When does $\frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$ give the centre of mass of a quadrilateral?

Having done the formula $\frac{1}{3}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c})$ for the centre of mass of a uniform triangular lamina with vertices at position vectors $\boldsymbol{a}$, $\boldsymbol{b}$, $\boldsymbol{c}$, I was recently asked in class whether $\frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$ was the corresponding result for a uniform quadrilateral lamina. It is easy to give examples where the formula does work (squares, rectangles, parallelograms), but equally clear from examples such as the trapezium in Figure 1, where the centre of mass is located below the centre line on which $\frac{1}{4}(\boldsymbol{a} + \boldsymbol{b} + \boldsymbol{c} + \boldsymbol{d})$ lies, that it does not always work.

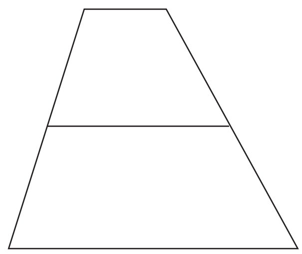

FIGURE 1

Here, we use a short vector argument to show that the formula only works for parallelograms. For the non-crossing quadrilateral shown in Figure 2(a), we can introduce non-parallel vectors $\boldsymbol{a}$, $\boldsymbol{b}$ so that its vertices have position vectors $\boldsymbol{0}$, $\boldsymbol{a}$, $\lambda\boldsymbol{a} + \mu\boldsymbol{b}$, $\boldsymbol{b}$ with $\lambda, \mu > 0$.

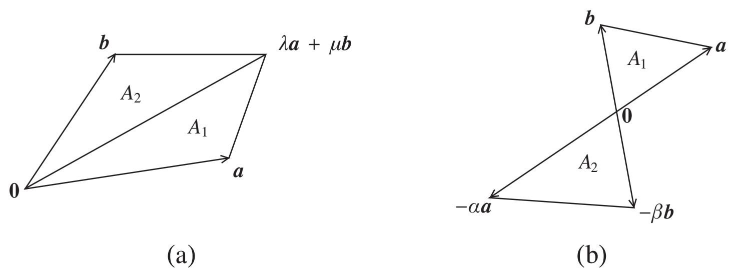

FIGURE 2: (a) Non-crossing quadrilateral; (b) Crossing quadrilateral

The quadrilateral is composed of two triangles with areas given by

$$A_1 = \tfrac{1}{2}\left|\boldsymbol{a} \times (\lambda\boldsymbol{a} + \mu\boldsymbol{b})\right| = \tfrac{1}{2}\mu\left|\boldsymbol{a} \times \boldsymbol{b}\right|$$

and

$$A_2 = \tfrac{1}{2}\left|(\lambda\boldsymbol{a} + \mu\boldsymbol{b}) \times \boldsymbol{b}\right| = \tfrac{1}{2}\lambda\left|\boldsymbol{a} \times \boldsymbol{b}\right|.$$

Let $\boldsymbol{g}_1$ be the centre of mass of the triangle with area A_1 and $\boldsymbol{g}_2$ that of the other triangle. Then the centre of mass of the quadrilateral is located at $\boldsymbol{g}$ where

$$(A_1 + A_2)\boldsymbol{g} = A_1\boldsymbol{g}_1 + A_2\boldsymbol{g}_2$$

https://doi.org/10.1017/mag.2023.30 Published online by Cambridge University Press

so that

$$\boldsymbol{g} = \frac{A_1}{A_1 + A_2} \cdot \frac{1}{3}\left[(1 + \lambda)\boldsymbol{a} + \mu\boldsymbol{b}\right] + \frac{A_1}{A_1 + A_2} \cdot \frac{1}{3}\left[\lambda\boldsymbol{a} + (1 + \mu)\boldsymbol{b}\right]$$

$$= \frac{1}{3(\lambda + \mu)}\left[\left(\mu + \lambda\mu + \lambda^2\right)\boldsymbol{a} + \left(\lambda + \lambda\mu + \mu^2\right)\boldsymbol{b}\right].$$

This coincides with the putative formula $\boldsymbol{g} = \frac{1}{4}\left[(1 + \lambda)\boldsymbol{a} + (1 + \mu)\boldsymbol{b}\right]$ if

$$\frac{1 + \lambda}{4} = \frac{1}{3(\lambda + \mu)}\left(\mu + \lambda\mu + \lambda^2\right) = \frac{\mu}{3(\lambda + \mu)} + \frac{\lambda}{3}$$

$$\text{and } \frac{1 + \mu}{4} = \frac{1}{3(\lambda + \mu)}\left(\lambda + \lambda\mu + \mu^2\right) = \frac{\lambda}{3(\lambda + \mu)} + \frac{\mu}{3},$$

from which (1) $3 = \lambda + \dfrac{4\mu}{\lambda + \mu}$ and (2) $3 = \mu + \dfrac{4\lambda}{\lambda + \mu}$.

Adding (1) and (2) gives $\lambda + \mu = 2$. Subtracting (2) from (1) gives $\left(\lambda - \mu\right)\left(\lambda + \mu - 4\right) = 0$, from which we deduce that $\lambda = \mu = 1$, so that the quadrilateral is indeed a parallelogram.

For a crossed quadrilateral and notation as in Figure 2(b) with $\alpha, \beta > 0$, a very similar argument shows that $\alpha = \beta = 1$, corresponding to a crossed parallelogram.

10.1017/mag.2023.30
Published by Cambridge University Press on behalf of The Mathematical Association

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