_1T_2T_4^2M_2^2M_1M_2^{-1}$
p_0	T_3^{-1}	$T_2(p_0)$	$T_0T_3^{-1}T_4M_1M_2M_1M_2$
p_0	$T_0^{-1}T_1$	$T_0^{-1}T_1T_2T_4(p_0)$	$T_0^3T_1^{-1}T_2^{-1}T_4^{-2}M_2^2M_1M_2^{-1}$
p_0	$T_0^{-1}T_5^{-1}$	p_{31}	$T_5^{-1}M_1$
p_0	$T_0T_1^{-1}T_2T_3^{-1}$	$T_0^{-1}T_2(p_{32})$	$T_3T_4^{-1}M_2^{-2}M_1M_2^{-2}$
p_0	$T_0^{-2}T_1T_2$	$T_0^{-1}T_1T_2(p_{31})$	$T_0^2T_5M_1$
p_0	$T_0T_1^{-1}T_4^{-1}$	$M_1M_2^{-1}M_1M_2(p_{42})$	$T_0T_2^{-1}T_3^{-1}M_1$
p_0	$T_1T_4^{-1}$	$M_1M_2T_0^{-1}T_4(p_{42})$	$T_0^{-1}T_2T_3^{-1}M_2^{-1}M_1M_2^{-1}$
p_0	$T_0T_1^{-1}T_3^{-1}T_6$	$M_1M_2M_1M_2^2T_0^{-1}T_4(p_{43})$	$T_1T_2^{-1}T_3M_2M_1M_2^{-2}$
p_0	$T_0^{-1}T_2T_3^{-1}$	$M_2^2T_0^{-1}T_4(p_{43})$	$T_4M_1M_2^2M_1M_2^{-1}$
p_0	$T_0T_1^{-1}T_3^{-1}$	$M_1M_2^{-1}(p_{42})$	$T_0^2T_2^{-1}T_4^{-1}M_2M_1M_2$
p_0	$T_0^{-1}T_1T_3$	$M_2^{-1}M_1M_2T_0^{-1}T_4(p_{42})$	$T_0T_1^{-1}T_3^{-1}T_6(M_1M_2^3)^2$
p_0	$T_0^{-1}T_1T_3^{-1}$	$M_1M_2^2M_1M_2T_0^{-2}T_3T_4(p_{43})$	$T_0T_4^{-1}M_2M_1M_2^{-2}$
p_0	$T_0^2T_1^{-1}T_3^{-1}$	$M_1M_2(p_{42})$	$T_0T_2^{-1}T_4^{-1}M_2^{-1}M_1M_2^{-1}$
p_{31}	T_3^{-1}	$T_2T_3^{-1}T_4(p_0)$	$T_0^{-1}T_3T_4^{-1}M_2^{-2}M_1M_2^{-2}$
p_{31}	$T_0^{-1}T_2T_3^{-1}$	$T_2M_2T_0^{-1}(p_{43})$	$T_0^{-1}T_4M_2^3$
p_{31}	$T_0T_1^{-1}T_3^{-1}$	$M_1M_2^{-2}(p_{42})$	$M_1M_2^{-1}M_1M_2^{-1}M_1T_1^{-1}$
p_{32}	T_0^{-1}	$T_0T_5(p_0)$	$T_0^{-1}T_5^{-1}M_1$
p_{32}	$T_0^{-2}T_6$	$T_6(p_0)$	$T_0T_5M_1$
p_{32}	$T_0T_1^{-1}T_3^{-1}$	$M_1M_2^2(p_{42})$	$T_0T_2^{-1}T_4^{-1}M_2M_1M_2$
p_{32}	$T_0^{-1}T_2T_3^{-1}$	$M_2^{-1}T_4^{-1}(p_{43})$	$T_0^{-1}T_4M_1M_2^2M_1M_2^{-1}$
p_{41}	$T_0^{-1}T_2T_3^{-1}$	$T_0^{-1}T_2(p_{43})$	$T_0^{-1}T_4M_1M_2^{-2}M_1M_2$
p_{41}	$T_0^{-2}T_6$	$M_2M_1M_2(p_{43})$	$T_0T_5(M_1M_2^2)^2$
p_{41}	T_0^{-1}	$M_2M_1M_2T_0^{-2}T_3T_4(p_{43})$	$M_1M_2^2M_1M_2^2$
p_{41}	T_3^{-1}	$M_1M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{42})$	$T_0^{-2}T_6M_2^2M_1M_2^{-1}$
p_{41}	$T_0T_1^{-1}T_3^{-1}$	$T_1^{-1}(p_{43})$	$T_0^2T_3^{-1}T_5M_1M_2M_1M_2^{-2}$
p_{42}	T_3^{-1}	$T_2M_1M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{42})$	$T_2^{-1}T_6M_2^3M_1M_2^3$
p_{42}	T_1^{-1}	$M_1T_0^{-1}T_4(p_{42})$	$T_0^{-1}T_1^{-1}T_2T_6M_1M_2^{-1}M_1M_2^2$
p_{42}	$T_0T_1^{-1}T_3^{-1}$	$M_1M_2^{-1}M_1M_2^2T_0^{-1}T_4(p_{43})$	$T_4M_1M_2^3M_1$
p_{42}	T_0^{-1}	$T_0^{-1}T_1T_2(p_{41})$	$T_6M_2^3M_1M_2^3$
p_{42}	$T_0^{-1}T_5^{-1}$	p_{41}	$T_0M_2^3M_1M_2^3$
p_{42}	$T_0^{-1}T_4T_5^{-1}$	$M_2^{-1}M_1M_2^2(p_{42})$	$T_0T_2^{-1}$
p_{42}	$T_1^{-1}T_6$	$T_0T_4^{-1}(p_{42})$	$M_2^2M_1M_2^{-1}$
p_{43}	T_0^{-1}	$M_2^{-1}M_1M_2^{-1}T_0^{-1}(p_{43})$	$M_2M_1M_2$
p_{43}	$T_0^{-1}T_5^{-1}$	$M_2^2M_1M_2^2T_0^{-2}T_3T_4(p_{43})$	$T_0^{-1}T_1T_2M_2^{-2}M_1M_2^{-2}$
p_{43}	$T_0T_1^{-1}T_3^{-1}$	$M_1M_2M_1M_2T_0^{-1}T_4(p_{42})$	$T_0^{-1}T_2T_3^{-1}M_1M_2^{-2}M_1M_2^{-2}$

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Table 8. *Let* $\gamma = N_{(z_1,z_2,t)}$ *be a Heisenberg translation. If* $A_a(\gamma p_b) = W(p_c)$*, then* $A_c^{-1}W^{-1}A_a\gamma A_b = W'$*. The case* $A_a = A_{43}$

p_b	γ	$W(p_c)$	W'
p_0	Id	$T_0(p_0)$	T_6
p_0	T_0T_5	$T_1T_2(p_0)$	$T_0^2T_6^{-1}$
p_0	$T_0T_1T_6^{-1}$	$T_1T_3(p_0)$	$T_0^2T_1^{-1}T_2T_3^{-2}M_2M_1M_2^{-2}$
p_0	T_1	$T_0T_1T_4^{-1}(p_0)$	$T_1^{-1}T_2T_4^2M_2^{-2}M_1M_2$
p_0	T_1T_3	$T_2(p_0)$	$T_1T_2^{-1}T_3T_4^{-1}(M_1M_2^{-1})^2$
p_0	T_1T_2	$T_0^{-1}T_1T_2(p_{32})$	$T_0^{-1}T_6M_1$
p_0	$T_0T_6^{-1}$	p_{32}	$T_0T_6^{-1}M_1$
p_0	$T_0^{-1}T_2T_3$	$T_2(p_{31})$	$T_0^2T_1^{-1}T_2T_3^{-1}T_4M_2^2M_1M_2^2$
p_0	$T_0^{-1}T_3$	$M_1M_2^{-1}(p_{43})$	$T_4M_2M_1M_2$
p_0	$T_0^2T_3^{-1}T_5$	$M_1M_2^2T_4(p_{42})$	$T_0^2T_2^{-1}T_4^{-1}M_2^3M_1$
p_0	$T_1^{-1}T_2T_4$	$M_2^{-1}M_1M_2(p_{43})$	$T_1^{-1}T_2T_4$
p_0	T_3T_5	$M_1M_2^{-1}M_1M_2T_0^{-1}T_4(p_{43})$	$T_0T_4^{-1}M_2^3M_1M_2^3$
p_0	T_3	$M_1M_2(p_{43})$	$T_0^{-1}T_4M_2^{-1}M_1M_2^{-1}$
p_0	$T_0T_3^{-1}$	$M_2M_1M_2^2T_4(p_{42})$	$T_0T_1^{-1}T_4^{-1}M_1M_2^2M_1M_2^{-1}$
p_0	$T_1^2T_3$	$M_2^2M_1M_2T_0^{-1}T_3T_4(p_{42})$	$T_0T_1^{-1}T_3^{-1}T_6M_1M_2^2M_1M_2^{-1}$
p_0	$T_0^3T_3^{-1}T_5$	$M_1M_2T_0^{-1}T_4(p_{43})$	$T_0^2T_4^{-1}T_5M_2^{-1}M_1M_2^{-1}$
p_{31}	Id	$T_6(p_0)$	$T_6^{-1}M_1$
p_{31}	T_0T_5	$T_0T_5(p_0)$	$T_0^{-2}T_6M_1$
p_{31}	$T_0^{-1}T_3$	$M_1M_2T_0T_2^{-1}(p_{42})$	$T_0^{-1}T_2T_3^{-1}M_2M_1M_2^{-2}$
p_{31}	$T_0^2T_3^{-1}T_5$	$M_1M_2T_2T_4(p_{42})$	$T_0T_2^{-1}T_4^{-1}M_2M_1M_2^{-2}$
p_{32}	$T_0^{-1}T_2T_4$	$T_0T_2T_3^{-1}T_4(p_0)$	$T_0T_1T_2^{-1}T_3T_4^{-1}M_2^2M_1M_2^2$
p_{32}	$T_0^{-1}T_3$	$M_1M_2^2(p_{43})$	$T_0^{-1}T_4M_2M_1M_2$
p_{32}	$T_0^2T_3^{-1}T_5$	$M_1M_2^{-1}T_0T_4^{-1}(p_{42})$	$T_0T_2^{-1}T_4^{-1}M_2^3M_1$
p_{41}	$T_0^{-1}T_3$	$M_1M_2^3(p_{43})$	$T_0^{-1}T_4M_2^3M_1M_2^3$
p_{41}	$T_0^{-1}T_1T_3$	$M_2M_1M_2T_4(p_{42})$	$T_4^{-1}(M_1M_2^3)^2$
p_{41}	Id	$M_2^2M_1M_2^2T_0^{-1}T_3T_4(p_{42})$	$T_0^{-2}T_6M_1M_2M_1M_2$
p_{41}	T_0T_5	$M_2^{-1}M_1M_2^{-1}(p_{42})$	$T_0^{-2}T_6M_1M_2^{-2}M_1M_2^{-2}$
p_{41}	$T_0^2T_4^{-1}T_5$	$T_2(p_{42})$	$T_0T_2^{-1}T_4^{-1}M_1M_2^2M_1M_2^{-1}$
p_{42}	$T_0^{-1}T_3$	$M_1M_2^{-1}M_1M_2^2T_4(p_{42})$	$T_0^2T_2^{-1}T_4^{-1}M_1M_2M_1M_2^{-2}$
p_{42}	Id	$M_2M_1M_2T_0(p_{42})$	$M_2^{-1}M_1M_2^{-1}$
p_{42}	T_6^{-1}	$M_2M_1M_2T_0^{-1}T_3T_4(p_{42})$	$T_5^{-1}M_2^{-1}M_1M_2^{-1}$
p_{43}	$T_0^{-1}T_3$	$M_1M_2M_1M_2T_0^{-1}T_4(p_{43})$	$T_0^2T_4^{-1}T_5M_1M_2^{-2}M_1M_2^{-2}$
p_{43}	T_1^{-1}	$T_0^2T_1^{-1}T_3^{-1}(p_{43})$	$T_0T_1^2T_6^{-1}M_2M_1M_2^{-2}$
p_{43}	$T_0^{-1}T_1T_3$	$T_2M_2M_1M_2T_0^{-1}T_4(p_{42})$	$T_1^{-1}T_6M_1M_2^2M_1M_2^{-1}$
p_{43}	Id	$T_0^{-1}T_1T_2(p_{41})$	$T_0T_5M_2^3M_1M_2^3$
p_{43}	T_6^{-1}	p_{41}	$M_2^3M_1M_2^3$
p_{43}	$T_0^{-1}T_2^{-1}T_3$	$M_2M_1M_2^{-2}(p_{43})$	T_2^{-1}
p_{43}	$T_2T_6^{-1}$	$T_0^{-1}T_2T_4(p_{43})$	$M_2^{-2}M_1M_2$

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Table 9. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation and $M \in \langle M_1, M_2\rangle$. If $A_a(\gamma M p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma MA_b = W'$. The case $A_a = A_{42}$ and $p_b = p_{42}$*

M	γ	$W(p_c)$	W'
M_2	T_1^{-1}	$T_0^{-1}T_1M_1T_4(p_{32})$	$M_2^{-2}M_1M_2T_1^{-1}T_4T_0$
	T_3^{-1}	$T_2T_3^{-1}(p_{32})$	$M_2M_1M_2^{-2}M_1T_1T_3$
M_2^3	T_4^{-1}	$(M_1M_2)^2T_0^{-1}T_3(p_{42})$	$M_2^2M_1M_2^{-1}M_1T_4T_2T_0^{-2}$
	$T_1^{-1}T_6$	$(M_1M_2^2)^2T_0^{-2}T_4T_3(p_{43})$	$M_2^2M_1M_2^{-1}M_1T_4$
	$T_2T_0^{-1}$	$(M_1M_2^{-1})^2T_0^{-1}(p_{43})$	$M_2^{-1}M_1M_2^2M_1T_3$
	$T_2T_3^{-1}T_1^{-1}$	$(M_1M_2)^2T_0^{-1}T_4(p_{42})$	$M_2^2M_1M_2^{-1}M_1T_4T_1^{-1}$
M_2^4	T_4^{-1}	$T_0^{-1}T_1T_3(p_0)$	$M_2^{-2}M_1M_2M_1T_3T_1T_0^{-2}$
	$T_0T_4^{-1}T_3^{-1}T_1^{-1}$	$M_1T_1M_1T_3^{-1}(p_0)$	$M_2^{-2}M_1M_2M_1T_4T_1^{-1}$
	$T_2T_3^{-1}T_1^{-1}$	$(M_1M_2)^2T_0^{-1}T_4(p_{42})$	$M_2^2M_1M_2^{-1}M_1T_4T_1^{-1}$
M_2^5	$T_0T_2^{-1}T_4^{-1}$	$T_0^{-1}T_4T_1(p_0)$	$M_2^2M_1M_2^5T_1T_4T_3^2T_0^{-1}$
	T_4^{-1}	$T_0^{-1}T_1^2T_3T_2(p_0)$	$M_2^5M_1M_2^2T_2T_3T_0^{-1}$
	$T_5^{-1}T_0^{-1}$	$T_0^2T_1^{-1}T_3^{-1}(p_0)$	$M_2^5M_1M_2^2T_4T_2^{-1}T_0$
	$T_2T_5^{-1}T_0^{-2}$	$T_2T_4^{-1}(p_0)$	$M_2^2M_1M_2^{-1}T_1^{-1}T_3^{-1}T_0^2$
$M_2M_1M_2$	T_4^{-1}	$T_4^{-1}T_1(p_{41})$	$M_2^2M_1M_2^{-1}T_1T_3$
	$T_3T_0^{-1}$	$T_0^{-1}T_1T_3(p_{41})$	$M_2^{-1}M_1M_2^2T_2^{-1}T_3T_0$
$M_2M_1M_2^2$	T_3^{-1}	$T_2T_3^{-1}(p_{31})$	$M_2^3M_1T_2T_4$
	$T_1T_3T_0^{-1}$	$T_0^{-1}T_1T_3(p_{31})$	$M_2^3M_1T_2^{-1}T_3T_0$
$M_2M_1M_2^3$	T_3^{-1}	$T_2T_3^{-1}(p_0)$	$M_2M_1M_2^4T_4T_2T_0^{-2}$
	$T_1T_0^{-1}$	$T_0^{-1}T_1T_3(p_0)$	$M_2M_1M_2^4T_2^{-1}T_3$
$M_2M_1M_2^4$	$T_3T_3^{-1}T_0^{-1}$	$(M_1M_2)^2T_5T_1^{-1}T_0(p_{43})$	$(M_2^3M_1)^2T_3$
	$T_1^{-1}T_3^{-2}T_0^2$	$M_2^2M_1M_2^{-1}M_1T_4T_1^{-1}(p_{42})$	$M_2^{-2}M_1M_2M_1T_3T_4T_2^{-1}T_1T_0^{-1}$
	$T_3^{-1}T_1$	$(M_1M_2)^2T_0^{-2}T_4T_1T_3(p_{43})$	T_3
	$T_3^{-1}T_4^{-1}T_2^{-1}T_0$	$T_0^{-1}T_1M_1M_2^2M_1T_2^{-1}M_2^2T_4(p_{43})$	$(M_2^2M_1)^2T_2^{-1}T_0$
	$T_3^{-1}T_4T_2T_0^{-2}$	$T_0^{-1}T_2T_1M_1M_2^{-1}M_1M_2^2T_2^{-1}T_3(p_{42})$	$M_2M_1M_2^{-2}M_1T_3T_1^{-1}T_4T_2T_0^{-1}$
	$T_3^{-1}T_1^{-1}$	$M_1M_2^{-2}M_1M_2(p_{42})$	$M_2^{-1}M_1M_2^2M_1T_3T_1T_4T_0^{-2}$
	$T_1T_0^{-1}$	$T_0^{-1}T_1M_1M_2^5M_1M_2^{-1}(p_{43})$	$(M_2^2M_1)^2T_1^{-1}T_0$
	$T_0T_3^{-1}T_2^{-1}$	$M_1M_2M_1M_2^4(p_{42})$	$M_2^{-1}M_1M_2^2M_1T_2^{-1}T_0$
	T_3^{-1}	q_∞	$M_2^2M_1M_2^5T_3$
$M_2M_1M_2^5$	$T_3^{-1}T_1^{-1}$	$T_1T_3(p_0)$	$M_2^3M_1T_3T_2^{-1}T_0$
	T_3^{-1}	$T_0T_2T_3^{-1}(p_0)$	$M_2^3M_1T_2T_4$
$M_2^2M_1M_2$	T_4^{-1}	$T_1T_4^{-1}(p_{31})$	$M_1M_2^3T_1T_3$
	$T_4T_2T_0^{-2}$	$T_0^{-1}T_2T_4(p_{31})$	$M_1M_2^3T_4T_1^{-1}T_0$

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Table 9. *Continued*

$M_2^2M_1M_2^2$	$T_4^{-1}T_1T_0^{-1}$	$T_0^{-1}T_1(M_1M_2)^2(p_{43})$	$M_1M_2^{-2}M_1M_2T_4T_3T_1T_0^{-2}$
	$T_2T_1T_0^{-2}$	$T_0^{-2}T_2M_1M_2M_1M_2^4T_3(p_{43})$	$M_2^{-1}M_1M_2^2T_4T_3T_1T_0^{-2}$
	$T_1T_2T_3T_0^{-2}$	$T_0^{-1}T_1T_3(p_{42})$	$M_2M_1M_2^{-2}M_1T_1^{-1}T_3$
	Id	$(M_1M_2)^2(p_{42})$	$M_2M_1M_2$
	T_4^{-1}	$T_0^2T_2^{-1}T_4^{-1}(p_{42})$	$M_2^{-2}M_1M_2M_1T_3T_1T_0^{-2}$
	$T_1T_0^{-1}$	$T_0^{-1}T_1(p_{41})$	$M_2^{-2}M_1M_2^4M_1T_4$
	$T_1T_3T_0^{-1}$	$T_0^{-1}T_1(M_1M_2^{-2})^2(p_{43})$	$M_2M_1M_2^{-2}M_1T_2^{-1}T_0$
$M_2^2M_1M_2^3$	$T_2T_3^{-1}T_0^{-1}$	$T_2T_4^{-1}(p_0)$	$M_2^{-1}M_1M_2^2M_1T_2T_4^2T_3T_0^{-1}$
	$T_2T_1T_0^{-2}$	$T_0^{-1}T_2T_1^2T_3(p_0)$	$M_2^2M_1M_2^{-1}M_1T_3T_1^{-1}T_0$
	T_3^{-1}	$T_0^2T_1^{-1}T_3^{-1}(p_0)$	$M_2^2M_1M_2^{-1}M_1T_4T_1T_0^{-1}$
	$T_1T_0^{-1}$	$T_0^{-1}T_4T_1(p_0)$	$M_2^{-1}M_1M_2^2M_1T_4^{-1}T_2^{-1}T_0$
$M_2^2M_1M_2^4$	$T_2T_3^{-1}T_0^{-1}$	$T_0T_4^{-1}T_1(p_0)$	$M_2^3M_1T_4T_1^{-1}T_0$
	T_3^{-1}	$T_2T_4(p_0)$	$M_2^3M_1T_3T_1T_0^{-1}$
$M_2^2M_1M_2^5$	$T_3^{-1}T_1^{-1}T_2T_0^{-1}$	$T_1M_1M_2^4M_1M_2(p_{42})$	$M_2^2M_1M_2^2T_4T_3T_0^{-1}$
	$T_3^{-1}T_2T_0^{-1}$	$T_0^{-1}T_2T_1(M_1M_2^2)^2(p_{42})$	$M_2M_1M_2^4T_4$
	T_3^{-1}	$T_2M_1M_2M_1M_2^4(p_{42})$	$M_2^{-1}M_1M_2^{-1}$
	$T_3^{-1}T_1^{-1}$	$(M_1M_2^{-1})^2(p_{42})$	$M_2^{-2}M_1M_2T_3$

Table 10. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation and $M \in \langle M_1, M_2\rangle$. If $A_a(\gamma M p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma MA_b = W'$. The case $A_a = A_{42}$ and $p_b = p_{42}$*

M	γ	$W(p_c)$	W'
$M_2^3M_1M_2$	T_4^{-1}	$T_4^{-1}T_1(p_0)$	$M_2^{-2}M_1M_2T_3T_1T_0^{-2}$
	$T_2T_0^{-1}$	$T_0^{-1}T_2T_4(p_0)$	$M_2^{-2}M_1M_2T_4T_1^{-1}$
$M_2^3M_1M_2^2$	$T_4^{-1}T_1T_0^{-1}$	$T_1T_3^{-1}(p_0)$	$M_2^2M_1M_2^{-1}M_1T_4T_3^2T_1T_0^{-3}$
	$T_1T_2T_0^{-2}$	$T_0^{-1}T_2^2T_1T_4(p_0)$	$M_2^{-1}M_1M_2^2M_1T_4T_2^{-1}T_0$
	T_4^{-1}	$T_0^2T_2^{-1}T_4^{-1}(p_0)$	$M_2^{-1}M_1M_2^2M_1T_3T_2T_0^{-1}$
	$T_0^{-1}T_2$	$T_0^{-1}T_2T_3(p_0)$	$M_2^2M_1M_2^{-1}M_1T_3^{-1}T_1^{-1}T_0$
$M_2^3M_1M_2^3$	$T_3^{-1}T_4^{-1}$	p_{41}	T_4T_3
	$T_0^{-1}T_1T_3^{-1}T_4^{-1}$	$T_0T_3^{-1}(p_{42})$	$M_2^2M_1M_2^{-1}M_1T_4T_3T_0^{-1}$
	$T_1T_4^{-1}T_0^{-1}$	$T_0^{-1}T_1M_1M_2^2M_1M_2^{-1}(p_{43})$	$M_1M_2^3T_3$
	$T_2T_3^{-1}T_4^{-1}T_0^{-1}$	$T_0T_4^{-1}(p_{42})$	$M_2^{-1}M_1M_2^2M_1T_3T_4T_0^{-1}$
	$T_1T_2T_4^{-1}T_0^{-2}$	$T_1T_2T_0^{-1}M_2^{-4}M_1M_2^{-1}M_1(p_{42})$	$T_1^{-1}T_0$
	$T_3^{-1}T_2T_0^{-1}$	$T_0^{-1}T_2M_2^{-4}M_1M_2^{-1}M_1(p_{43})$	$M_2^3M_1T_4$

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Table 10. *Continued*

	$T_2T_1T_0^{-2}$	$T_0^{-1}T_1T_2(p_{41})$	T_0
	T_4^{-1}	$M_1M_2^5M_1M_2^{-4}(p_{42})$	$(M_2^3M_1)^2T_3T_4T_2T_0^{-2}$
	$T_1T_0^{-1}$	$T_0^{-2}T_2^{-1}T_4T_2(p_{42})$	$M_2^{-1}M_1M_2^2M_1T_2^{-1}T_4^{-1}T_3T_1$
$M_2^3M_1M_2^4$	$T_2T_3^{-1}T_4^{-1}T_0^{-1}$	$T_4^{-1}T_1(p_{32})$	$M_2M_1M_2^{-2}T_1^{-1}T_4T_0$
	T_3^{-1}	$T_0^{-2}T_4T_2(p_{32})$	$M_2M_1M_2^{-2}T_1T_3$
$M_2^4M_1M_2$	$T_4^{-1}T_1T_0^{-1}$	$T_0^{-1}T_2T_1M_1M_2M_1M_2^4(p_{42})$	$M_2^5M_1M_2^2M_1T_4T_3T_1T_0^{-2}$
	$T_4^{-2}T_2^{-1}$	$T_0^{-1}T_1(M_1M_2^2)^2(p_{43})$	$(M_2^5M_1)^2T_2T_4T_3T_0^{-1}$
	$T_4^{-1}T_2$	$T_0^{-1}T_1T_2M_1M_2^4M_1M_2(p_{42})$	$M_2^5M_1M_2^2M_1T_2^{-1}T_0$
	$T_4^{-1}T_3^{-1}T_1^{-1}T_0$	$T_0^{-1}T_2(M_1M_2^2)^2(p_{43})$	$(M_2^2M_1)^2T_1^{-1}T_0$
	$T_4^{-1}T_3T_1T_0^{-2}$	$T_0^{-1}T_1(M_1M_2^5)^2(p_{43})$	$(M_2^{-1}M_1)^2T_1T_3T_4T_0^{-1}$
	$T_4^{-1}T_2^{-1}$	$M_1M_2M_1M_2^{-2}(p_{42})$	$M_2^2M_1M_2^{-1}M_1T_3T_4T_2T_0^{-2}$
	$T_2T_0^{-1}$	$T_0^{-1}T_2(M_1M_2^5)^2(p_{43})$	$(M_2^2M_1)^2T_2^{-1}T_0$
	$T_4^{-1}T_1^{-1}T_0$	$M_1M_2^4M_1M_2(p_{42})$	$M_2^2M_1M_2^{-1}M_1T_1^{-1}T_0$
	T_4^{-1}	q_∞	$M_2^{-1}M_1M_2^{-4}T_4$
$M_2^4M_1M_2^2$	$T_4^{-1}T_1T_0^{-1}$	$T_0T_2T_3^{-1}(p_0)$	$M_1M_2^3T_2^{-1}T_3T_0$
	T_4^{-1}	$T_1T_3(p_0)$	$M_1M_2^3T_4T_2T_0^{-1}$
$M_2^4M_1M_2^3$	$T_0^{-1}T_1T_3^{-1}T_4^{-1}$	$T_2T_3^{-1}(p_{32})$	$M_2^4M_1M_2T_2^{-1}T_3T_0$
	T_4^{-1}	$T_0^{-1}T_1T_3(p_{32})$	$M_2^4M_1M_2T_2T_4$
$M_2^4M_1M_2^4$	$T_4^{-2}T_3^{-1}T_0$	$T_1T_4^{-1}(p_{41})$	$M_2^{-1}M_1M_2^2T_1^{-1}T_4T_0$
	T_4^{-1}	$T_0^{-1}T_1T_3(p_{41})$	$M_2^{-4}M_1M_2^{-1}T_2T_4$
$M_2^4M_1M_2^5$	T_4^{-1}	$T_0^{-1}T_1T_3(p_{31})$	$M_1M_2^3T_2T_4$
	$T_4^{-1}T_3^{-2}T_1^{-1}T_0$	$T_3^{-1}T_2(p_{31})$	$M_1M_2^3T_2^{-1}T_3T_0$
$M_2^5M_1M_2$	$T_4^{-1}T_2^{-1}$	$T_2T_4(p_0)$	$M_1M_2^3T_4T_1^{-1}T_0$
	T_4^{-1}	$T_0T_1T_4^{-1}(p_0)$	$M_1M_2^3T_3T_1T_0^{-1}$
$M_2^5M_1M_2^2$	$T_4^{-1}T_1T_2^{-1}T_0^{-1}$	$T_2M_1M_2M_1M_2^{-2}(p_{42})$	$M_2^2M_1M_2^2T_4T_3T_0^{-1}$
	$T_4^{-1}T_1T_0^{-1}$	$T_0^{-1}T_2T_1(M_1M_2^2)^2(p_{42})$	$M_2^{-2}M_1M_2T_3$
$M_2^5M_1M_2^2$	$T_4^{-1}T_2^{-1}$	$(M_1M_2^5)^2(p_{42})$	$M_2M_1M_2^4T_4$
	T_4^{-1}	$T_1M_1M_2^4M_1M_2(p_{42})$	$M_2^5M_1M_2^5$
$M_2^5M_1M_2^4$	T_3^{-1}	$T_0^{-1}T_2T_4(p_{31})$	$M_2^3M_1T_3T_1T_0^{-1}$
	$T_3^{-1}T_4^{-2}T_2^{-1}T_0$	$T_4^{-1}T_1(p_{31})$	$M_2^3M_1T_4T_1^{-1}T_0$

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Table 10. Continued

$M_2^5M_1M_2^5$	$T_2^{-1}T_4^{-1}T_0$	$T_0^{-1}T_1(M_1M_2^4)^2(p_{43})$	$M_2^4M_1M_2M_1T_2T_4T_3T_0^{-1}$
	$T_1^{-1}T_3^{-1}T_4^{-2}T_2^{-1}T_0^2$	$T_0^2T_2^{-1}T_4^{-1}(p_{42})$	$M_2M_1M_2^4M_1T_1^{-1}T_4$
	$T_3^{-1}T_4^{-1}T_2^{-1}T_0$	$T_0^{-1}T_1(p_{41})$	$(M_2M_1)^2T_3$
	T_4^{-1}	$T_0^{-1}T_1T_3(p_{42})$	$M_2^4M_1M_2M_1T_4T_1T_0^{-1}$
	$T_4^{-2}T_3^{-1}T_2^{-1}T_0$	$T_0^{-1}T_1(M_1M_2)^2(p_{43})$	$M_2^4M_1M_2M_1T_1^{-1}T_0$
	$T_4^{-1}T_3^{-1}T_1^{-1}T_2^{-1}T_0$	$(M_1M_2)^2(p_{42})$	$M_2^{-2}M_1M_2^{-2}T_3T_4T_0^{-1}$
	$T_4^{-1}T_3^{-1}T_0$	$T_0^{-1}T_1T_2(M_1M_2^4)^2(p_{42})$	$M_2M_1M_2$

Table 11. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation and $M \in \langle M_1, M_2\rangle$. If $A_a(\gamma Mp_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma MA_b = W'$. The case $A_a = A_{42}$ and $p_b = p_{43}$*

M	γ	$W(p_c)$	W'
M_2	T_0^{-1}	$T_0^{-1}T_1T_3(p_0)$	T_3^{-1}
	$T_0T_1^{-1}T_3^{-1}$	$T_2T_3^{-1}(p_0)$	$T_0^{-2}T_3T_5^{-1}$
M_2^2	$T_0T_1^{-1}T_3^{-1}$	$T_0^2T_1^{-1}T_3^{-1}(p_0)$	$T_0^{-1}T_1^2T_2T_3M_2^{-2}M_1M_2^{-2}$
	$T_1^{-1}T_2T_3^{-1}$	$T_2T_4^{-1}(p_0)$	$T_0^{-1}T_2^{-1}T_4M_2M_1M_2$
	T_0^{-1}	$T_0^{-1}T_1T_4(p_0)$	$T_0T_1^{-1}T_4^{-1}M_2M_1M_2$
	$T_0^{-2}T_2$	$T_0^{-1}T_1^2T_2T_3(p_0)$	$T_0T_1^{-1}T_3^{-1}M_2^{-2}M_1M_2^{-2}$
M_2^3	$T_0^{-2}T_2$	$T_0^{-1}T_1T_2(p_{41})$	$T_0T_3^{-1}M_2^2M_1M_2^{-1}$
	$T_0^{-2}T_2T_4^{-1}$	$T_1M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{42})$	$T_0T_2^{-1}T_4^{-1}M_1M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_4^{-1}$	$M_2^{-1}M_1M_2^2T_0^{-1}T_4(p_{43})$	$M_2^{-1}M_1M_2^{-1}$
	$T_0^{-1}T_1^{-1}T_6$	p_{41}	$T_4M_2^2M_1M_2^{-1}$
	$T_1^{-1}T_4^{-1}$	$M_2^2M_1M_2^{-1}(p_{42})$	$T_0T_1^{-1}T_3^{-1}M_1M_2^{-1}M_1M_2^2$
	T_0^{-1}	$T_0^{-1}T_4(p_{42})$	$T_1^{-1}T_6M_2M_1M_2$
	$T_1^{-1}T_2T_3^{-1}$	$M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{43})$	$T_0T_6^{-1}M_2^2M_1M_2^2$
	$T_0^{-1}T_1^{-1}T_2$	$T_0^{-1}T_3(p_{42})$	$T_0^{-3}T_1T_6M_2^{-2}M_1M_2^{-2}$
	$T_0^{-1}T_2T_3^{-1}$	$T_2M_2^{-1}M_1M_2^2T_0^{-1}T_4(p_{42})$	$T_0T_1^{-1}T_4^{-1}M_1M_2^2M_1M_2^{-1}$
	$T_0T_1^{-1}T_3^{-1}$	$M_2^{-1}M_1M_2^2(p_{42})$	$T_0T_2^{-1}T_3^{-1}M_1M_2^{-1}M_1M_2^2$
M_2^4	$T_1^{-1}T_4^{-1}$	$T_0^{-1}T_1T_3(p_{32})$	$T_0^2T_3^{-1}T_5(M_1M_2^3)^2$
	$T_0^{-2}T_6$	$T_2T_3^{-1}(p_{32})$	$T_3(M_1M_2^3)^2$
$M_2M_1M_2$	T_0^{-1}	q_∞	$M_2M_1M_2$
	$T_0^{-2}T_1$	$T_1M_2M_1M_2T_0^{-2}T_3(p_{43})$	$M_2^{-1}M_1M_2^2$
	$T_0^{-2}T_1T_3$	$T_1M_2^{-1}M_1M_2^{-1}T_0^{-1}(p_{43})$	$T_0^{-1}T_4M_2M_1M_2^{-2}$

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Table 11. *Continued*

	$T_0T_1^{-1}T_3^{-1}$	$M_2^2M_1M_2^2T_0^{-1}T_3(p_{43})$	$T_1^{-1}T_2T_4M_2^{-2}M_1M_2$
	T_1^{-1}	$M_2M_1M_2^{-2}(p_{42})$	$T_0^{-1}T_3T_5^{-1}M_2M_1M_2$
$M_2M_1M_2^2$	$T_0^{-2}T_1$	$T_0T_2T_3^{-1}(p_0)$	$T_0T_2^{-1}T_5M_2^2M_1M_2^2$
	T_0^{-1}	$T_1T_3(p_0)$	$T_0T_1^{-1}M_2^2M_1M_2^2$
$M_2M_1M_2^3$	T_0^{-1}	$T_0^{-1}T_1T_3(p_{32})$	$T_0T_3^{-1}M_1$
	$T_0^{-2}T_1T_3^{-1}$	$T_2T_3^{-1}(p_{32})$	$T_0^{-1}T_3T_5^{-1}M_1$
$M_2M_1M_2^4$	$T_0^{-2}T_6$	$T_1T_4^{-1}(p_{41})$	$T_0^2T_2^{-1}T_4^{-1}M_2M_1M_2$
	T_3^{-2}	$T_2T_3^{-1}(p_{41})$	$T_2T_3^{-1}M_2^{-2}M_1M_2^{-2}$
	T_0^{-1}	$T_0^{-1}T_1T_3(p_{41})$	$T_1T_3M_2^{-2}M_1M_2^{-2}$
	$T_0^{-1}T_3^{-1}T_4$	$T_0^{-1}T_2T_4(p_{41})$	$T_1^{-1}T_4M_2M_1M_2$
$M_2M_1M_2^5$	T_0^{-1}	$T_0^{-1}T_1T_3(p_{31})$	$T_0T_1^{-1}M_2^2M_1M_2^2$
	$T_0^2T_1^{-1}T_3^{-2}$	$T_2T_3^{-1}(p_{31})$	$T_0T_2^{-1}T_5M_2^2M_1M_2^2$
$M_2^2M_1M_2$	T_0^{-1}	$T_2T_4(p_0)$	$T_0T_2^{-1}M_2^2M_1M_2^2$
	$T_0^{-2}T_2$	$T_0T_1T_4^{-1}(p_0)$	$T_0T_1^{-1}T_5M_2^2M_1M_2^2$
$M_2^2M_1M_2^2$	$T_0^{-2}T_2$	$M_2M_1M_2T_0^{-1}T_4(p_{43})$	$T_0T_2^{-1}T_5(M_1M_2^2)^2$
	T_0^{-1}	$M_2^{-1}M_1M_2^{-1}(p_{42})$	$(M_1M_2^3)^2$
$M_2^2M_1M_2^4$	$T_0^{-2}T_6$	$T_1T_4^{-1}(p_{31})$	$T_2M_2^{-1}M_1M_2^{-1}$
	$T_0^{-1}T_2T_3^{-1}T_4$	$T_0^{-1}T_2T_4(p_{31})$	$T_2^{-1}T_6M_2^{-1}M_1M_2^{-1}$
$M_2^2M_1M_2^5$	$T_0^{-2}T_2$	$T_0^{-1}T_1T_3(p_{42})$	$T_3^{-1}(M_1M_2^3)^2$
	$T_0^{-1}T_1^{-1}T_6$	$T_0^2T_2^{-1}T_4^{-1}(p_{42})$	T_3^{-1}
	$T_0^{-2}T_6$	$T_1M_2M_1M_2T_0^{-1}(p_{43})$	$T_0T_2^{-1}T_5M_2^3M_1M_2^3$
	T_0^{-1}	$M_2^{-2}M_1M_2T_0^{-1}T_4(p_{42})$	$T_1^{-1}T_6M_2^{-1}M_1M_2^2$
	$T_0T_1^{-1}T_2T_3^{-2}$	$T_2M_2M_1M_2T_0^{-1}(p_{43})$	$T_1^{-1}M_1$
	$T_1^{-1}T_2T_3^{-1}$	$T_0^{-1}T_2(p_{41})$	$M_1M_2^3M_1$
	$T_0^{-1}T_2T_3^{-1}$	$M_2M_1M_2T_0^{-2}T_3T_4(p_{42})$	$T_0^{-1}T_2T_3^{-1}M_2^3$
	$T_1^{-1}T_2T_3^{-1}T_4$	$M_2M_1M_2^{-2}T_0^{-1}T_3(p_{42})$	$T_0^2T_2^{-1}T_4^{-2}M_2^{-1}M_1M_2^2$
	$T_0^{-1}T_2T_3^{-1}T_4$	$T_0^{-1}T_2T_4(p_{42})$	$T_0^{-1}T_4^{-1}T_5^{-1}$
	$T_0^2T_1^{-1}T_3^{-2}$	$T_0^2T_1^{-1}T_3^{-1}(p_{42})$	$T_0T_4^{-1}T_5(M_1M_2^3)^2$
	$T_0T_1^{-1}T_3^{-1}$	$M_2M_1M_2(p_{42})$	$T_0T_1^{-1}T_3^{-1}M_2^3$
	T_3^{-1}	$T_0^{-1}T_1(p_{41})$	$T_0T_1T_2^{-1}T_3T_4^{-1}M_1M_2^3M_1$

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Table 12. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation and $M \in \langle M_1, M_2\rangle$. If $A_a(\gamma M p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma M A_b = W'$. The case $A_a = A_{42}$ and $p_b = p_{43}$*

M	γ	$W(p_c)$	W'
$M_2^3M_1M_2$	$T_0^{-2}T_2T_4^{-1}$	$T_1T_4^{-1}(p_{32})$	$T_0^{-1}T_4T_5^{-1}M_1$
	T_0^{-1}	$T_0^{-1}T_2T_4(p_{32})$	$T_0T_4^{-1}M_1$
$M_2^3M_1M_2^3$	$T_0^{-2}T_6$	$M_2^2M_1M_2^2T_0^{-2}T_3T_4(p_{43})$	$(M_1M_2)^2$
	$T_0^{-1}T_2T_3^{-1}$	$M_2M_1M_2T_0^{-1}T_3(p_{42})$	$T_0T_2^{-1}T_4^{-1}(M_1M_2)^2$
	$T_0^{-2}T_1T_2$	$M_2^{-1}M_1M_2^{-1}T_0^{-1}(p_{43})$	$T_0^{-1}T_1T_2(M_1M_2^{-2})^2$
$M_2^3M_1M_2^4$	$T_0^{-2}T_6$	$T_1T_4^{-1}(p_0)$	$T_0^{-1}T_4M_2^3M_1M_2^3$
	$T_0^{-1}T_2T_3^{-1}$	$T_0^{-1}T_2T_4(p_0)$	$T_0T_4^{-1}T_5M_2^3M_1M_2^3$
$M_2^3M_1M_2^5$	$T_0^{-1}T_2T_3^{-1}$	$T_0^{-1}T_1T_2^2T_4(p_0)$	$T_0^3T_1^{-1}T_2^{-2}T_4^{-1}(M_1M_2)^2$
	$T_1^{-1}T_2T_3^{-1}$	$T_0^{-1}T_2T_3(p_0)$	$T_0^{-1}T_1T_3^{-1}(M_1M_2^{-2})^2$
	$T_0^{-1}T_1^{-1}T_6$	$T_0^2T_2^{-1}T_4^{-1}(p_0)$	$T_0^{-1}T_2T_4(M_1M_2)^2$
	$T_0^{-2}T_6$	$T_1T_3^{-1}(p_0)$	$T_0^{-1}T_2T_3(M_1M_2^{-2})^2$
$M_2^4M_1M_2$	$T_0^{-2}T_6$	$T_2T_3^{-1}(p_{41})$	$T_0^2T_1^{-1}T_3^{-1}M_2M_1M_2$
	$T_0^{-1}T_3T_4^{-1}$	$T_0^{-1}T_1T_3(p_{41})$	$T_2^{-1}T_3M_2M_1M_2$
	T_4^{-2}	$T_1T_4^{-1}(p_{41})$	$T_1T_4^{-1}M_2^{-2}M_1M_2^{-2}$
	T_0^{-1}	$T_0^{-1}T_2T_4(p_{41})$	$T_2T_4M_2^{-2}M_1M_2^{-2}$
$M_2^4M_1M_2^2$	$T_0^{-2}T_6$	$T_2T_3^{-1}(p_{31})$	$T_1M_2^{-1}M_1M_2^{-1}$
	$T_0^{-1}T_1T_3T_4^{-1}$	$T_0^{-1}T_1T_3(p_{31})$	$T_1^{-1}T_6M_2^{-1}M_1M_2^{-1}$
$M_2^4M_1M_2^3$	$T_0^{-2}T_6$	$T_2T_3^{-1}(p_0)$	$T_0^{-1}T_3M_2^3M_1M_2^3$
	$T_0^{-1}T_1T_4^{-1}$	$T_0^{-1}T_1T_3(p_0)$	$T_0T_3^{-1}T_5M_2^3M_1M_2^3$
$M_2^4M_1M_2^4$	$T_0^{-1}T_1T_4^{-1}$	$T_1M_2^{-1}M_1M_2^{-1}T_0^{-1}(p_{43})$	$T_1^{-1}T_2T_4M_2^{-2}M_1M_2$
	$T_0^{-3}T_1T_6$	$T_1M_2M_1M_2T_0^{-2}T_3(p_{43})$	$T_3^{-1}T_4M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_1^{-1}T_6$	$M_2M_1M_2^{-2}(p_{42})$	$T_4^{-1}M_2^{-2}M_1M_2^{-2}$
	$T_1^{-1}T_3^{-1}T_6$	$M_2^2M_1M_2^2T_0^{-1}T_3(p_{43})$	$T_0^{-1}T_4M_2M_1M_2^{-2}$
	$T_0^{-2}T_6$	q_∞	$T_0^{-1}T_1T_2M_2^{-2}M_1M_2^{-2}$
$M_2^4M_1M_2^5$	$T_0^{-2}T_6$	$T_0T_2T_3^{-1}(p_0)$	$T_1M_2^{-1}M_1M_2^{-1}$
	$T_0^{-1}T_1^{-1}T_6$	$T_1T_3(p_0)$	$T_0^{-2}T_2T_5^{-1}M_2^{-1}M_1M_2^{-1}$
$M_2^5M_1M_2$	T_0^{-1}	$T_0^{-1}T_2T_4(p_{31})$	$T_0T_2^{-1}M_2^2M_1M_2^2$
	$T_0^2T_2^{-1}T_4^{-2}$	$T_1T_4^{-1}(p_{31})$	$T_0T_1^{-1}T_5M_2^2M_1M_2^2$
$M_2^5M_1M_2^2$	$T_0^2T_2^{-1}T_4^{-2}$	$T_0^2T_2^{-1}T_4^{-1}(p_{42})$	$T_0T_3^{-1}T_5(M_1M_2^3)^2$
	$T_0T_2^{-1}T_4^{-1}$	$M_2M_1M_2(p_{42})$	$T_0T_2^{-1}T_4^{-1}M_1M_2^3M_1$
	T_4^{-1}	$T_0^{-1}T_2(p_{41})$	$T_0T_1^{-1}T_2T_3^{-1}T_4M_2^3$
	T_0^{-1}	$M_2M_1M_2^{-2}T_0^{-1}T_3(p_{42})$	$T_2^{-1}T_6M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_1T_4^{-1}$	$M_2M_1M_2T_0^{-2}T_3T_4(p_{42})$	$T_0^{-1}T_1T_4^{-1}M_1M_2^3M_1$
	$T_0^{-1}T_1T_3T_4^{-1}$	$T_0^{-1}T_1T_3(p_{42})$	$T_0^{-1}T_3^{-1}T_5^{-1}$
	$T_0^{-2}T_6$	$T_2M_2M_1M_2T_0^{-1}(p_{43})$	$T_0T_1^{-1}T_5M_2^3M_1M_2^3$

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Table 12. *Continued*

	$T_1T_2^{-1}T_3T_4^{-1}$	$M_2^{-2}M_1M_2T_0^{-1}T_4(p_{42})$	$T_0^2T_1^{-1}T_3^{-2}M_2^2M_1M_2^{-1}$
	$T_1T_2^{-1}T_4^{-1}$	$T_0^{-1}T_1(p_{41})$	M_2^3
	$T_0^{-1}T_2^{-1}T_6$	$T_0^2T_1^{-1}T_3^{-1}(p_{42})$	T_4^{-1}
	$T_0T_1T_2^{-1}T_4^{-2}$	$T_1M_2M_1M_2T_0^{-1}(p_{43})$	$T_2^{-1}M_1$
	$T_0^{-2}T_1$	$T_0^{-1}T_2T_4(p_{42})$	$T_4^{-1}(M_1M_2^3)^2$
$M_2^5M_1M_2^3$	$T_0^{-1}T_2^{-1}T_6$	$T_0^2T_1^{-1}T_3^{-1}(p_0)$	$T_0^{-1}T_1T_3(M_1M_2)^2$
	$T_1T_2^{-1}T_4^{-1}$	$T_0^{-1}T_1T_4(p_0)$	$T_0^{-1}T_2T_4^{-1}(M_1M_2^{-2})^2$
	$T_0^{-2}T_6$	$T_2T_4^{-1}(p_0)$	$T_0^{-1}T_1T_4(M_1M_2^{-2})^2$
	$T_0^{-1}T_1T_4^{-1}$	$T_0^{-1}T_1^2T_2T_3(p_0)$	$T_0^3T_1^{-2}T_2^{-1}T_3^{-1}(M_1M_2)^2$
$M_2^5M_1M_2^4$	$T_0^{-2}T_6$	$T_0T_1T_4^{-1}(p_0)$	$T_2M_2^{-1}M_1M_2^{-1}$
	$T_0^{-1}T_2^{-1}T_6$	$T_2T_4(p_0)$	$T_2^{-1}T_6M_2^{-1}M_1M_2^{-1}$
$M_2^5M_1M_2^5$	$T_0^{-1}T_2^{-1}T_6$	$M_2M_1M_2T_0^{-1}T_3(p_{43})$	$T_2(M_1M_2^{-1})^2$
	$T_0^{-2}T_5^{-1}$	$M_2^{-1}M_1M_2^{-1}(p_{42})$	$T_0^{-1}T_5^{-1}$
	$T_0^{-2}T_6$	$M_2^2M_1M_2^2T_0^{-1}T_3T_4(p_{42})$	$(M_1M_2^3)^2$

Table 13. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation and $M \in \langle M_1, M_2\rangle$. If $A_a(\gamma M p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma MA_b = W'$. The case $A_a = A_{43}$ and $p_b = p_{42}$*

M	γ	$W(p_c)$	W'
M_2^2	$T_1^{-1}T_2T_4$	$T_0T_3^{-1}(p_{31})$	$T_0T_1T_4^{-1}(M_1M_2^3)^2$
	T_0T_5	$T_0^{-1}T_1T_2T_3(p_{31})$	$T_0^2T_1^{-1}T_3^{-1}(M_1M_2^3)^2$
M_2^3	$T_0^{-1}T_3$	$M_2M_1M_2^{-2}(p_{43})$	$T_1T_2^{-1}T_3M_1M_2M_1M_2^{-2}$
	Id	$M_2^3T_4(p_{42})$	$T_1^{-1}T_6M_2M_1M_2^{-2}$
	$T_1^{-1}T_2$	$M_1M_2^3T_4(p_{42})$	$T_0^3T_2^{-1}T_4^{-2}M_1M_2M_1M_2^{-2}$
	$T_0^{-1}T_2T_3$	$M_2M_1M_2T_0^{-1}T_3(p_{43})$	$T_0^{-1}T_1T_4M_2^{-1}M_1M_2^2$
	$T_0^2T_4^{-1}T_5$	$T_2M_2M_1M_2T_0^{-1}T_4(p_{42})$	$T_0T_1^{-1}T_3^{-1}T_6(M_1M_2^3)^2$
	$T_1^{-1}T_2T_4$	$M_2^{-2}M_1M_2(p_{43})$	$T_3M_1M_2M_1M_2^{-2}$
	$T_0T_2T_6^{-1}$	p_{41}	$T_0^2T_2^{-1}T_4^{-1}M_2^{-2}M_1M_2$
	T_0T_5	$T_0^{-1}T_1T_3(p_{43})$	$T_2M_2^2M_1M_2^2$
	$T_0^{-1}T_2^2T_3T_4$	$T_0^{-1}T_2T_4(p_{43})$	$T_0T_2^{-1}M_2^{-1}M_1M_2^{-1}$
	T_2	$T_0^{-1}T_1T_2(p_{41})$	$T_1T_3M_2^{-2}M_1M_2$
	T_2T_4	$M_2M_1M_2T_0^{-1}T_4(p_{43})$	$T_1T_3M_2^{-1}M_1M_2^2$
	$T_2^2T_4$	$T_1(M_1M_2)^2T_0^{-1}T_4(p_{42})$	$T_0^2T_1^{-1}T_3^{-1}M_1$
M_2^4	$T_0^{-1}T_3$	$T_3(p_0)$	$T_0^{-1}T_1T_2T_3^{-1}M_2^2M_1M_2^2$
	T_2	$T_1T_2T_3^{-1}(p_0)$	$T_3M_2^2M_1M_2^2$

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Table 13. *Continued*

	$T_0^{-1}T_2T_3$	$T_2^2T_4(p_0)$	$T_0^3T_2^{-2}T_4^{-1}M_2^{-1}M_1M_2^{-1}$
	Id	$T_0^2T_1T_2^{-1}T_4^{-1}(p_0)$	$T_1^{-1}T_2T_4M_2^{-1}M_1M_2^{-1}$
M_2^5	Id	$T_0^2T_3^{-1}(p_0)$	T_1T_3
	$T_0^{-1}T_3$	$T_0^3T_4^{-1}T_5(p_0)$	$T_0T_1^{-1}T_4$
$M_2M_1M_2$	$T_0T_6^{-1}$	$M_2M_1M_2(p_{43})$	$T_0T_6^{-1}$
	$T_0T_2^{-1}T_5$	$M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{43})$	$T_2M_1M_2^{-1}M_1M_2^2$
	T_0T_5	$M_2M_1M_2T_0^{-2}T_3T_4(p_{43})$	T_0T_5
$M_2M_1M_2^2$	T_0T_5	$T_0^{-1}T_1T_2T_4(p_0)$	$T_0^{-1}T_2M_2M_1M_2$
	$T_0T_2^{-1}T_5$	$T_0T_4^{-1}(p_0)$	$T_0T_1T_6^{-1}M_2M_1M_2$
$M_2M_1M_2^3$	$T_0T_2^{-1}T_5$	$T_3(p_0)$	$T_0T_3^{-1}(M_1M_2^{-1})^2$
	T_0T_5	$T_2^2T_4(p_0)$	$T_0T_1T_2^{-1}T_4^{-1}(M_1M_2^2)^2$
	$T_0^2T_3^{-1}T_5$	$T_1T_2T_3^{-1}(p_0)$	$T_1^{-1}T_2^{-1}T_3(M_1M_2^{-1})^2$
	$T_0^{-1}T_1T_4$	$T_0^2T_1T_2^{-1}T_4^{-1}(p_0)$	$T_2^2T_4(M_1M_2^2)^2$
$M_2M_1M_2^4$	$T_0^{-1}T_4$	$M_2^{-1}M_1M_2^{-1}(p_{43})$	$T_4M_1M_2^3M_1$
	$T_0^{-1}T_2T_4$	$T_2(p_{41})$	$T_0T_3T_4^{-1}M_2^3$
	$T_0^{-1}T_2T_4^2$	$M_1T_0^{-1}T_3(p_{43})$	$T_0T_2T_3^{-1}M_2^3M_1M_2^3$
	Id	$M_2^2M_1M_2^2T_3(p_{42})$	$T_2^{-1}T_6M_2^3M_1M_2^3$
	$T_3^{-1}T_4$	$M_2^2M_1M_2^2T_4(p_{42})$	$T_0T_1^{-1}M_1$
	$T_0^{-1}T_1T_4$	$T_1(p_{41})$	$T_0M_2^3$
	$T_0T_2^{-1}T_5$	$T_0^{-1}T_3(p_{43})$	T_2T_4
	$T_2T_3^{-1}T_4$	$T_0T_3^{-1}(p_{43})$	$T_0^{-1}T_1T_3^2T_4$
	$T_0^2T_3^{-1}T_5$	$M_2^2M_1M_2^2T_0^{-1}T_3T_4(p_{43})$	$T_3M_2^3$
	T_0T_5	$T_2M_2^{-1}M_1M_2^{-1}(p_{42})$	$T_1^{-1}T_6M_2^3M_1M_2^3$
	$T_0^{-1}T_1T_2T_4^2$	$T_1M_2^{-1}M_1M_2^{-1}(p_{42})$	$T_0T_2^{-1}M_1$
	T_1	$T_0T_4^{-1}(p_{43})$	$T_2T_4(M_1M_2^3)^2$
$M_2M_1M_2^5$	$T_0^{-1}T_2T_4^2$	$T_0^2T_3^{-1}T_5(p_{32})$	$T_0T_1^{-1}T_6M_2^{-2}M_1M_2^{-2}$
	Id	$T_0T_4^{-1}(p_{32})$	$T_0T_2^{-1}M_2^{-2}M_1M_2^{-2}$
$M_2^2M_1M_2$	$T_0T_2T_6^{-1}$	$T_0T_3^{-1}(p_0)$	$T_0T_1^{-1}T_5M_2M_1M_2$
	T_0T_5	$T_0^2T_4^{-1}T_5(p_0)$	$T_0^{-1}T_1M_2M_1M_2$
$M_2^2M_1M_2^2$	$T_0T_2^{-1}T_5$	$M_2^2M_1M_2^{-1}(p_{43})$	$T_1T_3M_2^2M_1M_2^2$
	$T_0^2T_4^{-1}T_5$	$T_2M_2M_1M_2(p_{42})$	$T_0T_2^{-1}T_3^{-1}M_2^{-1}M_1M_2^2$
	T_3T_5	$M_2M_1M_2^{-2}T_0^{-1}T_3(p_{43})$	T_4M_1
	$T_0T_1T_5$	$T_1M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{43})$	$T_1^{-1}T_4M_2^{-1}M_1M_2^{-1}$
	T_0T_5	q_∞	$T_1T_2M_2^2M_1M_2^2$

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Table 14. *Let* $\gamma = N_{(z_1,z_2,t)}$ *be a Heisenberg translation and* $M \in \langle M_1, M_2\rangle$. *If* $A_a(\gamma M p_b) = W(p_c)$, *then* $A_c^{-1}W^{-1}A_a\gamma MA_b = W'$. *The case* $A_a = A_{43}$ *and* $p_b = p_{42}$

M	γ	$W(p_c)$	W'
$M_2^2M_1M_2^3$	T_0T_5	$T_0^3T_4^{-1}T_5(p_0)$	$T_0^2T_1^{-1}T_3^{-1}M_2^3M_1M_2^3$
	$T_0^2T_3^{-1}T_5$	$T_0^2T_3^{-1}(p_0)$	$T_0T_1T_4^{-1}M_2^3M_1M_2^3$
$M_2^2M_1M_2^4$	T_0T_5	$T_0^2T_4^{-1}T_5(p_{32})$	$T_1M_2M_1M_2$
	$T_2T_3^{-1}T_4$	$T_0T_3^{-1}(p_{32})$	$T_0^2T_2T_6^{-1}M_2M_1M_2$
$M_2^2M_1M_2^5$	T_0T_5	$T_0^2T_4^{-1}T_5(p_{41})$	$T_3M_2^{-1}M_1M_2^{-1}$
	$T_0T_1^{-1}T_2T_3^{-1}T_4$	$T_0T_3^{-1}(p_{41})$	$T_4T_5^{-1}M_2^{-1}M_1M_2^{-1}$
	$T_2^2T_4^2$	$T_0^2T_3^{-1}T_5(p_{41})$	$T_0^3T_3^{-1}T_5M_2^2M_1M_2^2$
	Id	$T_0T_4^{-1}(p_{41})$	$T_0T_4^{-1}M_2^2M_1M_2^2$
$M_2^3M_1M_2$	$T_0^{-1}T_2T_3$	$T_0^2T_1^{-1}T_2T_3^{-1}(p_0)$	$T_1^2T_3(M_1M_2^2)^2$
	$T_0^2T_4^{-1}T_5$	$T_1T_2T_4^{-1}(p_0)$	$T_1^{-1}T_2^{-1}T_4(M_1M_2^{-1})^2$
	T_0T_5	$T_1^2T_3(p_0)$	$T_0T_1^{-1}T_2T_3^{-1}(M_1M_2^2)^2$
$M_2^3M_1M_2^2$	T_0T_5	$T_0^3T_3^{-1}T_5(p_0)$	$T_0^2T_2^{-1}T_4^{-1}M_2^3M_1M_2^3$
	$T_0^2T_4^{-1}T_5$	$T_0^2T_4^{-1}(p_0)$	$T_0T_2T_3^{-1}M_2^3M_1M_2^3$
$M_2^3M_1M_2^3$	$T_0^2T_4^{-1}T_5$	$M_2^2M_1M_2^{-1}T_3(p_{42})$	$T_0^2T_2^{-1}T_4^{-1}M_1M_2M_1M_2^{-2}$
	T_0T_5	$M_2M_1M_2T_0^{-1}T_3T_4(p_{42})$	$T_0^{-1}T_6(M_1M_2^2)^2$
	$T_0^{-1}T_1T_2$	$M_2M_1M_2T_0(p_{42})$	$T_0^{-1}T_1T_2(M_1M_2^2)^2$
$M_2^3M_1M_2^5$	Id	$T_0T_4^{-1}(p_{31})$	$T_2T_4M_1$
	$T_2^2T_4$	$T_0^{-1}T_1T_2T_4(p_{31})$	$T_0T_2^{-1}T_3M_1$
$M_2^4M_1M_2$	$T_0^{-1}T_3$	$M_2^{-1}M_1M_2^{-1}(p_{43})$	$T_3M_2^3$
	$T_0^{-1}T_1T_3$	$T_1(p_{41})$	$T_0T_3^{-1}T_4M_1M_2^3M_1$
	$T_0^{-1}T_1T_3^2$	$T_0^{-1}T_3(p_{43})$	$T_0T_2T_3^{-1}(M_1M_2^3)^2$
	$T_3T_4^{-1}$	$M_2^2M_1M_2^2T_3(p_{42})$	$T_0T_2^{-1}M_1$
	$T_1T_3T_4^{-1}$	$T_0T_4^{-1}(p_{43})$	$T_0^{-1}T_2T_3T_4^2$
	Id	$M_2^2M_1M_2^2T_4(p_{42})$	$T_1^{-1}T_6M_2^3M_1M_2^3$
	$T_0^{-1}T_2T_3$	$T_2(p_{41})$	$T_0M_1M_2^3M_1$
	$T_0^2T_4^{-1}T_5$	$M_2^2M_1M_2^2T_0^{-1}T_3T_4(p_{43})$	$T_4M_1M_2^3M_1$
	$T_0^{-1}T_1T_2T_3^2$	$T_2M_2^{-1}M_1M_2^{-1}(p_{42})$	$T_0T_1^{-1}M_1$
	$T_0T_2T_6^{-1}$	$T_0^{-1}T_4(p_{43})$	T_1T_3
	T_0T_5	$T_1M_2^{-1}M_1M_2^{-1}(p_{42})$	$T_2^{-1}T_6M_2^3M_1M_2^3$
	T_2	$T_0T_3^{-1}(p_{43})$	$T_1T_3(M_1M_2^3)^2$
$M_2^4M_1M_2^2$	$T_1T_3T_4^{-1}$	$T_0T_4^{-1}(p_{32})$	$T_0^2T_1T_6^{-1}M_2M_1M_2$
	T_0T_5	$T_0^2T_3^{-1}T_5(p_{32})$	$T_2M_2M_1M_2$

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Table 14. *Continued*

$M_2^4M_1M_2^4$	Id	$M_2M_1M_2(p_{43})$	$(M_1M_2^3)^2$
	T_1	$M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{43})$	$T_1M_1M_2^2M_1M_2^{-1}$
	T_1T_2	$M_2M_1M_2T_0^{-2}T_3T_4(p_{43})$	$T_1T_2(M_1M_2^3)^2$
$M_2^4M_1M_2^5$	T_2	$T_0^2T_3^{-1}T_5(p_0)$	$T_1^{-1}T_6M_2^{-2}M_1M_2^{-2}$
	Id	$T_0T_4^{-1}(p_0)$	$T_2^{-1}M_2^{-2}M_1M_2^{-2}$
$M_2^5M_1M_2$	Id	$T_0T_3^{-1}(p_{32})$	$T_0T_1^{-1}M_2^{-2}M_1M_2^{-2}$
	$T_0^{-1}T_1T_3^2$	$T_0^2T_4^{-1}T_5(p_{32})$	$T_0T_2^{-1}T_6M_2^{-2}M_1M_2^{-2}$
$M_2^5M_1M_2^2$	$T_0T_1T_2^{-1}T_3T_4^{-1}$	$T_0T_4^{-1}(p_{41})$	$T_0^3T_1^{-1}T_2^{-1}T_4^{-1}M_2^{-1}M_1M_2^{-1}$
	T_0T_5	$T_0^2T_3^{-1}T_5(p_{41})$	$T_4M_2^{-1}M_1M_2^{-1}$
	$T_1^2T_3^2$	$T_0^2T_4^{-1}T_5(p_{41})$	$T_0^3T_4^{-1}T_5M_2^2M_1M_2^2$
	Id	$T_0T_3^{-1}(p_{41})$	$T_0T_3^{-1}M_2^2M_1M_2^2$
$M_2^5M_1M_2^3$	Id	$T_0T_3^{-1}(p_{31})$	$T_1T_3M_1$
	$T_1^2T_3$	$T_0^2T_4^{-1}T_5(p_{31})$	$T_0T_1^{-1}T_4M_1$
$M_2^5M_1M_2^4$	T_1	$T_0^2T_4^{-1}T_5(p_0)$	$T_2^{-1}T_6M_2^{-2}M_1M_2^{-2}$
	Id	$T_0T_3^{-1}(p_0)$	$T_1^{-1}M_2^{-2}M_1M_2^{-2}$
$M_2^5M_1M_2^5$	Id	q_∞	$T_0M_2^{-1}M_1M_2^{-1}$
	T_1^{-1}	$M_2^{-1}M_1M_2^2(p_{43})$	$T_1^{-1}T_4M_2^{-1}M_1M_2^{-1}$
	$T_0^{-1}T_3$	$M_2M_1M_2^{-2}T_0^{-1}T_3(p_{43})$	$T_0T_3^{-1}M_2^3M_1M_2^3$
	$T_0T_3^{-1}$	$T_1M_2M_1M_2(p_{42})$	$T_0^2T_2^{-1}T_4^{-1}M_2^{-1}M_1M_2^2$
	T_1	$T_1M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{43})$	$T_2T_4M_2^2M_1M_2^2$

Table 15. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation and $M \in \langle M_1, M_2\rangle$. If $A_a(\gamma M p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma MA_b = W'$. The case $A_a = A_{43}$ and $p_b = p_{43}$*

M	γ	$W(p_c)$	W'
M_2	T_6^{-1}	$T_3(p_0)$	$T_0^{-1}T_1T_3^{-1}M_2M_1M_2^{-2}$
	$T_1^{-1}T_5$	$T_2^2T_4(p_0)$	$T_0T_2^{-1}T_4^{-1}M_2^{-2}M_1M_2$
	$T_0^{-1}T_4$	$T_0^2T_1T_2^{-1}T_4^{-1}(p_0)$	$T_1^{-1}T_2^2T_4M_2^{-2}M_1M_2$
	$T_0^{-1}T_2T_4$	$T_1T_2T_3^{-1}(p_0)$	$T_2^{-1}T_3M_2M_1M_2^{-2}$
M_2^2	$T_0^{-1}T_2T_4$	$T_0^2T_3^{-1}(p_0)$	$T_1M_1M_2^{-1}M_1M_2^2$
	$T_1^{-1}T_5$	$T_1T_2T_3(p_0)$	$T_0T_1^{-1}T_3^{-1}T_4M_1M_2^{-1}M_1M_2^2$
M_2^3	$T_1^{-1}T_5$	$M_2M_1M_2T_0^{-1}T_3T_4(p_{42})$	$T_1^{-1}T_6M_2^{-2}M_1M_2$

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Table 15. *Continued*

	$T_0^{-2}T_2T_3$	$M_2^2M_1M_2^{-1}T_3(p_{42})$	$T_0^{-1}T_5^{-1}M_1$
	$T_0^{-1}T_2$	$M_2M_1M_2T_0(p_{42})$	$T_0^{-1}T_2M_2^{-2}M_1M_2$
	$T_0^{-1}T_2T_4$	$M_2^{-1}M_1M_2^2T_4(p_{42})$	$M_2^3M_1M_2^3$
M_2^5	T_1^{-1}	$T_0T_3^{-1}(p_{31})$	$T_1T_3T_4^{-1}M_1M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_1T_3$	$T_0^2T_4^{-1}T_5(p_{31})$	$T_0T_1^{-1}M_1M_2^2M_1M_2^{-1}$
$M_2M_1M_2$	T_6^{-1}	$M_2^{-1}M_1M_2^{-1}(p_{43})$	$T_0T_6^{-1}(M_1M_2^2)^2$
	$T_2^{-1}T_5$	$T_1(p_{41})$	$T_0T_3^{-1}(M_1M_2^{-1})^2$
	$T_0^{-1}T_2^{-1}T_3T_5$	$T_0^{-1}T_3(p_{43})$	$T_2T_3^{-1}T_4M_1M_2^{-1}M_1M_2^2$
	T_5	$M_2^2M_1M_2^2T_0^{-1}T_3T_4(p_{43})$	$(M_1M_2^{-1})^2$
	$T_0^{-1}T_3T_5$	$T_2M_2^{-1}M_1M_2^{-1}(p_{42})$	$T_0T_1^{-1}T_3^{-1}M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_3$	$M_2^2M_1M_2^2T_3(p_{42})$	$T_0T_2^{-1}T_3^{-1}M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_1T_3$	$T_0T_4^{-1}(p_{43})$	$T_0T_2T_6^{-1}M_1M_2^2M_1M_2^{-1}$
$M_2M_1M_2^2$	$T_0^{-1}T_1T_3$	$T_0T_4^{-1}(p_{32})$	$T_4M_2^3M_1$
	$T_0^{-1}T_4T_5$	$T_0^2T_3^{-1}T_5(p_{32})$	$T_0T_1^{-1}T_2T_3^{-1}M_2^3M_1$
$M_2M_1M_2^4$	$T_0^{-1}T_4$	$M_2M_1M_2(p_{43})$	$T_0^{-1}T_4M_1M_2^{-1}M_1M_2^2$
	$T_0^{-1}T_2T_4$	$M_2^{-1}M_1M_2^2T_0^{-1}T_4(p_{43})$	$(M_1M_2)^2$
	$T_0^{-1}T_1T_4$	$M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{43})$	$T_0^{-1}T_1T_2(M_1M_2^{-2})^2$
	$T_0^2T_3^{-1}T_5$	$M_2M_1M_2T_6^{-1}(p_{43})$	$T_0^2T_3^{-1}T_5M_1M_2^{-1}M_1M_2^2$
$M_2M_1M_2^5$	$T_0^{-1}T_4$	$T_0T_4^{-1}(p_0)$	$T_0^{-1}T_1T_2^{-1}T_3M_1M_2^3$
	$T_0^{-1}T_2T_4$	$T_0^2T_3^{-1}T_5(p_0)$	$T_4^{-1}M_1M_2^3$
$M_2^2M_1M_2$	$T_0^{-1}T_2T_4$	$T_0T_3^{-1}(p_{32})$	$T_3M_1M_2^3$
	$T_0^{-1}T_3T_5$	$T_0^2T_4^{-1}T_5(p_{32})$	$T_0T_1T_2^{-1}T_4^{-1}M_1M_2^3$
$M_2^2M_1M_2^2$	$T_1T_3T_5$	$T_0^2T_4^{-1}T_5(p_{41})$	$T_0^{-1}T_2T_3M_2^{-2}M_1M_2$
	$T_0^{-1}T_1T_3$	$T_0T_4^{-1}(p_{41})$	$T_0^2T_2^{-1}T_4^{-1}M_2M_1M_2^{-2}$
$M_2^2M_1M_2^3$	$T_0^{-1}T_2T_4$	$T_0T_3^{-1}(p_{31})$	$T_1M_2^2M_1M_2^{-1}$
	$T_0T_1T_5$	$T_0^2T_4^{-1}T_5(p_{31})$	$T_0T_1^{-1}T_3^{-1}T_4M_2^2M_1M_2^{-1}$
$M_2^2M_1M_2^4$	$T_0^2T_3^{-1}T_5$	$T_0^2T_4^{-1}T_5(p_0)$	$T_1T_2^{-1}T_4^{-1}M_1M_2^3$
	$T_0^{-1}T_2T_4$	$T_0T_3^{-1}(p_0)$	$T_0^{-1}T_3M_1M_2^3$
$M_2^2M_1M_2^5$	$T_0^{-1}T_1^{-1}T_2T_4$	$M_2^{-1}M_1M_2^2(p_{43})$	$T_0^{-1}T_4M_2M_1M_2^{-2}$
	$T_1^{-1}T_5$	$M_2M_1M_2^{-2}T_0^{-1}T_3(p_{43})$	$M_2^{-1}M_1M_2^2$
	$T_0^{-1}T_4$	$M_2^2M_1M_2^{-1}(p_{43})$	$T_1T_2^{-1}T_3M_2M_1M_2^{-2}$
	$T_0^{-1}T_2$	$T_2M_2M_1M_2(p_{42})$	$T_3^{-1}M_2^{-2}M_1M_2^{-2}$

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Table 15. *Continued*

	$T_0^{-1}T_2T_4^2$	$M_2^{-2}M_1M_2T_0^{-1}T_4(p_{43})$	$T_3T_4^{-1}M_2^{-1}M_1M_2^2$
	$T_0^{-1}T_2^2T_4$	$T_2M_2^{-1}M_1M_2^2T_0^{-1}T_4(p_{43})$	$T_0^{-1}T_3M_2^{-2}M_1M_2$
	$T_2T_3^{-1}T_4$	$T_1M_2M_1M_2(p_{42})$	$T_0^{-1}T_3T_5^{-1}M_2M_1M_2$
	$T_0^2T_3^{-1}T_5$	$T_1M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{43})$	$T_1^{-1}T_2T_4M_2^{-2}M_1M_2$
	$T_0^{-1}T_2T_4$	q_∞	$T_1M_2M_1M_2^{-2}$
$M_2^3M_1M_2^2$	$T_0^{-1}T_1T_3$	$T_0T_4^{-1}(p_{31})$	$T_2M_2^{-1}M_1M_2^2$
	$T_0T_2T_5$	$T_0^2T_3^{-1}T_5(p_{31})$	$T_0T_2^{-1}T_3T_4^{-1}M_2^{-1}M_1M_2^2$

Table 16. *Let $\gamma = N_{(z_1,z_2,t)}$ be a Heisenberg translation and $M \in \langle M_1, M_2\rangle$. If $A_a(\gamma M p_b) = W(p_c)$, then $A_c^{-1}W^{-1}A_a\gamma MA_b = W'$. The case $A_a = A_{43}$ and $p_b = p_{43}$*

M	γ	$W(p_c)$	W'
$M_2^3M_1M_2^3$	$T_0^{-1}T_1T_3$	$M_2M_1M_2^{-2}(p_{43})$	$T_0T_1T_6^{-1}M_2^3M_1M_2^3$
	$T_0^{-1}T_1$	$M_2^3T_4(p_{42})$	$T_4^{-1}(M_1M_2^3)^2$
	$T_0^{-1}T_1T_2$	$T_0^{-1}T_1T_2(p_{41})$	Id
	$T_0^2T_4^{-1}T_5$	$M_2M_1M_2T_0^{-1}T_3(p_{43})$	$T_0^{-1}T_4(M_1M_2)^2$
	$T_0^{-1}T_1^2T_2T_3$	$T_2M_2M_1M_2T_0^{-1}T_4(p_{42})$	$T_0^2T_1^{-1}T_3^{-2}M_2^2M_1M_2^{-1}$
	T_0T_5	p_{41}	$T_0^{-1}T_5^{-1}$
	$T_0T_1T_5$	$T_0^{-1}T_1T_3(p_{43})$	$T_1^{-1}T_2M_1M_2M_1M_2^{-2}$
$M_2^3M_1M_2^4$	$T_0^2T_3^{-1}T_5$	$T_1^2T_3(p_0)$	$T_0^2T_1^{-2}T_2T_3^{-1}M_1M_2M_1M_2^{-2}$
	$T_0^{-1}T_1T_2$	$T_1T_2T_4^{-1}(p_0)$	$T_0^{-1}T_2^{-1}T_4M_1M_2^{-2}M_1M_2$
	$T_0^{-1}T_2$	$T_0^2T_1^{-1}T_2T_3^{-1}(p_0)$	$T_0^{-1}T_1T_3M_1M_2M_1M_2^{-2}$
	$T_0^{-1}T_2T_4$	$T_4(p_0)$	$T_1T_4^{-1}M_1M_2^{-2}M_1M_2$
$M_2^3M_1M_2^5$	$T_0^{-1}T_2T_4$	$T_0^3T_3^{-1}T_5(p_0)$	$T_0T_2^{-1}M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_2$	$T_0^2T_4^{-1}(p_0)$	$T_2T_3^{-1}T_4M_2^2M_1M_2^{-1}$
$M_2^4M_1M_2$	$T_0^{-1}T_3$	$M_2M_1M_2(p_{43})$	$T_0^{-1}T_3M_1M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_1T_3$	$M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{43})$	$(M_1M_2)^2$
	$T_0^{-1}T_2T_3$	$M_2^{-1}M_1M_2^2T_0^{-1}T_4(p_{43})$	$T_0^{-1}T_1T_2(M_1M_2^{-2})^2$
	$T_0^2T_4^{-1}T_5$	$M_2M_1M_2T_0^{-2}T_3T_4(p_{43})$	$T_0^2T_4^{-1}T_5M_1M_2^2M_1M_2^{-1}$
$M_2^4M_1M_2^2$	$T_0^2T_4^{-1}T_5$	$T_0^2T_3^{-1}T_5(p_0)$	$T_1^{-1}T_2T_3^{-1}M_2^3M_1$
	$T_0^{-1}T_1T_3$	$T_0T_4^{-1}(p_0)$	$T_0^{-1}T_4M_2^3M_1$
$M_2^4M_1M_2^3$	$T_0^{-1}T_1T_3$	$T_3(p_0)$	$T_2T_3^{-1}M_1M_2M_1M_2^{-2}$

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Table 16. *Continued*

	$T_0^{-1}T_1$	$T_0^2T_1T_2^{-1}T_4^{-1}(p_0)$	$T_0^{-1}T_2T_4M_1M_2^{-2}M_1M_2$
	$T_0^{-1}T_1T_2$	$T_1T_2T_3^{-1}(p_0)$	$T_0^{-1}T_1^{-1}T_3M_1M_2M_1M_2^{-2}$
	$T_0^2T_4^{-1}T_5$	$T_2^2T_4(p_0)$	$T_0^2T_1T_2^{-2}T_4^{-1}M_1M_2^{-2}M_1M_2$
$M_2^4M_1M_2^4$	T_0^{-1}	$M_2^{-1}M_1M_2^{-1}(p_{43})$	$(M_1M_2^{-1})^2$
	T_3^{-1}	$M_2^2M_1M_2^2T_4(p_{42})$	$T_0T_1^{-1}T_3^{-1}M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_1$	$T_1(p_{41})$	$T_4(M_1M_2^2)^2$
	$T_0^{-1}T_1T_3$	$T_0^{-1}T_3(p_{43})$	$T_2M_1M_2^2M_1M_2^{-1}$
	$T_2T_3^{-1}$	$T_0T_3^{-1}(p_{43})$	$T_0^{-1}T_1T_3^2M_1M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_1T_2$	$M_2^2M_1M_2^2T_0^{-1}T_3T_4(p_{43})$	$T_0T_6^{-1}(M_1M_2^2)^2$
	$T_0^2T_4^{-1}T_5$	$T_2M_2^{-1}M_1M_2^{-1}(p_{42})$	$T_0T_1^{-1}T_4^{-1}M_2^{-1}M_1M_2^2$
$M_2^4M_1M_2^5$	T_4^{-1}	$T_0T_4^{-1}(p_{32})$	$T_1T_2^{-1}T_3M_1M_2^3$
	$T_0^{-1}T_2T_4$	$T_0^2T_3^{-1}T_5(p_{32})$	$T_0T_4^{-1}M_1M_2^3$
$M_2^5M_1M_2$	$T_0^{-1}T_3$	$T_0T_3^{-1}(p_0)$	$T_0^{-1}T_1^{-1}T_2T_4M_2^3M_1$
	$T_0^{-1}T_1T_3$	$T_0^2T_4^{-1}T_5(p_0)$	$T_3^{-1}M_2^3M_1$
$M_2^5M_1M_2^2$	$T_0^{-1}T_1T_3$	q_∞	$T_2M_2^{-2}M_1M_2$
	$T_0^{-1}T_1T_2^{-1}T_3$	$M_2^2M_1M_2^{-1}(p_{43})$	$T_0^{-1}T_3M_2^{-2}M_1M_2$
	$T_2^{-1}T_5$	$M_2^{-2}M_1M_2T_0^{-1}T_4(p_{43})$	$M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_3$	$M_2^{-1}M_1M_2^2(p_{43})$	$T_1^{-1}T_2T_4M_2^{-2}M_1M_2$
	$T_0^{-1}T_1$	$T_1M_2M_1M_2(p_{42})$	$T_4^{-1}M_2^{-2}M_1M_2^{-2}$
	$T_0^{-1}T_1T_3^2$	$M_2M_1M_2^{-2}T_0^{-1}T_3(p_{43})$	$T_3^{-1}T_4M_2^2M_1M_2^{-1}$
	$T_0^{-1}T_1^2T_3$	$T_1M_2^2M_1M_2^{-1}T_0^{-1}T_3(p_{43})$	$T_0^{-1}T_4M_2M_1M_2^{-2}$
	$T_1T_3T_4^{-1}$	$T_2M_2M_1M_2(p_{42})$	$T_0^{-1}T_4T_5^{-1}M_2M_1M_2$
	$T_0^2T_4^{-1}T_5$	$T_2M_2^{-1}M_1M_2^2T_0^{-1}T_4(p_{43})$	$T_1T_2^{-1}T_3M_2M_1M_2^{-2}$
$M_2^5M_1M_2^3$	$T_0^{-1}T_1$	$T_0^2T_3^{-1}(p_0)$	$T_1T_3T_4^{-1}M_2^{-1}M_1M_2^2$
	$T_0^{-1}T_1T_3$	$T_0^3T_4^{-1}T_5(p_0)$	$T_0T_1^{-1}M_2^{-1}M_1M_2^2$
$M_2^5M_1M_2^4$	$T_0^{-1}T_1T_3$	$T_0^2T_4^{-1}T_5(p_{32})$	$T_0T_3^{-1}M_2^3M_1$
	T_3^{-1}	$T_0T_3^{-1}(p_{32})$	$T_1^{-1}T_2T_4M_2^3M_1$
$M_2^5M_1M_2^5$	$T_0T_1^{-1}T_3^{-1}$	$T_0T_3^{-1}(p_{41})$	$T_0T_2^{-1}T_3^{-1}M_2M_1M_2^{-2}$
	$T_0^{-1}T_1T_3$	$T_0^2T_4^{-1}T_5(p_{41})$	$T_1T_3M_2M_1M_2^{-2}$

Remark 4.5. *One can see that* $A_{31} = (T_0I_0)^{-2}(M_1M_2^3)^2$, $A_{32} = (T_0I_0)^2(M_1M_2^3)^2$, *and* $A_{41} = (T_0I_0)^3$ *from Table* 1. *From Tables* 7 *to* 8, *one can obtain that* $A_{42} = I_0T_5I_0$ *and* $A_{43} = T_0I_0T_6I_0$, *here* $T_5 = T_0^{-2}T_1T_2T_3T_4$, $T_6 = T_0^2T_3^{-1}T_4^{-1}$.

Remark 4.6. *In fact,* Γ *has a presentation with 15 generators and 591 relations. One can simplify the presentation in GAP(via the command* `IsomorphismSimplifiedFpGroup`*) to get a new presentation with 4 generators and 511 relations. The details of this presentation are available there* [15].

From the presentation of Γ, we get the following useful information by using GAP.

Corollary 4.7. *The abelianization* Γ' *of* Γ *is* $\mathbb{Z}/2\mathbb{Z}$.

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