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# Division Algebras of Prime Degree and Maximal Galois $p$-Extensions

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*Abstract.* Let $p$ be an odd prime number, and let $F$ be a field of characteristic not $p$ and not containing the group $\mu_p$ of $p$-th roots of unity. We consider cyclic $p$-algebras over $F$ by descent from $L = F(\mu_p)$. We generalize a theorem of Albert by showing that if $\mu_{p^n} \subseteq L$, then a division algebra $D$ of degree $p^n$ over $F$ is a cyclic algebra if and only if there is $d \in D$ with $d^{p^n} \in F - F^p$. Let $F(p)$ be the maximal $p$-extension of $F$. We show that $F(p)$ has a noncyclic algebra of degree $p$ if and only if a certain eigencomponent of the $p$-torsion of $\mathrm{Br}(F(p)(\mu_p))$ is nontrivial. To get a better understanding of $F(p)$, we consider the valuations on $F(p)$ with residue characteristic not $p$, and determine what residue fields and value groups can occur. Our results support the conjecture that the $p$ torsion in $\mathrm{Br}(F(p))$ is always trivial.

## Introduction

Let $p$ be an odd prime number, and let $F$ be a field with $\mathrm{char}(F) \neq p$ such that $F$ does not contain the group $\mu_p$ of $p$-th roots of unity. Let $L = F(\mu_p)$. The Galois field extensions of $L$ of degree $p$ are nicely described by Kummer theory, and the cyclic central simple algebras over $L$ of degree $p$ also have a nice description as symbol algebras. Such descriptions are lacking for the corresponding objects over $F$ because of the absence of roots of unity. However, the objects over $F$ can be described by descent in terms of those over $L$, a descent that is particularly tractable because $[L:F]$ is prime to $p$. In the case of cyclic field extensions of $F$ of degree $p$, the description by descent was given by Albert and he used this in his characterization of cyclic algebras of prime degree. The approach by descent was also used by Merkurjev [M] to prove that ${}_p\mathrm{Br}(F)$, the $p$-torsion in the Brauer group of $F$, is generated by algebras of degree $p$. Merkurjev (and, less explicitly, Albert), used the eigendecomposition for the action of the Galois group $H = \mathcal{G}(L/F)$ acting on abelian groups related to $L$ of exponent a power of $p$.

Here we take a closer look at the algebras of degree $p^n$ over $F$ in terms of the corresponding algebras over $L$, with particular attention to the question of cyclicity of algebras of degree $p$. If $A$ is a division algebra of degree $p^n$ over $F$, then $A$ is determined by $A \otimes_F L$ which is a division algebra of degree $p^n$ over $L$. When $n = 1$ and $p \geq 5$, it is a major open question whether $A$ must be a cyclic algebra. But conceivably it might be "harder" for $A$ to be a cyclic algebra than for $A \otimes_F L$, since $L$ has more cyclic field extensions of $p$ power degree than $F$. In Proposition 1.3 we will give a criterion for cyclicity of $A$ for a class of algebras for which $A \otimes_F L$ is cyclic.

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However, this criterion is not algorithmic, and we do not have any examples where this criterion actually produces a noncyclic $A$. It turns out that when $L$ contains the group $\mu_{p^n}$ of $p^n$-th roots of unity, then arguments using $H$-eigendecompositions work just as well for algebras of degree $p^n$; this allows us to generalize to $p^n$, Albert's criterion for cyclicity of algebras of degree $p$, (see Theorem 1.5 below).

Again with the cyclicity question in mind, we can pass from $F$ to its maximal $p$-extension $F(p)$. Since $F(p)$ has no cyclic field extensions of degree $p$, it certainly can have no cyclic division algebras of degree $p$. The cyclicity question for $F(p)$ reduces to the question of whether the $p$-torsion in its Brauer group, ${}_p\mathrm{Br}(F(p))$ can be nontrivial. If we let $J = F(p)(\mu_p)$, then ${}_p\mathrm{Br}(F(p)) \cong ({}_p\mathrm{Br}(J))^{\mathcal{G}(J/F(p))}$, so the cyclicity question for $F(p)$ reduces to the question whether $({}_p\mathrm{Br}(J))^{\mathcal{G}(J/F(p))}$ must always be trivial.

It is difficult to get at the arithmetic of $J$, since it is generally such a large extension of $L$. But we can obtain some information by valuation theory. We analyze in Section 2 the valuations on $F(p)$ and $J$ arising from valuations on $F$ with residue characteristic prime to $p$. We obtain in Theorem 2.7 a nice description of the residue fields and the value groups of such valuations. This allows us to produce many examples in which ${}_p\mathrm{Br}(J)$ is nontrivial. However, we will show that cyclic $p$-algebras in $({}_p\mathrm{Br}(J))^{\mathcal{G}(J/F(p))}$ coming from eigencomponents of $J^*/J^{*p}$ are all unramified with respect to each such valuation. So, this valuation theory does not help in finding noncyclic algebras of degree $p$ over $F(p)$. One can view this result as supporting a conjecture that for any field $K$, if $K$ has no cyclic field extension of degree $p$, then ${}_p\mathrm{Br}(K) = 0$. This is of course formally weaker than the assertion that every algebra of degree $p$ is cyclic.

## 1 Cyclic Algebras of Degree $p^n$

We work here with cyclic algebras and symbol algebras. Our notation for these is as follows: If $T$ is a cyclic Galois field extension of a field $K$ of degree $m$ with $\mathcal{G}(T/K) = \langle\tau\rangle$ and $a \in K^*$, we write $(T/K, \tau, a)$ for the $m^2$-dimensional cyclic $K$-algebra $\bigoplus_{i=0}^{m-1} Tx^i$, where $x^m = a$ and $xcx^{-1} = \tau(c)$ for $c \in T$. Recall that any $m^2$-dimensional central simple $K$-algebra containing $T$ has such a description. If $\mu_m \subseteq K$ (so $\mathrm{char}(K) \nmid m$) and $\zeta \in \mu_m^*$ (*i.e.,* $\zeta$ is a primitive $m$-th root of unity) and $a, b \in K^* = K - \{0\}$, we write $(a, b; K)_\zeta$ for the $m^2$-dimensional symbol algebra over $K$ with generators $i, j$ and relations $i^m = a$, $j^m = b$, and $ij = \zeta ji$. For $a \in F^*$, we write $[a]$ for the image of $a$ in $F^*/F^{*m}$.

Assume now that $p$ is an odd prime and that $\mu_p \not\subseteq F$. Let $L = F(\mu_p)$. We will look at cyclic field extensions and cyclic algebras over $F$ from the perspective of $L$. Let $H = \mathcal{G}(L/F)$. Let $s = |H|$, so $s \mid (p-1)$. Then $H$ acts on $L^*$ and on the Brauer group $\mathrm{Br}(L)$. (See [D, p. 50] for a description of the action on $\mathrm{Br}(L)$.) Let ${}_{p^n}\mathrm{Br}(F)$ denote the $p^n$-torsion subgroup of $\mathrm{Br}(F)$.

***Lemma 1.1*** *$F^*/F^{*p^n} \cong (L^*/L^{*p^n})^H$ and ${}_{p^n}\mathrm{Br}(F) \cong ({}_{p^n}\mathrm{Br}(L))^H$. Furthermore, the second isomorphism preserves the Schur index.*

**Proof** We have the succession of maps $F^*/F^{*p^n} \to (L^*/L^{*p^n})^H \to F^*/F^{*p^n} \to (L^*/L^{*p^n})^H$, where the outer maps arise from the canonical inclusion of $F$ in $L$ and the middle one is induced by the norm from $L$ to $F$. The composition of the first and second maps is the $s$-th power map, as is the composition of the second and the third. Since $s$ is prime to $p^n$, the $s$-th power map is an isomorphism of these $p^n$ torsion groups. This yields the first isomorphism of the lemma. The second isomorphism follows in exactly the same way, using the restriction and corestriction maps, since the composition in either order starting from ${}_{p^n}\mathrm{Br}(F)$ or $({}_{p^n}\mathrm{Br}(L))^H$ is the $s$-th power map (*cf.* [D, pp. 53–54]). Any central division algebra $D$ over $F$ of exponent a power of $p$ also has Schur index a power of $p$, *cf.* [P, Proposition b(ii), p. 261]. Therefore, as $s = [L{:}F]$ is prime to $p$, $D \otimes_F L$ has the same index as $D$, by [P, Proposition (vi), p. 243]. ∎

Lemma 1.1 suggests an interesting possibility: There may be a central simple division algebra $A$ over $L$ of degree $p$ with $[A] \in \mathrm{Br}(L)^H$, such that $A$ is a cyclic algebra over $L$, but the inverse image of $A$ in ${}_p\mathrm{Br}(F)$ is not a cyclic algebra. This possibility becomes more plausible when we recall that the cyclic field extensions of $F$ of degree $p$ correspond only to a portion of those of $L$, see Proposition 1.2 below.

We can put this into sharper focus using the eigendecomposition of $p^n$-torsion $H$-modules, which we now recall. Let $B$ be any $H$-module such that $B$ is $p^n$ torsion as an abelian group (*e.g.*, $B = L^*/L^{*p^n}$ or $B = {}_{p^n}\mathrm{Br}(L)$). We write $B$ additively. Let $\chi\colon H \to (\mathbb{Z}/p^n\mathbb{Z})^*$ be any character, *i.e.*, group homomorphism. Let $B^{(\chi)} = \{b \in B \mid \tau \cdot b = \chi(\tau) \cdot b \text{ for all } \tau \in H\}$; we call $B^{(\chi)}$ the *$\chi$-eigenmodule* of $B$ for the action of $H$ on $B$. If $B^{(\chi)} = B$, we say that $H$ acts on $B$ via $\chi$. There are in all $s = |H|$ distinct characters $\chi_1, \dots, \chi_s\colon H \to (\mathbb{Z}/p^n\mathbb{Z})^*$. Let $\gamma_i = \chi_i(\sigma)$ for some fixed generator $\sigma$ of the cyclic group $H$. Then $\{\gamma_1, \dots, \gamma_s\}$ is the unique cyclic subgroup of order $s$ in $(\mathbb{Z}/p^n\mathbb{Z})^*$. Because $\gamma_i - \gamma_j \in (\mathbb{Z}/p^n\mathbb{Z})^*$ whenever $i \neq j$, the ideals $(x - \gamma_1), \dots, (x - \gamma_s)$ are comaximal in the polynomial ring $\mathbb{Z}/p^n\mathbb{Z}[x]$. Therefore, the Chinese remainder theorem shows that for the groupring $R = \mathbb{Z}/p^n\mathbb{Z}[H]$,

$$\begin{aligned} R &\cong \mathbb{Z}/p^n\mathbb{Z}[x]/(x^s - 1) \cong \mathbb{Z}/p^n\mathbb{Z}[x]/\big((x-\gamma_1)\dots(x-\gamma_s)\big) \\ &\cong \bigoplus_{i=1}^{s} \mathbb{Z}/p^n\mathbb{Z}[x]/(x-\gamma_i) \cong \bigoplus_{i=1}^{s} \mathbb{Z}/p^n\mathbb{Z}. \end{aligned}$$

If the first map is induced by sending $\sigma$ to $x$, and $e_i$ is the primitive idempotent of $R$ associated with the $i$-th summand in this direct decomposition, then we have $\sigma \cdot e_i = \gamma_i \cdot e_i$, so $H$ acts on $e_i R$ via $\chi_i$. This direct decomposition yields the canonical eigendecomposition of any $p^n$-torsion $H$-module $B$,

$$B = \bigoplus_{i=1}^{s} e_i B = \bigoplus_{i=1}^{s} B^{(\chi_i)}. \tag{1.1}$$

This decomposition for $n = 1$ was used implicitly by Albert [A1, A3], and explicitly by Merkurjev [M]. Observe that, in this language, Lemma 1.1 says that ${}_{p^n}\mathrm{Br}(F) \cong$

$({}_{p^n}\mathrm{Br}(L))^{(\chi_1)}$, where $\chi_1$ is the trivial character. When $\mu_{p^n} \subseteq L$, we have the *cyclotomic character* $\alpha\colon H \to (\mathbb{Z}/p^n\mathbb{Z})^*$ defined by

$$\tau(\omega) = \omega^{\alpha(\tau)} \text{ for all } \tau \in H \text{ and all } \omega \in \mu_{p^n}. \tag{1.2}$$

Let $\alpha^{-1}$ denote the inverse to $\alpha$ in the group of characters from $H$ to $(\mathbb{Z}/p^n\mathbb{Z})^*$. So, for all $\tau \in H$, we have $\alpha^{-1}(\tau)\alpha(\tau) = 1$ in $\mathbb{Z}/p^n\mathbb{Z}$.

If $\mu_{p^n} \subseteq L$, then a central simple $L$-algebra $A$ of degree $p^n$ which is a cyclic algebra is a symbol algebra, $A \cong (a, b; L)_\omega$, where $\omega \in \mu_{p^n}^*$. So, $\tau[(a, b; L)_\omega] = [(\tau(a), \tau(b); L)_{\tau(\omega)}] = [(\tau(a), \tau(b); L)_\omega]^{\alpha^{-1}(\tau)}$ in $\mathrm{Br}(L)$. Note the complication introduced because $\tau$ acts on $\omega$, as well as on $a$ and $b$. It follows that if $\chi, \psi$ are characters: $H \to (\mathbb{Z}/p^n\mathbb{Z})^*$, then

(1.3)

$$\text{if } [a] \in (L^*/L^{*p^n})^{(\chi)} \text{ and } [b] \in (L^*/L^{*p^n})^{(\psi)}, \text{ then } [(a, b; L)_\omega] \in {}_{p^n}\mathrm{Br}(L)^{(\chi\psi\alpha^{-1})}.$$

Albert gave a characterization of the cyclic field extensions of $F$ of degree $p$ in terms of those of $L$ [A1, Theorem 2], [A2, p. 211, Theorem 15]. We need the following generalization of this:

***Proposition 1.2*** *Suppose $\mu_{p^n} \subseteq L$. Take any $[c] \in L^*/L^{*p^n}$, and let $K = L(\sqrt[p^n]{c})$. Then there is a cyclic extension $E$ of $F$ with $[E:F] = [K:L]$ and $E \cdot L = K$ if and only if $H$ acts on $\langle [c] \rangle$ via the cyclotomic character $\alpha$.*

**Proof** Let $C$ be the cyclic subgroup $\langle [c] \rangle$ of $L^*/L^{*p^n}$. We have the nondegenerate Kummer pairing $B\colon \mathcal{G}(K/L) \times C \to \mu_{p^n}$ given by $(\tau, [d]) \mapsto \delta/\tau(\delta)$ for any $\delta \in K$ with $\delta^{p^n} = d$; it is easy to check that $B$ is $H$-equivariant. Kummer theory shows that $K$ is Galois over $F$ if and only if $H$ maps $C$ to itself. Assume this is the case. Then there is a cyclic extension $E$ of $F$ with $[E:F] = [K:L]$ and $E \cdot L = K$ if and only if $\mathcal{G}(K/F)$ is abelian, if and only if $H$ acts trivially by conjugation on $\mathcal{G}(K/L)$. Indeed, observe that if $H$ acts trivially on $\mathcal{G}(K/L)$, then $\mathcal{G}(K/F) \cong \mathcal{G}(K/L) \times H$. Since $H$ acts on $\mu_{p^n}$ via $\alpha$, it follows by using the nondegenerate $H$-equivariant Kummer pairing $B$ that $H$ acting trivially on $\mathcal{G}(K/L)$ is equivalent to $H$ acts on $C$ via $\alpha$. ■

***Proposition 1.3*** *Let $\chi\colon H \to (\mathbb{Z}/p\mathbb{Z})^*$ be a character. Take any $a, b \in L^*$ with $[a] \in (L^*/L^{*p})^{(\chi)}$ and $[b] \in (L^*/L^{*p})^{(\alpha\chi^{-1})}$, and let $A = (a, b; L)_\omega$. Then there is a central simple algebra $B$ of degree $p$ over $F$ with $B \otimes_F L \cong A$. Furthermore, $B$ is a cyclic algebra if and only if there exist $a', b' \in L^*$ with $[a'] \in (L^*/L^{*p})^{(\alpha)}$ and $[b'] \in (L^*/L^{*p})^H$ such that $A \cong (a', b'; L)_\omega$.*

**Proof** By (1.3) and Lemma 1.1,

$$[A] \in ({}_p\mathrm{Br}(L))^{(\alpha^{-1}\chi(\alpha\chi^{-1}))} = ({}_p\mathrm{Br}(L))^H = \mathrm{im}({}_p\mathrm{Br}(F)).$$

Because the scalar extension map ${}_p\mathrm{Br}(F) \to {}_p\mathrm{Br}(L)$ is index-preserving (see Lemma 1.1), there is a central simple $F$-algebra $B$ of degree $p$ with $B \otimes_F L \cong A$. Suppose

$B$ is a cyclic algebra, say $B \cong (S/F, \rho, b')$ where $S$ is a cyclic field extension of $F$ of degree $p$, $\mathcal{G}(S/F) = \langle\rho\rangle$, and $b' \in F^*$. Let $T = S \cdot L$ which is a cyclic field extension of $L$ of degree $p$, and let $\rho' \in \mathcal{G}(T/L)$ be the generator such that $\rho'|_S = \rho$. We have $T = L(\sqrt[p]{a'})$ for some $a' \in L^*$, and $a'$ can be chosen so that $\rho'(\sqrt[p]{a'}) = \omega^{-1}\sqrt[p]{a'}$. By Proposition 1.2, $[a'] \in (L^*/L^{*p})^{(\alpha)}$, while $[b'] \in F^*/F^{*p} \cong (L^*/L^{*p})^H$. Thus, we have $A \cong B \otimes_F L \cong (T/L, \rho', b') \cong (a', b'; L)_\omega$, as desired.

Conversely, suppose $A \cong (a', b'; L)_\omega$, as in the proposition. Since $[a'] \in (L^*/L^{*p})^{(\alpha)}$, we know by Proposition 1.2 that there is a cyclic field extension $S$ of $F$ of degree $p$, such that $S \cdot L = L(\sqrt[p]{a'})$. Let $\rho'$ be the generator of $\mathcal{G}(S \cdot L/L)$ such that $\rho'(\sqrt[p]{a'}) = \omega^{-1}\sqrt[p]{a'}$, and let $\rho = \rho'|_S$. Since $[b'] \in (L^*/L^{*p})^H \cong F^*/F^{*p}$ (see Lemma 1.1), there is $c \in F^*$ with $[c] = [b']$ in $L^*/L^{*p}$. Then $B \otimes_F L \cong A \cong (S/F, \rho, c) \otimes_F L$, so $B \cong (S/F, \rho, c)$ since the map $\operatorname{Br}(F) \to \operatorname{Br}(L)$ is injective by Lemma 1.1. ■

***Remark 1.4*** Proposition 1.3 suggests a potential way of obtaining a noncyclic algebra of degree $p$ over $F$, but we must necessarily start with a character $\chi\colon H \to (\mathbb{Z}/p\mathbb{Z})^*$ different from $\alpha$ and the trivial character $\chi_1$. We would need

$$[a] \in (L^*/L^{*p})^{(\chi)} \quad \text{and} \quad [b] \in (L^*/L^{*p})^{(\alpha\chi^{-1})}$$

such that $A = (a, b; L)_\omega$ is a division algebra, but $A$ is not expressible as $(a', b'; L)_\omega$ for any $[a'] \in (L^*/L^{*p})^{(\alpha)}$ and $[b'] \in (L^*/L^{*p})^H$. If $[L:F] \le 2$, then there are not enough different characters, and the proposition is of no help. In this connection, recall Merkurjev's result [M, Theorem 1, Lemma 2] that if $[L:F] \le 3$, then ${}_p\operatorname{Br}(F)$ is generated by cyclic algebras of degree $p$.

The approach in Proposition 1.3 leads to a generalization of Albert's characterization of cyclic algebras of prime degree. This theorem has recently been proved independently by U. Vishne [V, Theorem 11.4].

***Theorem 1.5*** *Suppose $p \nmid [F(\mu_{p^n}):F]$. Let $D$ be a division algebra of degree $p^n$ over $F$. Then $D$ is a cyclic algebra over $F$ if and only if there is a $\gamma \in D$ with $\gamma^{p^n} \in F^* - F^{*p}$.*

**Proof** Suppose first that $D$ is a cyclic algebra, say $D \cong (C/F, \sigma, b)$. Then there is $\gamma \in D$ with $\gamma^{p^n} = b$ and $\gamma c \gamma^{-1} = \sigma(c)$ for each $c \in C$. If $b \in F^{*p}$, say $b = d^p$, then for $\delta = \gamma^{p^{n-1}} d^{-1}$ we have $\delta^p = 1$, and $\delta \notin F$ as $\delta$ is not central. So, $1 < [F(\delta):F] < p$, contradicting $[F(\delta):F] \mid \dim_F(D)$. Hence, $b \in F^* - F^{*p}$.

Conversely, suppose there is $\gamma \in D$ with $\gamma^{p^n} \in F^* - F^{*p}$, say $\gamma^{p^n} = c$. Let $L = F(\mu_p)$. The assumption that $p \nmid [F(\mu_{p^n}):F]$ implies that $\mu_{p^n} \subseteq L$. Let $E = D \otimes_F L$. Since $E$ contains the cyclic Galois field extension $L(\gamma)$ of degree $p^n$ over $L$, this $E$ must be a cyclic $L$-algebra; hence, $E \cong (a, c; L)_\omega$ for some $a \in L^*$ and $\omega \in \mu_{p^n}^*$. Let $\chi_1, \dots, \chi_s$ be the distinct characters mapping $H = \mathcal{G}(L/F) \to (\mathbb{Z}/p^n\mathbb{Z})^*$, with $\chi_1$ the trivial character, and let $\alpha$ be the cyclotomic character (see (1.2)). We have the eigendecompositions

$$(1.4) \qquad L^*/L^{*p^n} \cong \prod_{i=1}^{s} (L^*/L^{*p^n})^{(\chi_i)} \quad \text{and} \quad {}_{p^n}\operatorname{Br}(L) \cong \bigoplus_{i=1}^{s} ({}_{p^n}\operatorname{Br}(L))^{(\chi_i)}$$

as in (1.1) above. Write $[a] = \prod_{i=1}^{s}[a_i]$ in $L^*/L^{*p^n}$, where $[a_i] \in (L^*/L^{*p^n})^{(\chi_i)}$. Then, in ${}_{p^n}\mathrm{Br}(L)$, we have $E \cong (a, c; L)_\omega \sim \bigotimes_{i=1}^{s}(a_i, c; L)_\omega$. Also, $[c] \in (L^*/L^{*p^n})^{(\chi_1)}$ as $c \in F^*$; so each $(a_i, c; L)_\omega \in {}_{p^n}Br(L)^{(\chi_i\alpha^{-1})}$ by (1.3). Thus, each $(a_i, c; L)_\omega$ lies in a different direct summand of ${}_{p^n}\mathrm{Br}(L)$ in the eigendecomposition of (1.4). Since $[E] \in {}_{p^n}\mathrm{Br}(L)^H = {}_{p^n}\mathrm{Br}(L)^{(\chi_1)}$, we must have $E \sim (a_j, c; L)_\omega$ in ${}_{p^n}\mathrm{Br}(L)$, where $\chi_j\alpha^{-1} = \chi_1$, *i.e.*, $\chi_j = \alpha$; dimension count shows that $E \cong (a_j, c; L)_\omega$. So, as a cyclic algebra $E \cong (L(\sqrt[p^n]{a_j})/L, \tau', c)$, where $\tau'$ is the generator of $\mathcal{G}(L(\sqrt[p^n]{a_j})/L)$ mapping $\sqrt[p^n]{a_j}$ to $\omega\sqrt[p^n]{a_j}$. But, since $[a_j] \in (L^*/L^{*p^n})^{(\alpha)}$, the field $L(\sqrt[p^n]{a_j}) = S \cdot L$ for some cyclic field extension $S$ of $F$ of degree $p^n$, by Proposition 1.2. Then, if $\tau$ denotes the restriction of $\tau'$ to $S$, we have $(S/F, \tau, c) \otimes_F L \cong (S \cdot L/L, \tau', c) \cong E$. Since the map ${}_{p^n}\mathrm{Br}(F) \to {}_{p^n}\mathrm{Br}(L)$ is injective by Lemma 1.1, we have $D \cong (S/F, \tau, c)$, as desired. ■

***Remark 1.6*** Albert's result is the $n = 1$ case of Theorem 1.5 (see [A1, Theorem 5], [A4, p. 177, Theorem 4]), for which the condition $p \nmid [F(\mu_p):F]$ always holds. Our proof of Theorem 1.5 is similar to Albert's, though Albert used different terminology, which somewhat veiled his use of eigendecompositions. The theorem is false without the assumption that $p \nmid [F(\mu_{p^n}):F]$. Albert gave a counterexample with $p^n = 4$ [A3], and there are presumably examples with odd $p$ also.

## 2 Valuations on the Maximal $p$-Extension

Let $p$ be any prime number. Let $F_{\mathrm{sep}}$ be some fixed separable closure of $F$. We set $F(p)$ to be the union of all finite degree Galois extensions $S$ of $F$ in $F_{\mathrm{sep}}$ with $[S:F]$ a power of $p$. The following proposition gives a convenient characterization of the finite degree field extensions of $F$ within $F(p)$. It follows easily using Galois theory and standard properties of $p$-groups, and is certainly well known, though we could not find a reference for it.

***Proposition 2.1*** *Let $S$ be a field of any characteristic, and let $T$ be a field, $T \supseteq S$, $[T:S] < \infty$. Then the following are equivalent.*

(i) *The normal closure of $T$ over $S$ is Galois over $S$ of degree a power of $p$*, i.e., $T \subseteq S(p)$.
(ii) *There is a chain of fields $S = S_0 \subseteq S_1 \subseteq S_2 \subseteq \cdots \subseteq S_k = T$ with each $S_i$ Galois over $S_{i-1}$, and $[S_i:S_{i-1}] = p$.*

Proposition 2.1 provides an easy way to see that for any field $M$ with $F \subseteq M \subseteq F(p)$, we have $M(p) = F(p)$. In particular, $F(p)$ has no proper Galois $p$-extensions.

From now on, let $p$ be an odd prime number, let $F$ be a field with $\mathrm{char}(F) \neq p$ and $\mu_p \not\subseteq F$. Let $L = F(\mu_p)$ and let $J = F(p)(\mu_p) = F(p) \cdot L$. So, $J \subseteq L(p)$, as $\mathcal{G}(J/L) \cong \mathcal{G}(F(p)/F)$, but in general $J$ is much smaller than $L(p)$. We will see below that $J$ typically has many degree $p$ cyclic field extensions, and we can have ${}_p\mathrm{Br}(J)$ nontrivial. We identify $\mathcal{G}(J/F(p))$ with $\mathcal{G}(L/F)$, and call this group $H$.

***Proposition 2.2*** *If $({}_p\mathrm{Br}(J))^H \neq 0$, then there exists a division algebra of degree $p$ over $F(p)$ which is not a cyclic algebra.*

**Proof** By Lemma 1.1, with $F(p)$ replacing $F$, if $({}_p\mathrm{Br}(J))^H \neq 0$, then ${}_p\mathrm{Br}(F(p)) \neq 0$. By a theorem of Merkurjev [M, Theorem 2], the group ${}_p\mathrm{Br}(F(p))$ is generated by algebras of degree $p$. No such algebra can be a cyclic algebra, since $F(p)$ has no cyclic field extensions of degree $p$, as noted above. ■

***Remark 2.3*** Take any field $K$ with $L \subseteq K \subseteq J$ and $[K{:}L] = p$. Then, since $\mathcal{G}(J/L) \cong \mathcal{G}(F(p)/F)$ via the restriction map, there is a field $S$ with $F \subsetneq S \subsetneq F(p)$, $[S{:}F] = p$, and $S \cdot L = K$. By Proposition 2.1, $S$ is Galois over $F$. So by Albert's theorem (see Proposition 1.2 above), $K = L(\sqrt[p]{c})$, for $c \in L^*$ with $[c] \in (L^*/L^{*p})^{(\alpha)}$, where $\alpha\colon H \to (\mathbb{Z}/p\mathbb{Z})^*$ is the cyclotomic character, as in (1.2) above (with $n = 1$). By Kummer theory, the map $L^*/L^{*p} \to K^*/K^{*p}$ has kernel $\langle [c] \rangle \subseteq (L^*/L^{*p})^{(\alpha)}$. Consequently, for any character $\chi\colon H \to (\mathbb{Z}/p\mathbb{Z})^*$ with $\chi$ different from $\alpha$, the map $(L^*/L^{*p})^{(\chi)} \to (K^*/K^{*p})^{(\chi)}$ is injective. It follows by iteration and passage to the direct limit that the map $(L^*/L^{*p})^{(\chi)} \to (J^*/J^{*p})^{(\chi)}$ is injective for each $\chi \neq \alpha$. Thus, $(J^*/J^{*p})^{(\chi)}$ can be nontrivial for each $\chi \neq \alpha$, though necessarily $(J^*/J^{*p})^{(\alpha)} = 1$ by Albert's theorem (see Proposition 1.2 above), as $F(p)$ has no Galois extensions of degree $p$.

It is a more difficult question when or whether division algebras of degree $p$ over $L$ can remain division algebras after scalar extension to $J$. We will use valuation theory to show that this can occur. We use the following notation. Suppose $K$ is a field and $W$ is a valuation ring of (*i.e.,* with quotient field) $K$. Let $M_W$ denote the maximal ideal of $W$, let $\overline{W} = W/M_W$, the residue field of $W$; and let $\Gamma_W$ denote the value group of $W$ (written additively). For a field $K' \supseteq K$, an extension of $W$ to $K'$ is a valuation ring $W'$ of $K'$ such that $W' \cap K = W$.

***Example 2.4*** Let $k$ be any field with $\mathrm{char}(k) \neq p$ and $\mu_p \not\subseteq k$. Let $F$ be the twice iterated Laurent power series field $F = k((x))((y))$. Then $F$ has the Henselian valuation ring $V = k[[x]] + yk((x))[[y]]$, where $k[[x]]$ denotes the formal power series ring in $x$ over $k$. Also, $\overline{V} \cong k$. If $v\colon F^* \to \Gamma_V$ is the associated valuation, then $\Gamma_V = \mathbb{Z} \times \mathbb{Z}$, with right-to-left lexicographical ordering, with $v(x) = (1, 0)$ and $v(y) = (0, 1)$. Then $F(p) = k(p)((x))((y))$ and $J = k(p)(\mu_p)((x))((y))$, while $L(p) = \bigcup_{i=1}^{\infty} k(\mu_p)(p)((x^{1/p^i}))((y^{1/p^i}))$. (The descriptions of $F(p)$ and $J$ follow from Theorem 2.7 below, but can be seen more directly using the fact that since $\mu_p \not\subseteq k = \overline{V}$, there is no Galois field extension of $F$ of degree a power of $p$ which is totally ramified with respect to $V$, *cf.* [E, pp. 161–162, (20.11)] or, more explicitly, [JW, Cor. 2.4]. So Proposition 2.1 shows that every finite degree extension of $F$ within $F(p)$ must be unramified over $F$, hence $F(p)$ is unramified over $F$.) The unique extension of $V$ to $J$ is $Z = k(p)(\mu_p)[[x]] + yk(p)(\mu_p)((x))[[y]]$ with $\Gamma_Z = \Gamma_V = \mathbb{Z} \times \mathbb{Z}$; let $z\colon J^* \to \Gamma_Z$ be the associated valuation. For any $\omega \in \mu_p^*$, let $D = (x, y; J)_\omega$ (see §1 for the notation). Because the images of $z(x)$ and $z(y)$ in $\Gamma_Z/p\Gamma_Z$ are $\mathbb{Z}/p\mathbb{Z}$-independent, we know by [JW, Cor. 2.6] that $D$ is a division ring and $z$ extends to a valuation on $D$. Thus, ${}_p\mathrm{Br}(J)$ is nontrivial. Since $x$ and $y$ are $H$-stable, it is tempting to think that $[D]$ should be $H$-stable. But, in fact, $[D]$ lies in ${}_p\mathrm{Br}(J)^{(\alpha^{-1})}$ but not in ${}_p\mathrm{Br}(J)^H$, because of the nontrivial action of $H$ on $\mu_p$.

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To try to get more information about ${}_p\mathrm{Br}(J)$, we look at the valuations on $J$ with residue field of characteristic prime to $p$. We will consider a valuation ring on $J$ as an extension of one on $F$. For this, we now fix a valuation ring $V$ of $F$ with $\mathrm{char}(\overline{V}) \neq p$. Let $W_1, \dots, W_\ell$ be all the different extensions of $V$ to $L$. Let $T = W_1 \cap \cdots \cap W_\ell$, which is the integral closure of $V$ in $L$. This notation will be fixed for the rest of this section. Recall [E, pp. 95–96, Theorem (13.4)] that the maximal ideals of $T$ are $N_1, \dots, N_\ell$, where $N_i = M_{W_i} \cap T$, and that each $W_i$ is the localization $T_{N_i}$ of $T$ at $N_i$.

***Proposition 2.5*** *Let $V$ be a valuation ring of $F$ with* $\mathrm{char}(\overline{V}) \neq p$. *Let $W_1, W_2, \dots, W_\ell$ be all $\ell$ distinct valuation rings of $L$ extending $V$. Then each $\overline{W_i} \cong \overline{V}(\mu_p)$ and $\Gamma_{W_i} = \Gamma_V$. Also, $\ell\,[\overline{V}(\mu_p):\overline{V}] = [L:F]$.*

**Proof** Let $\omega \in \mu_p^* \subseteq L$, and let $f \in F[x]$ be the monic minimal polynomial of $\omega$ over $F$. Then $f \in V[x]$, as $\omega$ is integral over $V$, which is integrally closed. Also, $f \mid \sum_{i=0}^{p-1} x^i$ in $F[x]$, and hence in $V[x]$ by the division algorithm, as $f$ is monic. So the image $\overline{f}$ of $f$ in $\overline{V}[x]$ divides $\sum_{i=0}^{p-1} x^i$ in $\overline{V}[x]$. This shows that the roots of $\overline{f}$ are all primitive $p$-th roots of unity, and $\overline{f}$ has no repeated roots. So if $\overline{f} = \prod_{i=1}^{k} g_i$ is the irreducible monic factorization of $\overline{f}$ in $\overline{V}[x]$, then the $g_i$ are distinct and $\deg(g_i) = [\overline{V}(\mu_p):\overline{V}]$. Since $fF[x] \cap V[x] = fV[x]$ by the division algorithm, we have $V[\omega] \cong V[x]/fV[x]$, so

$$V[\omega]/M_V V[\omega] \cong V[x]/(M_V, f) \cong \overline{V}[x]/(\overline{f}) \cong \bigoplus_{i=1}^{k} \overline{V}[x]/(g_i),$$

a direct sum of fields. The inverse images in $V[\omega]$ of the $k$ maximal ideals of $V[\omega]/M_V V[\omega]$ are maximal ideals $P_1, \dots, P_k$ of $V[\omega]$ such that each $P_i \cap V = M_V$ and $V[\omega]/P_i \cong \overline{V}[x]/(g_i) \cong \overline{V}(\mu_p)$. Because $T$ is integral over $V[\omega]$, for each $P_i$ there is a maximal ideal $N_i$ of $T$ with $N_i \cap V[\omega] = P_i$. Then, for $W_i = T_{N_i}$, we have $\overline{W_i} \cong T/N_i \supseteq V[\omega]/P_i \cong \overline{V}(\mu_p)$. By the fundamental inequality, [E, p. 128, Cor. (17.8)] or [B, Ch. VI, §8.3, Theorem 1], we have

$$\begin{aligned} (2.1) \qquad [L:F] &\geq \textstyle\sum_{i=1}^{k} [\overline{W_i}:\overline{V}]\,|\Gamma_{W_i}:\Gamma_V| \geq \sum_{i=1}^{k} [\overline{W_i}:\overline{V}] \\ &\geq \textstyle\sum_{i=1}^{k} [\overline{V}(\mu_p):\overline{V}] = \sum_{i=1}^{k} \deg(g_i) = \deg(f) = [L:F]. \end{aligned}$$

Hence, equality must hold throughout (2.1). Therefore, each $\overline{W_i} = \overline{V}(\mu_p)$ and $\Gamma_{W_i} = \Gamma_V$, and $k = [L:F] \big/ [\overline{V}(\mu_p):\overline{V}]$. Furthermore, (2.1) and the fundamental inequality show that $W_1, \dots, W_k$ are all the extensions of $V$ to $L$; so $\ell = k$. ∎

***Remark 2.6*** Let $S$ be any Galois extension field of $F$ of degree $p$, and let $U$ be any extension of $V$ to $S$. Then $[\overline{U}:\overline{V}] \mid [S:F] = p$, as $S/F$ is Galois. Consequently, $\overline{U}$ and $\overline{V}(\mu_p)$ are linearly disjoint over $\overline{V}$, and hence $[\overline{U}(\mu_p):\overline{U}] = [\overline{V}(\mu_p):\overline{V}]$. It follows by Proposition 2.5 applied to $U$ in $S$ in place of $V$ in $F$ that the number of extensions of $U$ to $S(\mu_p)$ is $\ell$. Since any field $S'$ with $F \subseteq S' \subseteq F(p)$ and $[S':F] < \infty$ is obtainable from $F$ by a tower of degree $p$ Galois extensions (see Proposition 2.1), it follows by iteration that every extension of $V$ to $S'$ has exactly $\ell$ extensions to $S'(\mu_p)$. Because

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this holds for every finite degree extension $S'$ of $F$ in $F(p)$, it clearly holds for every field $S''$ with $F \subseteq S'' \subseteq F(p)$.

The main result of this section describes the residue field and value group of any extension of $V$ to $J$. In case $\mu_p \subseteq \overline{V}$ (*i.e.*, $\ell = [L{:}F]$, by Proposition 2.5), this will require looking at two pieces of $\Gamma_V$. For this, let $P$ be the union of all prime ideals $\mathfrak{P}$ of $V$ such that $V/\mathfrak{P}$ contains no primitive $p$-th root of unity. Since the prime ideals of $V$ are linearly ordered by inclusion, it is clear that $P$ is a prime ideal of $V$ (possibly $P = (0)$), and $P$ is maximal with the property that $\mu_p \not\subseteq V/P$. (Note also that for every prime ideal $Q \subseteq P$, we have $\mu_p \not\subseteq V/Q$. For, if $\mu_p \subseteq V/Q$, then $\mu_p \subseteq V/P$, as $\operatorname{char}(V/P) \neq p$.) The localization $V_P$ of $V$ at $P$ is a valuation ring of $F$ (a "coarsening" of $V$); let $\widetilde{V} = V/P$, which is a valuation ring of $\overline{V_P}$. Recall [B, Ch. VI, §4.3, Remark] that there is a canonical short exact sequence of value groups:

$$0 \longrightarrow \Gamma_{\widetilde{V}} \longrightarrow \Gamma_V \longrightarrow \Gamma_{V_P} \longrightarrow 0. \tag{2.2}$$

***Theorem 2.7*** *Let $V$ be a valuation ring of $F$ with $\operatorname{char}(\overline{V}) \neq p$, and let $\ell$ be the number of extensions of $V$ to $L$. Let $Y$ be any extension of $V$ to $F(p)$. Then $\overline{Y} \cong \overline{V}(p)$. If $\mu_p \not\subseteq \overline{V}$, then $\Gamma_Y = \Gamma_V$. If $\mu_p \subseteq \overline{V}$, let $P$ be the prime ideal of $V$ maximal such that $\mu_p \not\subseteq V/P$, as above, and let $Q$ be the prime ideal of $Y$ with $Q \cap V = P$; let $\widetilde{Y} = Y/Q$. Then $\Gamma_{Y_Q} = \Gamma_{V_P}$, while $\Gamma_{\widetilde{Y}} = \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} \Gamma_{\widetilde{V}}$. Furthermore, $Y$ has exactly $\ell$ different extensions $Z_1, Z_2, \dots, Z_\ell$ to $J$, and each $\overline{Z_i} \cong \overline{Y}(\mu_p)$ and $\Gamma_{Z_i} = \Gamma_Y$.*

Note that in view of the exact sequence like (2.2) for $\Gamma_Y$, the theorem determines $\Gamma_Y$ completely. It says that when we view $\Gamma_Y$ as in the divisible hull $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma_V$ of $\Gamma_V$, then $\Gamma_Y$ is the subgroup generated by $\mathbb{Z}[1/p] \otimes_{\mathbb{Z}} \Gamma_{\widetilde{V}}$ (the $p$-divisible hull of $\Gamma_{\widetilde{V}}$) and $\Gamma_V$.

To prove the theorem we will analyze the range of possibilities for value groups and residue fields of extensions of $V$ to degree $p$ Galois field extensions of $F$. This will be done in terms of the corresponding extensions of $L$, where we can invoke Kummer theory. To facilitate the analysis, we need some information on the eigencomponents of induced modules, which is given in the next proposition.

Let $H = \langle \sigma \rangle$ be a cyclic group of finite order $s$, and let $\overline{H} = \langle \sigma^m \rangle$ for some $m \mid s$. Let $A$ be any $\overline{H}$-module, and let $B$ be the induced $H$-module, $B = \operatorname{Ind}_{\overline{H} \to H} A = \mathbb{Z}[H] \otimes_{\mathbb{Z}[\overline{H}]} A$. So, as abelian groups $B = \bigoplus_{i=0}^{m-1} \sigma^i \otimes A$, where each $\sigma^i \otimes A \cong A$. The left action of $H$ on $B$ arises from the multiplication action of $H$ on $\mathbb{Z}[H]$. That is,

$$\begin{aligned} \sigma \cdot \big( \operatorname{id} \otimes a_0 + \sigma \otimes a_1 + \cdots + \sigma^{m-1} \otimes a_{m-1} \big) &= \\ \operatorname{id} \otimes \sigma^m \cdot a_{m-1} + \sigma \otimes a_0 + \sigma^2 \otimes a_1 &+ \cdots + \sigma^{m-1} \otimes a_{m-2}\,. \end{aligned} \tag{2.3}$$

***Proposition 2.8*** *With $H = \langle \sigma \rangle$ and $\overline{H} = \langle \sigma^m \rangle$ as above, let $A$ be an $\overline{H}$-module which is $e$-torsion for some integer $e$. Let $B = \operatorname{Ind}_{\overline{H} \to H} A$, as above. Let $\chi\colon H \to (\mathbb{Z}/e\mathbb{Z})^*$ be any character. Then the projection map $\pi\colon B \to A$ given by $\sum_{i=0}^{m-1} \sigma^i \otimes a_i \mapsto a_0$ maps $B^{(\chi)}$ bijectively onto $A^{(\chi|_{\overline{H}})}$, where $\chi|_{\overline{H}}$ is the restriction of $\chi$ to $\overline{H}$.*

**Proof** Let $b = \sum_{i=0}^{m-1} \sigma^i \otimes a_i \in B$. Note that since $\sigma^m \in \overline{H}$, we have $\sigma^m \cdot b = \sum_{i=0}^{m-1} \sigma^i \otimes \sigma^m(a_i)$. Now, $b \in B^{(\chi)}$ if and only if $\sigma \cdot b = \chi(\sigma)\, b$, if and only if

(2.4) $a_0 = \chi(\sigma)a_1, a_1 = \chi(\sigma)a_2, \dots, a_{m-2} = \chi(\sigma)a_{m-1},$ and $\sigma^m \cdot a_{m-1} = \chi(\sigma)a_0$.

If $b \in B^{(\chi)}$, then $\sigma^m(a_0) = \pi(\sigma^m \cdot b) = \pi(\chi(\sigma)^m b) = \chi(\sigma^m)(a_0)$. Hence, $a_0 \in A^{(\chi|_{\overline{H}})}$. Furthermore, if $a_0 = 0$, then (2.4) shows that each $a_i = \chi(\sigma)^{-i} a_0 = 0$; so $\pi$ maps $B^{(\chi)}$ injectively to $A^{(\chi|_{\overline{H}})}$. On the other hand, if we take any $a_0 \in A^{(\chi|_{\overline{H}})}$, then $\sigma^m \cdot a_0 = \chi(\sigma^m)a_0 = \chi(\sigma)^m a_0$; so if we choose $a_1 = \chi(\sigma)^{-1}a_0, \dots, a_i = \chi(\sigma)^{-i}a_0, \dots, a_{m-1} = \chi(\sigma)^{-(m-1)}a_0$, then $\sigma^m \cdot a_{m-1} = \sigma^m \cdot (\chi(\sigma)^{-(m-1)}a_0) = \chi(\sigma)^{-(m-1)}\sigma^m \cdot a_0 = \chi(\sigma)a_0$, so the equations in (2.4) are satisfied, showing that $a_0 \in \pi(B^{(\chi)})$. Thus, $\pi\colon B^{(\chi)} \to A^{(\chi|_{\overline{H}})}$ is a bijection. ■

**Proof of Theorem 2.7** It was noted in Remark 2.6 that $Y$ has exactly $\ell$ extensions to $J$. The assertions about $\overline{Z_i}$ and $\Gamma_Z$ follow by applying Proposition 2.5 to $Y$ in $F(p)$ in place of $V$ in $F$. It remains to analyze $\overline{Y}$ and $\Gamma_Y$. For this, we look closely at what can happen with Galois $p$-extensions of $F$. These are difficult to get at directly, so we look at the corresponding extensions of $L$.

Let us now select and fix one of the $\ell$ extensions of $V$ to $L$; call it $W$. Let $w\colon L^* \to \Gamma_W$ be the associated valuation. Now, let $c \in L^* - L^{*p}$ with $[c] \in (L^*/L^{*p})^{(\alpha)}$, and let $K = L(\sqrt[p]{c})$. Let $S = F(p) \cap K$, which we know by Proposition 1.2 is a degree $p$ Galois extension of $F$. (Moreover, all such Galois extensions of $F$ arise this way.) Let $R$ be a valuation ring of $K$ with $R \cap L = W$, let $r\colon K^* \to \Gamma_R$ be its valuation, and let $U = R \cap S$, which is a valuation ring of $S$ with $U \cap F = V$. The description of $R$ and $U$ breaks down into three possible cases:

***Case 1*** $(w(c) \notin p\Gamma_W)$ Then, since $r(\sqrt[p]{c}) = \frac{1}{p}w(c) \in \Gamma_R$, the fundamental inequality implies that $\Gamma_R = \langle \frac{1}{p}w(c)\rangle + \Gamma_W$. By Proposition 2.5 applied to $U$ in $S$ instead of $V$ in $F$, we have $\Gamma_U = \Gamma_R = \langle \frac{1}{p}w(c)\rangle + \Gamma_V$. So $|\Gamma_U{:}\Gamma_V| = p = [S{:}F]$, and the Fundamental Inequality shows that $\overline{U} = \overline{V}$ and $U$ is the unique extension of $V$ to $S$.

***Case 2*** $(w(c) \in p\Gamma_W)$ Then, by modifying $c$ by a $p$-th power in $L$, we may assume that $w(c) = 0$. Let $\overline{c}$ be the image of $c$ in $\overline{W}$. For this case, assume that $\overline{c} \notin \overline{W}^{*p}$. Then $\overline{R}$ contains $\overline{\sqrt[p]{c}} = \sqrt[p]{\overline{c}}$ which is not in $\overline{W}$. So, the fundamental inequality implies that $\overline{R} = \overline{W}(\sqrt[p]{\overline{c}})$. Because $p = [\overline{R}{:}\overline{W}] \mid [\overline{R}{:}\overline{V}]$ but $p \nmid [\overline{R}{:}\overline{U}]$ by Proposition 2.5 applied to $U$ in $S$, we have $p \bigm| [\overline{U}{:}\overline{V}]$. The fundamental inequality implies that $[\overline{U}{:}\overline{V}] = p$, $\Gamma_U = \Gamma_V$, and $U$ is the unique extension of $V$ to $S$. We noted earlier that $\overline{U}$ is Galois over $\overline{V}$. A comparison of degrees over $\overline{V}$ shows that $\overline{R} = \overline{U} \cdot \overline{W}$ so $\overline{R}$ is abelian Galois over $\overline{V}$. Thus, $\overline{U}$ is the unique cyclic Galois extension of $\overline{V}$ of degree $p$ within $\overline{R}$.

***Case 3*** $(w(c) \in p\Gamma_W$, and so we may assume $w(c) = 0)$ For this case, assume that $\overline{c} \in \overline{W}^{*p}$. We claim that there are $p$ different valuation rings of $K$ extending $W$. Consider the subring $W[\sqrt[p]{c}]$ of $K$. Since $x^p - c$ is the minimal polynomial of $\sqrt[p]{c}$ over $L$, we have $W[\sqrt[p]{c}] \cong W[x]\big/(W[x] \cap (x^p - c)L[x]) = W[x]/(x^p - c)W[x]$, where the last equality follows by the division algorithm for monic polynomials in

$W[x]$. Hence, $W[\sqrt[p]{c}]/M_W\, W[\sqrt[p]{c}] \cong W[x]/(M_W, x^p - c) \cong \overline{W}[x]/(x^p - \overline{c})$. Because $\overline{c} \in \overline{W}^{*p}$ and $\mu_p \subseteq \overline{W}$, $x^p - \overline{c}$ factors into distinct linear terms in $\overline{W}[x]$, say $x^p - \overline{c} = (x - d_1)\ldots(x - d_p)$. Then the Chinese remainder theorem shows that $\overline{W}[x]/(x^p - \overline{c}) \cong \bigoplus_{i=1}^{p} \overline{W}[x]/(x - d_i)$. Because $W[\sqrt[p]{c}]/M_W W[\sqrt[p]{c}]$ thus has $p$ maximal ideals, $W[\sqrt[p]{c}]$ has at least $p$ maximal ideals. Let $C$ be the integral closure of $W$ in $K$. Since $C$ is integral over $W[\sqrt[p]{c}]$, $C$ has at least $p$ different maximal ideals, say $N_1, \ldots, N_p$. Each localization $R_i = C_{N_i}$ is a different valuation ring of $K$ with $R_i \cap L = W$. The fundamental inequality shows that there must be exactly $p$ of the $R_i$, as claimed.

Now, since $\mathcal{G}(S/F)$ acts transitively on the valuation rings of $S$ extending $V$ [E, p. 105, (14.1)], the number of such extensions is either 1 or $p$. There are at least $p$ extensions of $V$ to $K$ (namely, the $R_i$), but every extension of $V$ to $S$ has $\ell \le p-1$ extensions to $K$ by Proposition 2.5 applied over $S$. Hence, there must be more than one, so exactly $p$ extensions of $V$ to $S$, call them $U_1, \ldots, U_p$. The fundamental inequality shows that each $\overline{U_i} = \overline{V}$ and $\Gamma_{U_i} = \Gamma_V$. This completes Case 3.

We must still see what constraints are imposed by the condition that $[c] \in (L^*/L^{*p})^{(\alpha)}$. For this, let $H = \mathcal{G}(L/F) = \langle\sigma\rangle$, as usual, and let $\overline{H} = \{\tau \in H \mid \tau(W) = W\}$, the decomposition group of $W$ over $V$. Because $H$ acts transitively on the set of extensions of $V$ to $L$ and there are $\ell$ such extensions, $|H{:}\overline{H}| = \ell$, so $\overline{H} = \langle\sigma^\ell\rangle$. Each $\tau \in \overline{H}$ maps $W$ to itself, so induces an automorphism $\overline{\tau}$ of $\overline{W}$. Recall ([E, p. 147, (19.6)] or [ZS, p. 69, Theorem 21]) that the map $\overline{H} \to \mathcal{G}(\overline{W}/\overline{V})$ given by $\tau \mapsto \overline{\tau}$ is a group epimorphism. By Proposition 2.5 we have $|\overline{H}| = |H|/\ell = [L{:}F]/\ell = |\mathcal{G}(\overline{W}/\overline{V})|$, and therefore the map $\overline{H} \to \mathcal{G}(\overline{W}/\overline{V})$ is an isomorphism. Also, because $\overline{\tau}$ acts on the $p$-th roots of unity in $\overline{W}$ according to the action of $\tau$ on the $p$-th roots of unity in $L$, the cyclotomic character $\overline{\alpha}$ for $\mathcal{G}(\overline{W}/\overline{V})$ corresponds to the restriction $\alpha|_{\overline{H}}$.

Observe that the distinct extensions of $V$ to $L$ are $\sigma^i(W)$ for $0 \le i \le \ell - 1$. Each $\Gamma_{\sigma^i(W)}$ is canonically identified with $\Gamma_W$ inside the divisible hull of $\Gamma_V$, and for the associated valuation $w_i$ of $\sigma^i(W)$ we have $w_i = w \circ \sigma^{-i}$. Likewise, for $0 \le i \le \ell - 1$ we identify $\overline{\sigma^i(W)}$ with $\overline{W}$ using the isomorphism $\overline{\sigma^i}\colon \overline{W} \to \overline{\sigma^i(W)}$ induced by $\sigma^i\colon W \to \sigma^i(W)$. So, for $c \in \sigma^i(W)$, we have $\overline{c} \in \overline{\sigma^i(W)}$ corresponds to $\overline{\sigma^{-i}(c)}$ in $\overline{W}$.

We can now determine $\overline{Y}$. View $\overline{W}^*$ as an $\overline{H}$-module, where $\tau \in \overline{H}$ acts by $\overline{\tau}$. Let $\operatorname{Ind}_{\overline{H}\to H}\overline{W}^*$ be the induced $H$-module described before Proposition 2.8, with $m = \ell$. Recall that $T$ denotes the integral closure of $V$ in $L$, so $T = \bigcap_{i=0}^{\ell-1}\sigma^i(W)$ [E, p. 95, Theorem 3.3.(b)]. Let $\gamma\colon T^* \to \operatorname{Ind}_{\overline{H}\to H}\overline{W}^*$ be the map given by $\gamma(t) = \sum_{i=0}^{\ell-1}\sigma^i \otimes \overline{\sigma^{-i}(t)}$ (the bar denotes image in $\overline{W}^*$). The surjectivity of $\gamma$ is equivalent to the assertion that for every $r_0, \ldots, r_{\ell-1} \in \overline{W}^*$, there is $t \in T^*$ with $\overline{\sigma^{-i}(t)} = r_i$ in $\overline{W}$ for each $i$, *i.e.*, $\overline{t} = \overline{\sigma^i}(r_i)$ in $\overline{\sigma^i(W)}$. This holds by the Approximation Theorem ([E, p. 79, Theorem (11.14)] or [ZS, p. 30, Lemma 2]). (For this the valuation rings $\sigma^0(W), \ldots, \sigma^{\ell-1}(W)$ need not be independent, just incomparable. This result uses only the Chinese remainder theorem applied to $T$.) Also, since $\sigma^\ell \cdot \overline{\sigma^{-(\ell-1)}(t)} = \overline{\sigma^\ell(\sigma^{-(\ell-1)}(t))} = \overline{\sigma(t)}$, we have $\sigma \cdot \gamma(t) = \gamma(\sigma(t))$, so $\gamma$ is an $H$-module epimorphism. Therefore, the corresponding map $T^*/T^{*p} \to \operatorname{Ind}_{\overline{H}\to H}(\overline{W}^*/\overline{W}^{*p})$ is an $H$-

module epimorphism. So, $(T^*/T^{*p})^{(\alpha)}$ maps onto $\left(\operatorname{Ind}_{\overline{H}\to H}(\overline{W}^*/\overline{W}^{*p})\right)^{(\alpha)}$, which by Proposition 2.8 projects onto $(\overline{W}^*/\overline{W}^{*p})^{(\overline{\alpha})}$. That is, for any $a \in \overline{W}^* - \overline{W}^{*p}$ such that $[a] \in (\overline{W}^*/\overline{W}^{*p})^{(\overline{\alpha})}$ there is $t \in T^*$ with $[\overline{t}] = [a]$ in $\overline{W}^*/\overline{W}^{*p}$. If we choose $c = t$, then for the resulting $K = L(\sqrt[p]{c})$ we are in Case 2 above, with $\overline{R} = \overline{W}(\sqrt[p]{\overline{t}}) = \overline{W}(\sqrt[p]{a})$, and $\overline{U}$ is the degree $p$ Galois extension of $\overline{V}$ within $\overline{R}$. Since we can do this for any $[a] \in (\overline{W}^*/\overline{W}^{*p})^{(\overline{\alpha})}$, Proposition 1.2 shows that every Galois extension of $\overline{V}$ of degree $p$ is realizable as some $\overline{U}$, and so lies in $\overline{Y}$.

Now, $F(p)$ is the direct limit of finite towers of Galois extensions of degree $p$ starting with $F$ (see Proposition 2.1). If $S'$ is the top field in such a tower, then $\overline{Y \cap S'}$ is obtained from $\overline{V}$ by a succession of Galois extensions of degree 1 or $p$. Hence $\overline{Y \cap S'} \subseteq \overline{V}(p)$ for each $S'$, and therefore $\overline{Y} \subseteq \overline{V}(p)$. But, iteration of the argument in the preceding paragraph shows that any finite degree extension of $\overline{V}$ within $\overline{V}(p)$ is obtainable as $\overline{Y \cap S'}$ for a suitably built $S'$. Hence, $\overline{Y} = \overline{V}(p)$, as desired.

We now determine $\Gamma_Y$. For the trivial $\overline{H}$-module $\Gamma_W$, we have the induced $H$-module $\operatorname{Ind}_{\overline{H}\to H}\Gamma_W$. Let $\beta\colon L^* \to \operatorname{Ind}_{\overline{H}\to H}\Gamma_W$ be the map given by $d \mapsto \sum_{i=0}^{\ell-1}\sigma^i \otimes w(\sigma^{-i}(d))$. Since $\sigma^\ell \cdot w(\sigma^{-(\ell-1)}(d)) = w(\sigma(d))$, as $w \circ \sigma^\ell = w$ and $\sigma^\ell$ acts trivially on $\Gamma_W$, this $\beta$ is an $H$-module homomorphism. By reducing mod $p$ we obtain an $H$-module homomorphism $\overline{\beta}\colon L^*/L^{*p} \to \operatorname{Ind}_{\overline{H}\to H}(\Gamma_W/p\Gamma_W)$. So, for our $c \in L^*$ used to define $K$, since $[c] \in (L^*/L^{*p})^{(\alpha)}$, we have $\overline{\beta}[c] \in (\operatorname{Ind}_{\overline{H}\to H}\Gamma_W/p\Gamma_W)^{(\alpha)}$, so Proposition 2.8 shows that $w(c) + p\Gamma_W \in (\Gamma_W/p\Gamma_W)^{(\alpha|_{\overline{H}})}$.

Suppose first that $\mu_p \not\subseteq \overline{V}$. Then $\ell < s = [L{:}F]$, by Proposition 2.5. So $\overline{H}$, of order $s/\ell$, is nontrivial. Since the cyclotomic character $\alpha$ has order $s$, its restriction $\alpha|_{\overline{H}}$ has order $|\overline{H}|$, so is nontrivial. Since $\overline{H}$ acts trivially on $\Gamma_W$, it follows that $(\Gamma_W/p\Gamma_W)^{(\alpha|_{\overline{H}})} = (0)$. Now, the only way we could have $\Gamma_U$ larger than $\Gamma_V$ is if our $c$ is in Case 1 above. But then we would have $w(c) \notin p\Gamma_W$, yielding a nontrivial element in the trivial group $(\Gamma_W/p\Gamma_W)^{(\alpha|_{\overline{H}})}$. Since this cannot occur, we see that Case 1 never arises when $\mu_p \not\subseteq \overline{V}$. Therefore, $\Gamma_U = \Gamma_V$ for every degree $p$ Galois extension $S$ of $F$. It follows by iteration and passage to the direct limit that $\Gamma_Y = \Gamma_V$, as asserted.

Now suppose instead that $\mu_p \subseteq \overline{V}$. Proposition 2.5 shows that $\ell = s$, *i.e.,* there are $s$ different extensions $W_1, \dots, W_s$ of $V$ to $L$. Consider first the extreme case where $\mu_p \subseteq V/\mathfrak{p}$ for each nonzero prime ideal $\mathfrak{p}$ of $V$. For any such $\mathfrak{p}$, the extensions of the localizations $V_\mathfrak{p}$ to $L$ are the localizations $W_{1\mathfrak{p}}, \dots, W_{s\mathfrak{p}}$. (Each $W_{i\mathfrak{p}}$ coincides with the localization of $W_i$ at its prime ideal lying over $\mathfrak{p}$.) Since $\mu_p \subseteq \overline{V_\mathfrak{p}}$, which is the quotient field of $V/\mathfrak{p}$, Proposition 2.5 applied to $V_\mathfrak{p}$ shows that $V_\mathfrak{p}$ has $s$ different extensions to $L$. (The Proposition applies, as $\operatorname{char}(\overline{V_\mathfrak{p}}) \neq p$.) So, $W_{i\mathfrak{p}} \neq W_{j\mathfrak{p}}$ for $i \neq j$. Now, for each $i$, the rings between $W_i$ and $L$ are the $W_{i\mathfrak{p}}$ as $\mathfrak{p}$ ranges over the nonzero prime ideals of $V$. Since $W_{i\mathfrak{p}} \neq W_{j\mathfrak{p}}$ for $i \neq j$, it follows that the valuation rings $W_1, \dots, W_s$ are pairwise independent, *i.e.,* there is no valuation ring of $L$ (smaller than $L$ itself) containing both $W_i$ and $W_j$ for any $i \neq j$. Because of this independence, the Approximation Theorem (see [E, p. 80, (11.16)]) applies, and shows that our map $\beta\colon L^* \to \operatorname{Ind}_{\overline{H}\to H}\Gamma_W$ is surjective; so $\overline{\beta}\colon L^*/L^{*p} \to \operatorname{Ind}_{\overline{H}\to H}(\Gamma_W/p\Gamma_W)$ is also surjective, so it is also surjective when restricted to the $\alpha$-eigencomponents. By Proposition 2.8 $(\operatorname{Ind}_{\overline{H}\to H}(\Gamma_W/p\Gamma_W))^{(\alpha)}$ projects onto $(\Gamma_W/p\Gamma_W)^{(\alpha|_{\overline{H}})}$, which here is all of $\Gamma_W/p\Gamma_W$ since $|\overline{H}| = 1$ as $\ell = s$. This means that for any $\varepsilon \in \Gamma_W - p\Gamma_W$

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there is $c \in L^*$ such that $[c] \in (L^*/L^{*p})^{(\alpha)}$ and $w(c) \equiv \varepsilon \mod p\Gamma_W$). If we let $K = L(\sqrt[p]{c})$ for this choice of $c$, then we are in Case 1 above which shows that $\Gamma_U = \Gamma_R = \langle \frac{1}{p}\varepsilon \rangle + \Gamma_W$. Since this is true for any $\varepsilon \in \Gamma_W - p\Gamma_W$, it follows by iteration and passage to the direct limit that $\Gamma_Y = \varinjlim \frac{1}{p^n}\Gamma_V = \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} \Gamma_V$. This is what is asserted in the theorem, since in the extreme case we are now considering $P = (0)$, so $\widetilde{V} = V$ and $\widetilde{Y} = Y$.

We handle the general situation by combining the cases previously considered. Suppose $\mu_p \subseteq \overline{V}$. For the prime ideal $P$ defined in the theorem, we have $\mu_p \not\subseteq \overline{V_P}$, which is the quotient field of $V/P$. Now, $Y_Q$ is an extension of $V_P$ to $F(p)$. Since $\mu_p \not\subseteq \overline{V_P}$ and $\operatorname{char}(\overline{V_\mathfrak{p}}) \neq p$, by applying to $V_\mathfrak{p}$ the argument given previously for $V$ we obtain $\Gamma_{Y_Q} = \Gamma_{V_P}$, as desired. Furthermore, $\overline{Y_Q} \cong \overline{V_P}(p)$. Thus, $\widetilde{Y} = Y/Q$ can be viewed as an extension of $\widetilde{V} = V/P$ from $\overline{V_P}$ to $\overline{V_P}(p)$. By the choice of $P$, the extreme case considered in the previous paragraph applies to $\widetilde{V}$. Hence, $\Gamma_{\widetilde{Y}} = \mathbb{Z}[1/p] \otimes_{\mathbb{Z}} \Gamma_{\widetilde{V}}$. ■

***Example 2.9*** Let $F_0 = \mathbb{Q}(x, y)$, the rational function field in two variables over $\mathbb{Q}$. Let $V_0 = \mathbb{Q}[x]_{(x)} + y\mathbb{Q}(x)[y]_{(y)}$. Here we are localizing first with respect to the prime ideal $(x)$ of $\mathbb{Q}[x]$, and second with respect to the prime ideal $(y)$ of $\mathbb{Q}(x)[y]$. Then $V_0$ is a valuation ring of $F_0$ with $\overline{V_0} \cong \mathbb{Q}$ and $\Gamma_{V_0} = \mathbb{Z} \times \mathbb{Z}$. If $v_0\colon F_0^* \to \Gamma_{V_0}$ is the associated valuation, then $v_0(x) = (1, 0)$ and $v_0(y) = (0, 1)$. Note that $V_0$ is the intersection with $F_0$ of the standard Henselian valuation ring on $\mathbb{Q}((x))((y))$ described in Example 2.4 above. For any odd prime $p$, let $F = F_0(\sqrt[p]{1 + x})$. To see how $V_0$ extends to $F$, let $T$ be the integral closure of $V_0$ in $F$, and let $S = V_0[\sqrt[p]{1 + x}] \subseteq T$. Since $S \cong V_0[t]/\big(t^p - (1 + x)\big)$, we have

$$\begin{aligned} S/M_{V_0}S &\cong \overline{V_0}[t]/\big(t^p - (1 + \overline{x})\big) \cong \mathbb{Q}[t]/(t^p - 1) \\ &\cong \mathbb{Q}[t]/(t - 1) \oplus \mathbb{Q}[t]/(t^{p-1} + \cdots + 1) \cong \mathbb{Q} \oplus \mathbb{Q}(\mu_p). \end{aligned}$$

So $T$, being integral over $S$, has at least two maximal ideals $N_1$ and $N_2$, with $\mathbb{Q} \subseteq T/N_1$ and $\mathbb{Q}(\mu_p) \subseteq N_2$. The fundamental inequality shows that for the extensions $V_i = T_{N_i}$ of $V_0$ to $F$, we have $\overline{V_1} \cong \mathbb{Q}$, $\overline{V_2} \cong \mathbb{Q}(\mu_p)$, and $\Gamma_{V_1} = \Gamma_{V_2} = \Gamma_{V_0} = \mathbb{Z} \times \mathbb{Z}$. Furthermore, $V_1$ and $V_2$ are the only extensions of $V_0$ to $F$. If $Y_i$ is any extension of $V_i$ to $F(p)$, then Theorem 2.7 shows that $\overline{Y_1} \cong \mathbb{Q}(p)$ and $\Gamma_{Y_1} = \mathbb{Z} \times \mathbb{Z}$. Let $\mathfrak{p}$ be the prime ideal $yV_2$. Then $V_2/\mathfrak{p} \cong \mathbb{Q}(x)(\sqrt[p]{1 + x})$, which does not contain $\mu_p$. So $\mathfrak{p}$ is the prime ideal $P$ of Theorem 2.7 for $V_2$. Since $\Gamma_{V_2/\mathfrak{p}} = \mathbb{Z} \times 0$, Theorem 2.7 shows that $\Gamma_{Y_2} = \mathbb{Z}[1/p] \times \mathbb{Z}$ while $\overline{Y_2} \cong \mathbb{Q}(\mu_p)(p)$.

***Remark 2.10*** By using Theorem 2.7 we can now see that one cannot construct an example of a nonsplit algebra of degree $p$ in ${}_p\operatorname{Br}(J)^H$ by using only the valuations $V$ on $L$ with $\operatorname{char}(\overline{V}) \neq p$. This provides support for a conjecture that ${}_p\operatorname{Br}(F(p)) = 0$. For this, let $V$ be a valuation ring of $F$ with $\operatorname{char}(\overline{V}) \neq p$, let $W$ be an extension of $V$ to $L$ with associated valuation $w\colon L^* \to \Gamma_W$, and let $Z$ be an extension of $W$ to $J$. There are three types of symbol algebras $A = (a, b; L)_\omega$ (with $a, b \in L^*$ and $\omega \in \mu_p^*$) for which it is known that $w$ extends to a valuation on $A$, and hence $A$ is a division algebra:

(1) If $w(a)$ and $w(b)$ map to $\mathbb{Z}/p\mathbb{Z}$-independent elements of $\Gamma_W/p\Gamma_W$, then, *cf.* [JW, Corollary 2.6], the valuation ring of $A$ is tame and totally ramified over $W$, with residue division algebra $\overline{V}$ and value group $\langle \frac{1}{p}w(a), \frac{1}{p}w(b)\rangle + \Gamma_W$.

(2) If $w(a) \notin p\Gamma_W$ and $w(b) = 0$, and for the image $\overline{b}$ of $b$ in $\overline{W}$ we have $\overline{b} \notin \overline{W}^{*p}$, then *cf.* [JW, Corollary 2.9], the valuation ring of $A$ is semiramified over $W$, with residue division algebra $\overline{W}(\sqrt[p]{\overline{b}})$ and value group $\langle \frac{1}{p}w(a)\rangle + \Gamma_W$.

(3) If $w(a) = w(b) = 0$ and $(\overline{a}, \overline{b}; \overline{W})_{\overline{\omega}}$ is a division ring, then the valuation ring on $A$ is unramified over $V$, with residue algebra $(\overline{a}, \overline{b}; \overline{W})_{\overline{\omega}}$ and value group $\Gamma_W$.

For, if $i$ and $j$ are standard generators of $A = (a, b; L)_\omega$, then it is easy to check that the map $u\colon A - \{0\} \to \Gamma_W$ given by $u(\sum_{r=0}^{p-1}\sum_{s=0}^{p-1} c_{rs}\, i^r j^s) = \min\{w(c_{rs}) \mid c_{rs} \neq 0\}$ ($c_{rs} \in L$) is a valuation on $A$ with the specified residue algebra and value group. (The proof is similar to but easier than the proof of [JW, Theorem 2.5].) Since type (3) reduces the problem of obtaining a division algebra to the same problem over the residue field, it is not helpful for constructing examples, and we will not consider this type further.

Suppose we choose $a, b \in L^*$ so that for some character $\chi\colon H \to \mathbb{Z}/p\mathbb{Z}^*$ we have $[a] \in (L^*/L^{*p})^{(\chi)}$ and $[b] \in (L^*/L^{*p})^{(\alpha\chi^{-1})}$. Then $[A] \in {}_p\mathrm{Br}(L)^H$ by (1.3), so $[A \otimes_L J] \in {}_p\mathrm{Br}(J)^H$. But we will see that the valuation conditions that assure $A$ is a division algebra break down over $J$. Suppose first that $\mu_p \notin \overline{V}$; so, in the notation of Theorem 2.7 and its proof, $\ell < [L{:}F]$ and $\overline{H}$ is nontrivial. Suppose $w(a) \notin p\Gamma_W$. Then as in the proof of Theorem 2.7, Proposition 2.8 implies that the image of $w(a)$ is nontrivial in $\Gamma_W/p\Gamma_W{}^{(\chi|_{\overline{H}})}$. This forces $\chi|_{\overline{H}}$ to be trivial, as $\overline{H}$ acts trivially on $\Gamma_W$. Hence, $\alpha\chi^{-1}|_{\overline{H}} = \alpha|_{\overline{H}}$, which is nontrivial and is identified with the cyclotomic character $\overline{\alpha}$ for $\mathcal{G}(\overline{W}/\overline{V})$. The nontriviality of $\alpha\chi^{-1}|_{\overline{H}}$ forces $w(b) \in p\Gamma_W$, so we may assume $w(b) = 0$. If $\underline{b} \notin \overline{W}^{*p}$, then $A$ is a division algebra of type (2). But Proposition 2.8 implies that $\overline{b}$ maps to

$$(\overline{W}^*/\overline{W}^{*p})^{(\alpha\chi^{-1}|_{\overline{H}})} = (\overline{W}^*/\overline{W}^{*p})^{(\overline{\alpha})}.$$

Hence, on passing to $J$ we find that $\overline{b} \in (\overline{Z}^*/\overline{Z}^{*p})^{(\overline{\alpha})}$, which is trivial as $\overline{Z} \cong \overline{V}(p)(\mu_p)$, (see Remark 2.3). This means that $\overline{b} \in \overline{Z}^{*p}$, and we have lost the conditions for type (2) for $A \otimes_L J$. Likewise, if $w(b) \notin p\Gamma_W$, then we are forced to have $w(a) \in p\Gamma_W$, and when we adjust $a$ so that $w(a) = 0$, the same argument as just given shows that $\overline{a} \in \overline{Z}^{*p}$. Thus, we have not been able to obtain a type (1) or a type (2) valued division algebra in ${}_p\mathrm{Br}(J)^H$ when $\mu_p \notin \overline{V}$.

Suppose instead that $\mu_p \subseteq \overline{V}$. Since $\overline{Z} = \overline{V}(p)(\mu_p) = \overline{V}(p)$ and $\mu_p \subseteq \overline{V}$, $\overline{Z}^*/\overline{Z}^{*p}$ is trivial. Therefore, we will not obtain any valued division algebras of degree $p$ of type (2) or type (3) over $J$. We are left to search for type (1) division algebras. Thus, we may assume that $w(a)$ and $w(b)$ are $\mathbb{Z}/p\mathbb{Z}$-independent in $\Gamma_W/p\Gamma_W$. Here $\overline{H}$ is trivial, but choose the prime ideal $P$ of $V$ as in Theorem 2.7, and let $\mathfrak{P}$ be the prime ideal of $W$ with $\mathfrak{P} \cap V = P$, and $\widetilde{H}$ the (nontrivial) decomposition group of $W_{\mathfrak{P}}$ over $V_P$; let $w_{\mathfrak{P}}$ be the valuation of $W_{\mathfrak{P}}$. We have an $H$-module homomorphism $\widetilde{\gamma}\colon L^*/L^{*p} \to \mathrm{ind}_{\widetilde{H}\to H}(\Gamma_{W_{\mathfrak{P}}}/p\Gamma_{W_{\mathfrak{P}}})$ so since

$[a] \in (L^*/L^{*p})^{(\chi)}$, we find that $\widetilde{\gamma}[a] \in (\operatorname{ind}_{\widetilde{H}\to H}(\Gamma_{W_{\mathfrak{P}}}/p\Gamma_{W_{\mathfrak{P}}}))^{(\chi)}$. By Proposition 2.8 it follows that $w_{\mathfrak{P}}(a) \in (\Gamma_{W_{\mathfrak{P}}}/p\Gamma_{W_{\mathfrak{P}}})^{(\chi|_{\widetilde{H}})}$. Since $\widetilde{H}$ acts trivially on $\Gamma_{W_{\mathfrak{P}}}$, this implies that $w_{\mathfrak{P}}(a) \in p\Gamma_{W_{\mathfrak{P}}}$ or $\chi|_{\widetilde{H}}$ is trivial. If $w_{\mathfrak{P}}(a) \in p\Gamma_{W_{\mathfrak{P}}}$, we can modify $a$ by a $p$-th power in $L^*$ to assume that $w_{\mathfrak{P}}(a) = 0$. But then for $\widetilde{W} = W/\mathfrak{P}$, the exact sequence like (2.2) for $\Gamma_W$ shows that $w(a) \in \Gamma_{\widetilde{W}}$. But then Theorem 2.7 shows that $w(a) \in p\Gamma_Z$, so that $(a, b; J)_\omega$ is not a type (1) valued division algebra over $J$. On the other hand, if $\chi|_{\widetilde{H}}$ is trivial, then $\alpha\chi^{-1}|_{\widetilde{H}} = \alpha|_{\widetilde{H}}$, which is nontrivial. Hence, the argument just given for $a$ now shows that $w(b) \in p\Gamma_Z$, so again we do not obtain a type (1) valued division algebra over $J$.

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