Canad. Math. Bull. Vol. **25** (4), 1982

# NATURAL COVERS AND $R$-QUOTIENT MAPS

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ABSTRACT. We extend the comprehensive treatment of $k$-spaces and sequential spaces provided by Franklin's refined notion of a natural cover to $k_R$-spaces and $s_R$-spaces. For this purpose, an apparently unstudied class of maps of topological spaces, the class of $R$-quotient maps, is introduced.

1. **Natural covers.** Our work is done in the category of topological spaces and continuous maps. We note that the greater part of the material of this section is a reproduction of parts of [6], to which the reader is referred for additional information. As in [6], a *natural cover* $\Sigma$ assigns to each topological space $X$ a cover $\Sigma_X$ of $X$ satisfying:

(i) if $S\in\Sigma_X$ and $S$ is homeomorphic to $T\subseteq Y$, then $T\in\Sigma_Y$,

(ii) if $f:X\to Y$ is continuous and $S\in\Sigma_X$, there is a $T\in\Sigma_Y$ with $f(S)\subseteq T$.

Given a topological space $(X, \tau)$ and a natural cover $\Sigma$, we define $(X, \tau)$ to be a:

(i) $\Sigma'$-*space*, if whenever $x\in\bar{E}$ in $X$, then $x\in \mathrm{Cl}_S(S\cap E)$, for some $S\in\Sigma_X$,

(ii) $\Sigma$-*space*, if whenever $F\cap S$ is closed in $S$ for each $S\in\Sigma_X$ (i.e., whenever $F$ is $\Sigma$-*closed* in $X$), then $F$ is closed in $X$,

(iii) $\Sigma_R$-*space*, if whenever $f$ is a real-valued function on $X$ whose restriction to each $S\in\Sigma_X$ is continuous (i.e., whenever $f$ is a $\Sigma$-*continuous* function on $X$), then $f$ is continuous on $X$.

In each of these cases, the topology on $X$ is in some way determined by the relative topologies on the elements of $\Sigma_X$. The closure operator is prescribed for $\Sigma'$-spaces, the closed sets are determined in $\Sigma$-spaces, and the $\Sigma_R$-spaces, at least if Tychonoff, have the weak topology generated by their $\Sigma$-continuous real-valued functions.

For several different natural covers $\Sigma$, the $\Sigma'$-spaces, $\Sigma$-spaces and $\Sigma_R$-spaces have received attention in the literature. We list some of these in tabular form in the next page.

Where a reference is supplied in the table, it serves to provide a reasonable source of further information about the class of spaces mentioned, as well as to indicate that the terminology is not our own (although it may not

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Received by the editors September 26, 1980 and in revised form February 20, 1981 and March 18, 1981.

1980 AMS Subject Classification: Primary: 54A10, 54C99 Secondary: 54D50, 54D55

| $\Sigma$ | countable sets | convergent sequences | compact spaces | countably compact spaces |
|---|---|---|---|---|
| $\Sigma'$-spaces | $c'$-spaces [11] | Frechet spaces [5] | $k'$-spaces [3], [4] | quasi-$k'$-spaces |
| $\Sigma$-spaces | $c$-spaces [7], [11] | sequential spaces [5] | $k$-spaces [3] | quasi $k$-spaces [12] |
| $\Sigma_R$-spaces | $c_R$-spaces | $s_R$-spaces [8], [10] | $k_R$-spaces [9], [10] | quasi $k_R$-spaces |

have originated with the author of the reference either). In this connection, we mention that Juhasz refers to our $c$-spaces as spaces of countable tightness in [7], while Schedler calls them cluster spaces in [11]. We should also mention that by a convergent sequence we mean the *range* of such a sequence together with its limit point.

Natural covers were used, in [6], to provide a common background for the many similar results about $k'$-spaces and $k$-spaces, on the one hand, and Frechet and sequential spaces on the other. Two important such results stated in the language of natural covers:

THEOREM. ([6], Propositions 2.10, 3.6). (i) *$X$ is a $\Sigma$-space if and only if the natural projection $\varphi$ from the disjoint union of the elements of $\Sigma_X$ to $X$ is a quotient mapping.*

(ii) *$X$ is a $\Sigma'$-space if and only if the natural projection $\varphi:\oplus\Sigma_X \to X$ is an hereditarily quotient mapping.*

THEOREM. ([6], Proposition 3.7). *If each subspace of $X$ is a $\Sigma$-space, then $X$ is a $\Sigma'$-space.*

Little of this sort has been done with $\Sigma_R$-spaces; it is the primary purpose of this paper to provide analogous results for $\Sigma_R$-spaces. As the table above indicates, $\Sigma_R$-spaces have been studied for some natural covers $\Sigma$. The most important result obtained is the deep theorem of Mazur ([8]), improved by Noble ([10]):

THEOREM. ([10], Theorem 5.1). (i) *Every weakly inaccesible*(1) *cardinal is non-sequential.*(2)

(ii) If *$X_\alpha$ is first countable and not indiscrete, for each $\alpha \in A$, then $\Pi X_\alpha$ is an $s_R$-space iff the cardinal of $A$ is non-sequential.*

(1) A cardinal $\aleph_\alpha$ is said to be *weakly inaccessible* iff $\alpha > 0$ is a limit ordinal and $\sum_{s\in S} m_s < \aleph_\alpha$ whenever $\|S\| < \aleph_\alpha$ and each $m_s < \aleph_\alpha$.

(2) A cardinal $\|A\|$ is said to be *non-sequential* iff there does not exist a non-zero real-valued sequentially continuous function $g: 2^A \to R$ which maps finite sets to zero.

2. ***R*-quotient maps.** A continuous map $f$ from $X$ onto $Y$ will be called an *$R$-quotient map* provided whenever $g: Y \to R$, $g$ is continuous if and only if $g \circ f$ is continuous. It follows immediately that $f$ is $R$-quotient if and only if $f$ is continuous and whenever $g: Y \to Z$ where $Z$ is a Tychonoff space, then $g$ is continuous if and only if $g \circ f$ is continuous. It is also obvious that every quotient map is an $R$-quotient map, and easily verified that composition of $R$-quotient maps yields an $R$-quotient map.

THEOREM 2.1. *Let $X$ be a topological space, $Y$ a set and $f$ a map from $X$ onto $Y$. Among the topologies on $Y$ making $f$ an $R$-quotient map, there is a finest $\sigma$ and a coarsest $\rho$.*

**Proof.** The quotient topology $\sigma$ on $Y$ induced by $f$ is clearly the finest topology on $Y$ with respect to which $f$ is an $R$-quotient map.

To construct the coarsest such topology $\rho$, let $g$ be the collection of all topologies on $Y$ which make $f$ an $R$-quotient map, and let $\rho = \bigcap g$. Then $\rho$ is a topology on $Y$ and clearly $f: X \to (Y, \rho)$ is continuous. Moreover, if $g: Y \to R$ and $g \circ f$ is continuous, then $g: (Y, \tau) \to R$ is continuous for each $\tau \in g$. It follows easily that $g: (Y, \rho) \to R$ is continuous. This establishes that $f: X \to (Y, \rho)$ is an $R$-quotient map.

COROLLARY. *If $f: X \to Y$ and $\rho$ and $\sigma$ are the topologies on $Y$ given above, then a topology $\tau$ on $Y$ makes $f$ an $R$-quotient map if and only if $\rho \subseteq \tau \subseteq \sigma$.*

The topology $\rho$ on $Y$ will be referred to as the *realquotient topology* on $Y$ induced by the map $f$, and when $Y$ carries this topology, $f$ will be called a *realquotient map.*

THEOREM 2.2. *The realquotient topology $\rho$ induced on $Y$ by $f: X \to Y$ can be characterized in the following ways:*

(i) *$\rho$ is the weak topology on $Y$ generated by the maps $g: Y \to R$ such that $g \circ f$ is continuous,*

(ii) *$\rho$ is the unique completely regular $R$-quotient topology on $Y$,*

(iii) *$\rho$ is the topology on $Y$ obtained by using as a base for the closed sets of $Y$ the zero sets of the quotient topology $\sigma$.*

**Proof.** (i) Let $\tau$ be the weak topology on $Y$ generated by the collection $\mathcal{F} = \{g: Y \to R \mid g \circ f \text{ is continuous}\}$. We claim that $\tau = \rho$, for which it suffices to show that $f: X \to (Y, \tau)$ is $R$-quotient and that if $\tau'$ is an $R$-quotient topology on $Y$, then $\tau \subseteq \tau'$.

To show that $f: X \to (Y, \tau)$ is $R$-quotient, we first establish continuity. If $U$ is a subbasic open set in $(Y, \tau)$, then $U = g^{-1}(V)$ for some $g \in \mathcal{F}$ and $V$ open in $R$. Then $f^{-1}(U) = (g \circ f)^{-1}(V)$ is open in $X$ since $g \circ f$ is continuous. Thus $f$ is continuous. Next, suppose $g: (Y, \tau) \to R$ has the property that $g \circ f$ is continu-

https://doi.org/10.4153/CMB-1982-065-1 Published online by Cambridge University Press

ous. Then $g \in \mathcal{F}$. so $g$ is continuous on $(Y, \tau)$. It follows that $f$ is an $R$-quotient map.

Suppose next that $\tau'$ is an $R$-quotient topology on $Y$. We will show that $\tau \subseteq \tau'$. If $U$ is a subbasic open set for the topology $\tau$, then $U = g^{-1}(V)$ for some open $V \subseteq R$ and map $g: Y \to R$ such that $g \circ f$ is continuous. But if $g \circ f$ is continuous, then $g: (Y, \tau') \to R$ is continuous, whence $U = g^{-1}(V)$ is open in $(Y, \tau')$. Thus $\tau \subseteq \tau'$.

(ii) Since, by (i), $\rho$ is the weak topology generated by a collection of real-valued maps, $\rho$ is completely regular. Suppose $\tau$ is a completely regular $R$-quotient topology on $Y$. Then $\tau$ is the weak topology generated by $\{g: Y \to R \mid g \text{ is } \tau\text{-continuous}\}$. But $g: Y \to R$ is $\tau$-continuous if and only if $g \circ f$ is continuous (since $\tau$ is an $R$-quotient topology) if and only if $g \in \mathcal{F}$, the generating collection for the topology $\rho$, by (i). Thus $\tau = \rho$.

(iii) Let $\tau$ be the topology generated by using the zero sets in $(Y, \sigma)$ as a base for the closed sets. Now since $\rho$ is completely regular, the zero sets in $(Y, \rho)$ form a base for the closed sets of $(Y, \rho)$ and these are still zero sets in $(Y, \sigma)$ and hence closed in $(Y, \tau)$. Thus $\rho \subseteq \tau$. Obviously $\tau \subseteq \sigma$, so $\tau$ is an $R$-quotient topology. But the continuous real-valued functions on $(Y, \rho)$ and $(Y, \tau)$ coincide, so the zero sets coincide, whence $\rho = \tau$.

COROLLARY. *If the quotient topology induced by the map $f$ on $Y$ is completely regular, it is the unique $R$-quotient topology on $Y$ induced by $f$.*

3. **$\Sigma_R$-spaces.** $R$-quotient maps can be used to characterize the $\Sigma_R$-spaces, in much the same way quotient and hereditarily quotient maps characterize the $\Sigma$-spaces and $\Sigma'$-spaces, respectively.

THEOREM 3.1. *If $X$ is a $\Sigma_R$-space and $f$ is an $R$-quotient map of $X$ onto $Y$, then $Y$ is a $\Sigma_R$-space.*

**Proof.** Let $g: Y \to R$ be a $\Sigma$-continuous map. If $S \in \Sigma_X$, find $T \in \Sigma_Y$ such that $f(S) \subseteq T$. Now $f$ is continuous on $S$ and $g$ is continuous on $T$, so $g \circ f$ is continuous on $S$. Thus $g \circ f$ is $\Sigma$-continuous. But then, $X$ being a $\Sigma_R$-space, $g \circ f$ is continuous and hence, $f$ being an $R$-quotient, $g$ is continuous. Thus $Y$ is a $\Sigma_R$-space.

THEOREM 3.2. *$X$ is a $\Sigma_R$-space if and only if the natural projection $\varphi$ of $\oplus \Sigma_X$ onto $X$ is an $R$-quotient map.*

**Proof.** If $\varphi$ is an $R$-quotient map, then since $\oplus \Sigma_X$ is clearly a $\Sigma_R$-space, $X$ is a $\Sigma_R$-space by 3.1.

Conversely, suppose $X$ is a $\Sigma_R$-space. The projection $\varphi$ is continuous in any case, so we need only show that it enjoys the composition property. But if $g: X \to R$ has the property that $g \circ \varphi$ is continuous, then easily $g$ is $\Sigma$-continuous on $X$, and hence continuous. Thus $\varphi$ is an $R$-quotient map.

https://doi.org/10.4153/CMB-1982-065-1 Published online by Cambridge University Press

COROLLARY. *With each $\Sigma_R$-space $X$ is associated a $\Sigma$-space $\Sigma X$ such that:*
(i) *$X$ is a continuous, one-one image of $\Sigma X$,*
(ii) *if $X$ is completely regular, $X$ is a $\Sigma$-space iff $\Sigma X$ is completely regular (in which case $X=\Sigma X$).*

**Proof.** Let $\tau$ be the topology on $X$, and $\varphi$ the natural projection from $\oplus\Sigma_X$ onto $X$. If $\sigma$ is the quotient topology induced on $X$ by $\varphi$, then $\tau\subseteq\sigma$ so $\Sigma X=(X, \sigma)$ is a $\Sigma$-space of which $(X, \tau)$ is a continuous one-one image. This proves (i).

To prove (ii), note first that if $X$ is a completely regular $\Sigma$-space, then the quotient topology induced on $X$ by the projection $\varphi$ is the unique $R$-quotient topology on $X$ and hence $X=\Sigma X$ (whence $\Sigma X$ is completely regular). On the other hand, if $\Sigma X$ is completely regular, then $\sigma$ is the unique $R$-quotient topology on $X$, so that $X=\Sigma X$, whence $X$ is a $\Sigma$-space.

Let us briefly illustrate the results above, using the natural cover by compact sets. Call two topologies $\tau$ and $\tau'$ on a fixed set $X$ *compact equivalent* iff $\tau$ and $\tau'$ determine the same compact subsets of $X$. Call $(X, \tau)$ *weakly minimal* $k_R$ iff $X$ is a $k_R$-space and no strictly weaker $T_2$ topology compact equivalent to $\tau$ makes $X$ a $k_R$-space.

The $k_R$ topologies on a fixed set $X$ which are compact equivalent to a given $k_R$ topology $\tau$ on $X$ form a lattice. The supremum of this lattice is the unique $k$-space topology on $X$ compact equivalent to $\tau$ (which makes $X$ into $\Sigma X$). The infimum of this lattice is the unique completely regular $k_R$-topology compact equivalent to $\tau$, i.e., the unique weakly minimal $k_R$-topology in the lattice (Theorems 2.2 and 3.2). Thus

THEOREM 3.3. *A $k_R$-space ($k$-space) is weakly minimal $k_R$ iff it is completely regular.*

4. **$\Sigma$-covering maps.** A continuous map $f$ of $X$ onto $Y$ will be called a *$\Sigma$-covering map* if for every $B\in\Sigma_Y$, there is some $\Lambda\in\Sigma_X$ such that $f(\Lambda)=B$. If $\Sigma$ is the natural cover by compact sets, the $\Sigma$-covering maps are the *compact-covering* maps (see [2] and [13]).

THEOREM 4.1. *If $\Sigma$ is a natural cover with the property that whenever $T\in\Sigma_T$ and $f$ is a continuous map on $T$, then $f$ is an $R$-quotient map, then a space $X$ is a $\Sigma_R$-space if and only if every $\Sigma$-covering map onto $X$ is an $R$-quotient map.*

**Proof.** Let $X$ be a $\Sigma_R$-space, $f$ a $\Sigma$-covering map of $S$ onto $X$. To show $f$ is an $R$-quotient map, let $g: X\to R$ be such that $g\circ f$ is continuous. It will suffice to show $g$ is $\Sigma$-continuous. Let $B\in\Sigma_X$. Find $A\in\Sigma_S$ such that $f(A)=B$. Then $f\mid A: A\to B$ is an $R$-quotient mapping so that, since $(g\mid B)\circ(f\mid A)$ is continuous, $g\mid B$ is continuous. Thus $g$ is $\Sigma$-continuous.

Conversely, suppose every $\Sigma$-covering map onto $X$ is an $R$-quotient map.

https://doi.org/10.4153/CMB-1982-065-1 Published online by Cambridge University Press

But the natural projection $\varphi:\oplus\Sigma_X\to X$ is $\Sigma$-covering, and hence an $R$-quotient map, whence $X$ is a $\Sigma_R$-space. The theorem above thus provides a characterization of both $k_R$-spaces and $s_R$-spaces, but fails to cover $c_R$-spaces, for example, since the natural cover by countable sets does not satisfy the required condition.

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