

ON THE COLOURINGS OF GRAPHS

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A graph G is defined by a set $V(G)$ of vertices, a set $E(G)$ of edges, and a relation of incidence which associates with each edge two distinct vertices called its ends. We consider only the case in which $V(G)$ and $E(G)$ are both finite.

An n -colouring of G is usually defined as a mapping f of $V(G)$ into the set of integers $\{1, 2, \dots, n\}$ which maps the two ends of any edge onto distinct integers. The integers 1 to n are the n "colours". Much work has been done on n -colourings in recent years, especially by G. A. Dirac.

Let us introduce a complication by assigning to each edge A a colour $g(A)$ and redefining an n -colouring of G as a mapping f of $V(G)$ into $\{1, 2, \dots, n\}$ such that the ends of an edge A are not both mapped into $g(A)$. The two ends of an edge can now receive the same colour, provided that it is not that of the edge. The old definition corresponds to the case in which any two vertices joined by an edge are joined by n edges of different colours.

Suppose a vertex v_0 is joined to vertices v_1, v_2, \dots, v_n by edges of colours 1, 2, \dots , n respectively. Then it is easily seen that there is no n -colouring of G in which v_1 has colour 1, v_2 has colour 2 and so on. It is found that from the original requirement that the ends of an edge A must not both have the colour $g(A)$ we can deduce a hierarchy of colour-propositions. The typical colour-proposition may be represented by a symbol $[S_1, S_2, \dots, S_n]$, where the S_i are subsets of $V(G)$. It asserts the falsity of the statement that G has an n -colouring in which every member of S_1 has colour 1, every member of S_2 has colour 2, and so on. All colour-propositions in which the S_i are not disjoint are true. We call them trivial.

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colour-proposition of rank $< n$. The process must of course terminate at some finite rank $\lambda(G)$. We denote the resulting hierarchy of basic colour-propositions by $H(G)$.

We are led to enquire whether every true and non-trivial colour-proposition is preceded by a member of $H(G)$. In what follows we answer this question in the affirmative. But first we make a generalization in order to simplify the proof.

We define a chromophore C as a finite set V of vertices for which certain non-trivial colour-propositions $[S_1, S_2, \dots, S_n]$ are asserted. An n -colouring of C is a mapping f of V into $\{1, 2, \dots, n\}$ which is consistent with all the asserted colour-propositions. Any one of the asserted colour-propositions which is not preceded by another is a basic colour-proposition of rank 1. The basic colour-propositions of ranks 2, 3, \dots , are defined successively as before. We thus obtain a hierarchy $H(C)$. We can say that a graph is a chromophore in which the colour-propositions of rank 1 have a special form.

THEOREM 1. Every true non-trivial colour-proposition of a chromophore C is preceded by a member of $H(C)$.

Proof. If possible let C be a chromophore for which the theorem fails and which has the least number of vertices consistent with this condition.

Let $Q = [Q_1, Q_2, \dots, Q_n]$ be any true non-trivial colour-proposition of C . Suppose first that

$$V = \bigcup_{i=1}^n Q_i.$$

Then Q is preceded by some basic colour-proposition of rank 1, for otherwise it would define a colouring of C in which each member of Q_i had colour i . This would contradict the definition of Q .

We may now suppose C has a vertex v belonging to no set Q_i . We define a chromophore C' on $V - \{v\}$. As its initially asserted colour-propositions we take all the basic colour-propositions $[P_1, P_2, \dots, P_n]$ of C such that v belongs to no P_i . We note that C' can have no basic colour-proposition of rank > 1 .

If Q is a colour-proposition of C' it is preceded by a basic colour-proposition of C' by the definition of C' , and therefore by a basic colour-proposition of C . We may therefore assume that there is a colouring f' of C' such that $f'(x) = i$ whenever $x \in Q_i$ ($i = 1, 2, \dots, n$). Let f_i denote that mapping of V into $\{1, 2, \dots, n\}$ which agrees with f' on $V - \{v\}$ and satisfies $f_i(v) = i$ ($i = 1, 2, \dots, n$).

Now f_i is not a colouring of C since Q is a colour-proposition of C . Hence there is a basic colour-proposition $P_i = [S_{i1}, S_{i2}, \dots, S_{in}]$ of C of rank 1 such that $f_i(x) = j$ for each $x \in S_{ij}$ ($j = 1, 2, \dots, n$). We must suppose $v \in S_{ii}$ since f' is a colouring of C' . The compound $R = \{P_1, P_2, \dots, P_n\}_v$ is non-trivial. It is therefore preceded by a basic colour-proposition $R' = [Z_1, Z_2, \dots, Z_n]$ of C of rank ≤ 2 . (Possibly $R' = R$.) But then the above definitions imply that R' is a basic colour-proposition of C' and that $f'(x) = i$ for each $x \in Z_i$ ($i = 1, 2, \dots, n$). This contradicts the definition of f' as a colouring of C' . The theorem follows.

Returning to the graph G we observe that it is permissible to assert of the null subset \emptyset of $V(G)$ that every member of \emptyset has the colour i ($i = 1, 2, \dots, n$). So the statement that G has no n -colouring is equivalent to the assertion that $[\emptyset, \emptyset, \dots, \emptyset]$ is a colour-proposition. Applying theorem 1 we obtain

THEOREM 2. G has no n -colouring if and only if $[\emptyset, \emptyset, \dots, \emptyset]$ is a member of the hierarchy $H(G)$.

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