Fourier Coefficients of Vector-valued Modular Forms of Dimension 2

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Abstract. We prove the following theorem. Suppose that $F = (f_1, f_2)$ is a 2-dimensional vector-valued modular form on $\mathrm{SL}_2(\mathbb{Z})$ whose component functions f_1, f_2 have *rational* Fourier coefficients with *bounded denominators*. Then f_1 and f_2 are classical modular forms on a congruence subgroup of the modular group.

1   Introduction

Let $\Gamma = \mathrm{SL}_2(\mathbb{Z})$ with $\rho\colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ a representation of Γ. For the purposes of the present paper, a *vector-valued modular form of integral weight k associated with ρ* is a column vector of functions $F(\tau) = {}^t\big(f_1(\tau), \ldots, f_n(\tau) \big)$ holomorphic in the upper half-plane and satisfying

$$F|_k\gamma(\tau) = \rho(\gamma)F(\tau), \quad \gamma \in \mathrm{SL}_2(\mathbb{Z}).$$

Moreover, each component function is assumed to have a left-finite q-expansion

$$f_i(\tau) = q^{m_i} \sum_{i=0}^{\infty} a_{ni}q^n,$$

where, as usual, $q = e^{2\pi i \tau}$ and τ is the coordinate on the upper half-plane. We are concerned here with vector-valued modular forms with the property that all of the Fourier coefficients a_{ni} are *rational numbers*. In this case it is known [1] that the exponents m_i are also rational.

Suppose that $a(\tau) = q^h \sum_n a_n q^n$ is a q-expansion with coefficients $a_n \in \mathbb{Q}$. We say that $a(\tau)$ has *bounded denominators* if there is an integer N such that $Na_n \in \mathbb{Z}$ for all n. Otherwise, $a(\tau)$ has *unbounded denominators*. If $F(\tau)$ is a vector-valued modular form whose components $f_i(\tau)$ have rational Fourier coefficients, we say that $F(\tau)$ has bounded (respectively unbounded) denominators if *each f_i* (respectively, *some f_i*) has bounded (respectively unbounded) denominators. The second author has conjectured (see [7] for the case of two-dimensional ρ) that the following are *equivalent*:

(i) $F(\tau)$ has rational Fourier coefficients with bounded denominators;
(ii)   each $f_i(\tau)$ is a modular form on a congruence subgroup of Γ.

The main result of the present paper is a proof of the conjecture for 2-dimensional representations ρ.

The theory of 2-dimensional vector-valued modular forms was developed in [6] and [7], and in particular the conjecture was proved in [7] for all but finitely many 2-dimensional ρ. We recall some of the ideas (under the assumption that ρ is irreducible) since they will play a rôle in the present paper. There is a unique normalized nonzero holomorphic vector-valued modular form F_0 of *least integral weight* k_0, and the components of F_0 constitute a fundamental system of solutions of the linear differential equation

$$(1.1) \qquad (D_{k_0+2} \circ D_{k_0} - k_1 E_4)f = 0.$$

Here and below, we use the following notation:[1] for an even integer $k \geq 2$, E_k is the usual weight k Eisenstein series with q-expansion

$$E_k(q) = 1 + \frac{2}{\zeta(1-k)} \sum_{n \geq 1} \sigma_{k-1}(n)q^n,$$

and for an integer k we have the weight 2 operator

$$D_k = q\frac{d}{dq} - \frac{k}{12}E_2.$$

Written in terms of q, (1.1) has a regular singular point at $q = 0$, the indicial roots are the exponents m_1, m_2, and

$$(1.2) \qquad k_0 = 6(m_1 + m_2) - 1 \in \mathbb{Z}, \quad k_1 = \frac{36(m_1 - m_2)^2 - 1}{144}.$$

The recursive formula for the Fourier coefficients a_n of f_1 shows that they are rational when f_1 is suitably normalized. Moreover, it is shown in [7] that for almost all ρ, there is a prime p (depending on ρ) such that the p-adic valuation of a_n is *strictly decreasing* for $n \to \infty$. Thus f_1 has unbounded denominators for such ρ. In the remaining exceptional cases (approximately 300 isomorphism classes of ρ) this method will fail because, as numerical computations show, there is no such prime p. Thus another device is needed to achieve unbounded denominators in these cases. (The exceptions include 54 classes of *modular* ρ for which the components are modular forms on a congruence subgroup, and for these cases one of course has bounded denominators.) Further techniques are used to show that if F_0 has unbounded denominators then every nonzero F with rational Fourier coefficients has the same property.

In the present paper we will show that for all choices of ρ, exceptional or not, the component functions f_1, f_2 can be described using Gauss's hypergeometric function $F(a, b; c; j^{-1})$ evaluated at the inverse of the absolute modular invariant j (*cf.* Proposition 2.2 below). This will allow us to show (in the nonmodular cases) that *infinitely many primes* occur in the denominators of the Fourier coefficients, and in particular that denominators are unbounded. In fact, more precise arithmetic information is available in this situation, as we will explain in due course.

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[1] The normalization of E_2 used here differs from that in [6], [7].

Hypergeometric series appear in the work of Bantay and Gannon [2] on vector-valued modular forms and the 'fundamental matrix'. In hindsight we observed that the technique of this paper is strongly suggested by the work of Bantay and Gannon, but we in fact drew our inspiration from an earlier paper of Kaneko and Zagier [4] on supersingular j-invariants. Kaneko and Zagier considered a special case of (1.1) for which one of the solutions is a modular form (the corresponding ρ is *indecomposable*), and they use a change of local variable to re-express the modular form in terms of hypergeometric series. Other papers, for example [3] and [8], have also studied connections between modular linear differential equations and hypergeometric differential equations.

We will prove the following results.

Theorem 1.1   *Let m_1 and m_2 be rational numbers such that $m_1 - m_2 = P/Q$, $\gcd(P, Q) = 1$, and $Q \geq 2$, and let k_0, k_1 be as in (1.2). Then (1.1) has two linearly independent solutions f_1, f_2 with rational q-expansions, and exactly one of the following is true:*

(i)   *at least one of f_1 or f_2 has unbounded denominators;*
(ii) $Q \leq 5$.

Theorem 1.2   *Let F denote any 2-dimensional vector valued modular form whose components have rational Fourier coefficients. Then exactly one of the following is true:*

(i)   *at least one of the components of F has unbounded denominators;*
(ii)   *both components of F are modular forms on a congruence subgroup.*

Remark 1.3   As we have explained, the components of the vector-valued modular form F_0 satisfy the assumptions (hence also the conclusions) of Theorem 1.1. However, most choices of m_1 and m_2 do *not* correspond to any ρ. Theorem 1.1 teaches us that the origin of the unbounded denominator phenomenon is not so much the representation ρ, but rather the differential equation (1.1) to which it is associated.

We shall actually prove a much more precise result than is stated in Theorem 1.1. If $Q \geq 6$ we will show that at least one of the following statements holds:

(1.3) (a) $m_1 > m_2$ and for every prime p in the arithmetic progression $Qn + P$, one of the Fourier coefficients of f_1 has p-adic valuation -1;

   (b) $m_1 < m_2$ and for every prime p in the arithmetic progression $Qn - P$, one of the Fourier coefficients of f_2 has p-adic valuation -1.

2   A Modular Change of Variable

In this section, m_1, m_2, k_0, k_1, P, Q are assumed to satisfy the conditions stated in Theorem 1.1. Let $\eta(q)$ denote Dedekind's eta function

$$\eta(q) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Lemma 2.1   *One has $D_1(\eta^2) = 0$.*

Proof Let $\Delta = \eta^{24}$ and recall the well-known identity $q\frac{d\Delta}{dq} = E_2\Delta$. This is equivalent with $D_{12}(\Delta) = 0$. The identity $D_1(\eta^2) = 0$ follows from this by application of the Leibniz rule. ∎

Let f denote a solution of (1.1). In order to study the q-expansion of f we introduce the change of variable $\widetilde{f} := f\eta^{-2k_0}$. Lemma 2.1 shows that f is a solution of (1.1) if and only if $\widetilde{f}$ satisfies

$$(2.1) \qquad D_2 \circ D_0\widetilde{f} - k_1 E_4\widetilde{f} = 0.$$

We will show that this is a hypergeometric equation when expressed in terms of the local parameter j^{-1}. To begin, let $\theta = q\frac{d}{dq}$. The differential equation (2.1) is equivalent with the equation

$$(2.2) \qquad \theta^2(\widetilde{f}) - \frac{1}{6}E_2\theta(\widetilde{f}) - k_1 E_4\widetilde{f} = 0.$$

We will re-express equation (2.2) in terms of $J := j/1728$. Note that

$$\frac{j}{j - 1728} = \frac{E_4^3}{E_6^2},$$

$$\frac{d\widetilde{f}}{dj} = -\frac{E_4}{jE_6}\theta(\widetilde{f}),$$

$$\frac{d^2\widetilde{f}}{dj^2} = \left(\frac{E_4}{jE_6}\right)^2\left(\theta^2(\widetilde{f}) - \frac{E_2}{6}\theta(\widetilde{f})\right) - \left(\frac{7j - 4\cdot 1728}{6j(j - 1728)}\right)\frac{d\widetilde{f}}{dj}.$$

Then (2.2) becomes

$$j(j - 1728)\frac{d^2\widetilde{f}}{dj^2} + \frac{7j - 4.1728}{6}\frac{d\widetilde{f}}{dj} - k_1\widetilde{f} = 0,$$

which is equivalent to the Gauss normal form

$$(2.3) \qquad J(1 - J)\frac{d^2\widetilde{f}}{dJ^2} + \left(\frac{4 - 7J}{6}\right)\frac{d\widetilde{f}}{dJ} + k_1\widetilde{f} = 0.$$

(Here, and below, we write $J = j/1728$.) The general Gauss normal form is expressed in terms of parameters a, b and c as

$$J(1 - J)\frac{d^2\widetilde{f}}{dJ^2} + \left(c - (a + b + 1)J\right)\frac{d\widetilde{f}}{dJ} - ab\widetilde{f} = 0.$$

This corresponds to (2.3) when

$$(2.4) \qquad a = \frac{1}{12} + \left(\frac{m_1 - m_2}{2}\right), \quad b = \frac{1}{12} - \left(\frac{m_1 - m_2}{2}\right), \quad c = \frac{2}{3}.$$

Observe that $a - b = m_1 - m_2$ is *not* an integer (because $Q \geq 2$). Thus (2.3) has two independent solutions at $J = \infty$ given by

$$J^{-a}F(a, 1 + a - c; 1 + a - b; J^{-1}), \quad J^{-b}F(b, 1 + b - c; 1 + b - a; J^{-1}),$$

where $F(a, b; c; z)$ is Gauss's hypergeometric function

$$(2.5) \qquad F(a, b; c; z) := 1 + \sum_{n \geq 1}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n.$$

This proves the following.

Proposition 2.2 *Assume that m_1, m_2, k_0, k_1, P, Q are as in the statement of Theorem 1.1. Then (1.1) has two linearly independent solutions given by the series*

$$f_1 = \eta^{2k_0} j^{-a} F(a, 1 + a - c; 1 + a - b; J^{-1}),$$
$$f_2 = \eta^{2k_0} j^{-b} F(b, 1 + b - c; 1 + b - a; J^{-1}),$$

where a, b, c are as in (2.4). ∎

Remark 2.3 The lowest terms in the q-expansions of f_1 and f_2 have exponents m_2 and m_1 respectively. Note that we have rescaled the hypergeometric series by j^{-a} and j^{-b}, rather than J^{-a} and J^{-b}, simply to avoid introducing a fractional power of 12. This choice ensures that the Fourier coefficients of f_1 and f_2 are rational.

Let us consider the first of these series, in particular the coefficients given by the corresponding Pochhammer symbols occurring in (2.5). For $n \geq 1$, the coefficients in question are

$$C_n := \frac{(a)_n (1 + a - c)_n}{(1 + a - b)_n (1)_n} = (144Q)^{-n} \prod_{k=0}^{n-1} \frac{(12Qk + Q + 6P)(12Qk + 5Q + 6P)}{(Qk + Q + P)(k + 1)}.$$

Suppose that $p = Qn + P$ is a *prime* that divides $(12Qk + Q + 6P)(12Qk + 5Q + 6P)$ for some k in the range $0 \leq k \leq n - 1$ ($n \geq 1$). If p divides the first factor, then it divides $12Qk + Q + 6P - 6(Qn + P) = Q(12k + 1 - 6n)$, so that

$$Qn + P | 12k + 1 - 6n \leq 6n - 11 \Rightarrow n(Q - 6) \leq -(P + 11).$$

Similarly, if p divides the second factor then

$$Qn + P | 12k + 5 - 6n \leq 6n - 7 \Rightarrow n(Q - 6) \leq -(P + 7).$$

In particular, if P is positive (*i.e.*, $m_1 > m_2$) and $Q \geq 6$ then neither of these conditions can hold, so p cannot divide the numerator of C_n. It is then evident that the p-adic valuation of C_n is exactly -1. For the second hypergeometric series we consider the coefficients

$$C_n' := \frac{(a)_n (1 + b - c)_n}{(1 + b - a)_n (1)_n} = (144Q)^{-n} \prod_{k=0}^{n-1} \frac{(12Qk + Q - 6P)(12Qk + 5Q - 6P)}{(Qk + Q - P)(k + 1)}.$$

We easily find results similar to those obtained in the first case, but now for primes $Qn - P$ and $P < 0$. Combining these results yields a proof of the following lemma.

Lemma 2.4 *Suppose that $Q \geq 6$. Then one of the following holds:*

(i) *$m_1 > m_2$ and every prime $p = Qn + P$ is such that the p-adic valuation of C_n is -1;*

(ii) *$m_2 > m_1$ and every prime $p = Qn - P$ is such that the p-adic valuation of C_n' is -1.* ∎

Proposition 2.5 *Let m_1, m_2, k_0, k_1, P, Q be as in the statement of Theorem 1.1, and assume further that $Q \geq 6$. Let f_1, f_2 be the two q-expansions in Proposition 2.2. Then one of the following holds.*

(i) $m_1 > m_2$ *and for every prime p in the arithmetic progression $Qn + P$ there is at least one Fourier coefficient of f_1 that has p-adic valuation -1;*

(ii) $m_2 > m_1$ *and for every prime p in the arithmetic progression $Qn - P$ there is at least one Fourier coefficient of f_2 that has p-adic valuation -1.*

Proof   Suppose that $m_1 > m_2$, and fix a prime $p = Qn + P$ ($n \geq 1$). We have

$$\eta^{-2k_0} j^a f_1 = 1 + \sum_{n \geq 1}(12)^{3n} C_n j^{-n} =: 1 + \sum_{n \geq 1} c_n q^n,$$

and by part (i) of Lemma 2.4 the p-adic valuation of C_n is -1. It follows that the p-adic valuation of c_n is -1 while that of c_m is *nonnegative* for $1 \leq m \leq n - 1$.

If all Fourier coefficients of f_1 have nonnegative p-adic valuation, the same is true of $\eta^{-2k_0} j^a f_1$. This is because the η-power has integral coefficients, while the only primes occurring in the denominators of coefficients of $j^a = 1728^a J^a$ divide $12Q$ (*cf.* (2.4)), which is coprime to p. This contradicts the previous paragraph, and thus shows that some Fourier coefficient of f_1 has negative p-adic valuation. A similar argument shows that the first such coefficient has p-adic valuation exactly -1, because that is true of the coefficients c_m. This completes the proof of the proposition in case $m_1 > m_2$. The proof in the case $m_2 > m_1$ is completely parallel.   ■

Notice that both (1.3) and Theorem 1.1 are consequences of Proposition 2.5.

3   Vector-valued Modular Forms

This section is devoted to the proof of Theorem 1.2. We first develop some general results concerning vector-valued modular forms whose components have rational Fourier coefficients. We use the following additional notation:

– $\mathfrak{M}$ is the algebra of (classical) holomorphic modular forms on Γ.
– $\mathfrak{M}_{\mathbb{Q}}$ is the $\mathbb{Q}$-algebra of holomorphic modular forms with Fourier coefficients in $\mathbb{Q}$.
– $\mathfrak{p} = \mathfrak{M}_{\mathbb{Q}}\Delta$ is the principal ideal of $\mathfrak{M}_{\mathbb{Q}}$ generated by the discriminant Δ.
– $\rho \colon \Gamma \to \mathrm{GL}_n(\mathbb{C})$ is an n-dimensional representation of Γ such that $\rho(T)$ is (similar to) a unitary matrix.
– $\mathcal{H}(\rho)$ is the $\mathbb{Z}$-graded space of holomorphic vector-valued modular forms associated to ρ; it is a free $\mathfrak{M}$-module of rank n [5].
– $\mathcal{H}(\rho)_{\mathbb{Q}}$ is the space of vector-valued modular forms in $\mathcal{H}(\rho)$, all of whose component functions have Fourier coefficients in $\mathbb{Q}$; it is a module over $\mathfrak{M}_{\mathbb{Q}}$.
– $\mathfrak{R}$ is the (noncommutative) polynomial ring $\mathfrak{M}[d]$ such that $df - fd = D(f)$ ($f \in \mathfrak{M}$); $\mathcal{H}(\rho)$ is a left $\mathfrak{R}$-module where $f \in \mathfrak{M}$ acts as multiplication by f and d acts as D. Similarly, $\mathfrak{R}_{\mathbb{Q}} = \mathfrak{M}_{\mathbb{Q}}[d]$ and $\mathcal{H}(\rho)_{\mathbb{Q}}$ is a left $\mathfrak{R}_{\mathbb{Q}}$-module [6].

The next result is technical, but very useful.

Proposition 3.1   *Assume that ρ is irreducible. If $I \subseteq \mathcal{H}(\rho)_{\mathbb{Q}}$ is a nonzero $\mathfrak{R}_{\mathbb{Q}}$-submodule, then there is an integer r such that $\mathfrak{p}^r \mathcal{H}(\rho)_{\mathbb{Q}} \subseteq I$. In other words, $\mathfrak{p}^r$ annihilates $\mathcal{H}(\rho)_{\mathbb{Q}}/I$.*

Proof Let $A = \text{Ann}_{\mathfrak{M}_{\mathbb{Q}}}\big(\mathcal{H}(\rho)_{\mathbb{Q}}/I\big)$. It is an ideal in $\mathfrak{M}_{\mathbb{Q}}$, and we have to show that $\mathfrak{p}^r \subseteq A$ for some r.

Choose any nonzero $F \in I$, and consider the vector-valued modular forms $D^j F$ ($0 \leq j \leq n - 1$). If they are linearly dependent over $\mathfrak{M}_{\mathbb{Q}}$, then the components of F satisfy a modular linear differential equation of order $\leq n - 1$, and hence are linearly dependent. Because ρ is irreducible this is not possible. It follows that the $D^j F$ span an $\mathfrak{M}_{\mathbb{Q}}$-submodule of $\mathcal{H}(\rho)_{\mathbb{Q}}$ of (maximal) rank n. So I is also a submodule of maximal rank n.

We claim that $\mathcal{H}(\rho)_{\mathbb{Q}}/I$ is a *torsion* $\mathfrak{M}_{\mathbb{Q}}$-module. If not, we can find $G \in \mathcal{H}(\rho)_{\mathbb{Q}}/I$ such that the annihilator of G in $\mathfrak{M}_{\mathbb{Q}}$ reduces to 0. Then the submodule generated by G is a rank 1 free module, call it J, and there is a short exact sequence

$$0 \to I \to K \to J \to 0$$

of $\mathfrak{M}_{\mathbb{Q}}$-modules. Because J is free the sequence splits and we obtain $K \cong I \oplus J$ which is free of rank $n + 1$. This is not possible because $\mathcal{H}(\rho)_{\mathbb{Q}}$ has rank n, whence no submodule has rank greater than n. This proves the claim.

Because $\mathcal{H}(\rho)_{\mathbb{Q}}/I$ is a torsion module and $\mathfrak{M}_{\mathbb{Q}}$ a domain, it follows that A is *nonzero*. It is also easy to see that A is a graded ideal of $\mathfrak{M}_{\mathbb{Q}}$. Furthermore, because I is a left $\mathfrak{R}_{\mathbb{Q}}$-submodule and D is a derivation, an easy calculation shows that d (aka D) leaves A invariant. Hence, A is a nonzero, graded, left $\mathfrak{R}_{\mathbb{Q}}$-submodule of $\mathfrak{M}_{\mathbb{Q}}$.

In Lemmas 2.6 and 2.7 of [5] it was proved that a nonzero graded $\mathfrak{R}$-submodule of $\mathfrak{M}$ contains $\mathfrak{M}\Delta^r$ for some r. A check of the proof shows that it still works if the base field $\mathbb{C}$ is replaced by $\mathbb{Q}$ and $\mathfrak{M}\Delta$ is replaced by $\mathfrak{p}$. So $\mathfrak{p}^r \subseteq A$ for some r, as required. This completes the proof of the proposition. ∎

Corollary 3.2 *Assume that ρ is irreducible. Then the following are equivalent:*

(i) $\mathcal{H}(\rho)_{\mathbb{Q}}$ *contains at least one nonzero vector-valued modular form with bounded denominators;*

(ii) *Every nonzero element in $\mathcal{H}(\rho)_{\mathbb{Q}}$ has bounded denominators.*

Proof The set of vector-valued modular forms in $\mathcal{H}(\rho)_{\mathbb{Q}}$ with bounded denominators is an $\mathfrak{R}_{\mathbb{Q}}$-submodule of $\mathcal{H}(\rho)_{\mathbb{Q}}$; call it I. If $I \neq 0$, then Proposition 3.1 applies. It tells us that $\Delta^r \mathcal{H}(\rho)_{\mathbb{Q}} \subseteq I$ for some r. Thus for any $F \in \mathcal{H}(\rho)_{\mathbb{Q}}$ we find that $\Delta^r F$ has bounded denominators, whence $F = \Delta^{-r}\Delta^r F$ does too. The corollary follows. ∎

We turn to the proof of Theorem 1.2 and suppose that $F = {}^t(g_1, g_2) \in \mathcal{H}(\rho)_{\mathbb{Q}}$ has weight k and that g_1 and g_2 are *not* both modular forms on a congruence subgroup of Γ. We have to show that F has unbounded denominators. If ρ is irreducible, it suffices by Corollary 3.2 to find *one* vector-valued modular form in $\mathcal{H}(\rho)_{\mathbb{Q}}$ with unbounded denominators. As explained in the introduction, $\mathcal{H}(\rho)$ has a unique (normalized) nonzero vector-valued modular form $F_0 = {}^t(f_1, f_2)$ of minimal weight, and F_0 has rational Fourier coefficients. Let $f_1 = q^{m_1} + \cdots$ and $f_2 = q^{m_2} + \cdots$[2] with notation as in Theorem 1.1; in particular, $m_1 - m_2 = P/Q$ with $\gcd(P, Q) = 1$,

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[2]The referee has pointed out that in order to ensure that the coefficients of q^{m_1} and q^{m_2} in the components of F_0 are both 1, one might need to replace ρ by an equivalent representation. This does not affect the argument, however.

$Q \geq 1$. These f_1 and f_2 form a fundamental system of solutions of the differential equation (1.1). If $Q = 1$, then $m_1 = m_2$ is an integer, and in this case $\rho(T)$ is a scalar (*cf.* [6]). Because ρ is irreducible this is not the case, so that $Q \geq 2$.

Now we can apply Theorem 1.1 to see that either $Q \leq 5$, or else one of f_1, f_2 has unbounded denominators. In the second case we are done. We show that $Q \leq 5$ leads to a contradiction. Indeed, in this case Proposition 3.2 of [7] tells us that $\ker \rho$ is a congruence subgroup of Γ. But then all components of all vector-valued modular forms in $\mathcal{H}(\rho)$ are modular forms on the same congruence subgroup, and this contradicts the existence of F. This completes the proof of Theorem 1.2 in the case that ρ is irreducible.

Now suppose that ρ is *not* irreducible. Then we may, and shall, assume that it is upper triangular:

$$\rho(\gamma) = \begin{pmatrix} \alpha(\gamma) & \beta(\gamma) \\ 0 & \delta(\gamma) \end{pmatrix} \quad \gamma \in \Gamma.$$

Note that in this situation, the analog of Corollary 3.2 is *false*. Indeed, $F = {}^t(f_1, 0) \in \mathcal{H}(\rho)_{\mathbb{Q}}$ has weight k and bounded denominators whenever f_1 is a modular form of weight k on Γ with character α and rational Fourier coefficients.

Let $I \subseteq \mathcal{H}(\rho)_{\mathbb{Q}}$ be the set of vector-valued modular forms with bounded denominators. It is an $\mathfrak{R}_{\mathbb{Q}}$-submodule, and by our preceding remarks it contains the space of functions $\mathfrak{M}' = \{{}^t(f_1, 0)\}$ described above. Suppose, by way of contradiction, that $F \in I$. Since $F \notin \mathfrak{M}'$ it follows that I has rank at least 2 considered as $\mathfrak{M}_{\mathbb{Q}}$-module. At this point, we can apply the proofs of Proposition 3.1 and Corollary 3.2 word-for-word to see that every nonzero vector-valued modular form in $\mathcal{H}(\rho)_{\mathbb{Q}}$ has bounded denominators.

We now apply results about indecomposable 2-dimensional ρ and their associated vector-valued modular forms obtained in [5, Section 4]. By Lemma 4.3 (*loc. cit.*), we always have $Q = 6$ in this case. Let $F_0 = {}^t(f_1, f_2) \in \mathcal{H}(\rho)_{\mathbb{Q}}$ be as before, *i.e.*, a nonzero vector-valued modular form of least weight k_0. As in [5] we distinguish two cases, according to whether $DF_0 = 0$ or not. If this does *not* hold,0 then f_1, f_2 are, once again, a fundamental system of solutions of the DE (1.1) and we can apply Theorem 1.1 immediately to conclude that F_0 has unbounded denominators, which is the desired contradiction in this case. Note that we may always choose $\rho(T)$ diagonal, in which case we have in this case [5, equation (27)]

$$(3.1) \qquad \rho(T) = \begin{pmatrix} e^{2\pi i m_1} & 0 \\ 0 & e^{2\pi i m_2} \end{pmatrix},$$
$$0 \leq m_2 < m_1 < 1, \quad m_1 - m_2 = 1/6 \text{ or } 5/6, \quad 12 m_i \in \mathbb{Z}.$$

Finally, suppose that $DF_0 = 0$. This holds for those indecomposable ρ' for which

$$\rho'(T) = \begin{pmatrix} e^{2\pi i m_1} & 0 \\ 0 & e^{2\pi i m_2} \end{pmatrix},$$
$$0 \leq m_1 < m_2 < 1, \quad m_2 - m_1 = 1/6 \text{ or } 5/6, \quad 12 m_i \in \mathbb{Z}.$$

In this case, consider the tensor product $\rho'' := \chi \otimes \rho'$, where $\chi \colon \Gamma \to \mathbb{C}^*$ is the character of Γ satisfying $\chi(T) = e^{-2\pi i m_2}$. Because the isomorphism classes of indecomposable 2-dimensional ρ are determined by $\rho(T)$ [5, Lemma 4.3], it follows that ρ'' is equivalent to that ρ in (3.1) for which $m_2 = 0$. Now we have already proved that $\mathcal{H}(\rho)_\mathbb{Q}$ contains some vector-valued modular form with unbounded denominator, so $\mathcal{H}(\rho'')_\mathbb{Q}$ also contains such a vector-valued modular form, say G. Then $\rho' = \chi^{-1} \otimes \rho''$, and if $m_2 = b/12$ ($b \in \mathbb{Z}$) then $\eta^{2b} G \in \mathcal{H}(\rho')_\mathbb{Q}$ also has unbounded denominators. This final contradiction completes the proof of Theorem 1.2 ∎

4 Final Remarks

Prior to the writing of this paper, the authors were quite mystified by the nature of the denominators of the Fourier coefficients of the coordinate functions of vector valued modular forms that are not themselves modular forms (that is, vector-valued modular forms corresonding to representations with infinite image). These denominators tend to be nearly squarefree and, with a few exceptional divisors, are divisble only by primes in at most two arithmetic progressions.[3] For example, when one takes $m_1 = 3/10$ and $m_2 = 2/10$, there is a unique irreducible representation

$$\rho \colon \operatorname{SL}_2(\mathbb{Z}) \to \operatorname{GL}_2(\mathbb{C})$$

such that

$$\rho \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{3\pi i/5} & 0 \\ 0 & e^{2\pi i/5} \end{pmatrix}.$$

Let F_0 be a nonzero vector valued modular form for ρ of lowest weight. Then F_0 may be rescaled to have rational Fourier coefficients such that the 1000-th Fourier coefficient of the first component function of F_0 has denominator equal to $3^2 \cdot 13$ times the product of every prime in the arithmetic progression $10n + 9$ in the range 0 through $10,000$. The denominator of the 1001-st Fourier coefficient of this same q-expansion is 3 times the product of every prime in the arithmetic progression $10n + 9$ in the range 0 through $10,009$. For the 1002-nd coefficient, however, the denominator is 13 times the product of all primes in the progression $10n + 9$ in the same range, except that 919 is omitted. While the connection with hypergeometric series does help to explain the origin of the arithmetic progressions, it is still somewhat mysterious why the denominators of these series tend to be very nearly squarefree products of all primes in one or two arithmetic progressions. One might naively expect far more cancellation to occur than seems to be the case. A deeper study of the arithmetic properties of these coefficients might prove interesting.

The authors hope that the techniques of this paper will lead to progress in understanding vector-valued modular forms associated to higher dimensional representations of the modular group. Initial computations suggest that they too are connected with higher order analogues of the hypergeometric differential equation.

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[3]Chris Marks has made similar observations.

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