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On an application of the complex nonlinear complementarity problem

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A theorem on the existence of a solution under feasibility assumptions to a convex minimization problem over polyhedral cones in complex space is given by using the fact that the problem of solving a convex minimization program naturally leads to the consideration of the following nonlinear complementarity problem: given $g : C^n \to C^n$, find z such that $g(z) \in S^*$, $z \in S$, and $\operatorname{Re}\langle g(z), z\rangle = 0$, where S is a polyhedral cone and S^* its polar.

1. Introduction

In [5] and [6], the authors have studied the existence and uniqueness of a solution to the following nonlinear complementarity problem: given $g : C^n \to C^n$, find z such that

$$\begin{aligned} &g(z) \in S^* , \quad z \in S , \\ &\operatorname{Re}\langle g(z), z\rangle = 0 , \end{aligned} \tag{1.1}$$

where S is a polyhedral cone in C^n and S^* the polar cone of S. The purpose of this paper is to apply the existence theorem of [5] to study the extent to which the existence of a feasible solution ensures the existence of an optimal solution to the following convex program:

$$\begin{aligned} &\text{MINIMIZE} \quad \operatorname{Re} f(z, \bar{z}) \\ &\text{SUBJECT TO} \quad g(z) \in L^* , \quad z \in P , \end{aligned} \tag{1.2}$$

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where L and P are polyhedral cones in C^m and C^n respectively, $g : C^n \to C^m$ is an analytic mapping concave with respect to L^* on P, and $f : Q \to C$ is an analytic mapping having a convex real part with respect to R_+ on $\{(z, \bar{z}) : z \in P\}$. Here the linear manifold Q is given by

$$Q = \{(z, w) \in C^n \times C^n : w = \bar{z}\} .$$

We show that the above problem has an optimal solution if the feasible region of the problem is bounded and has a nonempty interior. Further we show that the above conclusion also holds if the boundedness of the feasible region is relaxed, but in that case, we have to impose more restrictions on the growth of the mapping f.

2. Notations and definitions

Let C^n $[R^n]$ denote the n-dimensional complex [real] space with hermitian [euclidean] norm $\|\cdot\|$ and the usual inner product $\langle \cdot, \cdot \rangle$. R_+^n denotes the nonnegative orthant of R^n. If A is a matrix, then A^T, $\bar{A}$, A^H denote its transpose, complex conjugate, conjugate transpose. For any $x \in R_+^n$, $\|x\|_\infty = \max\{x_i : 1 \le i \le n\}$ denotes the l^∞-norm.

A nonempty set $S \subset C^n$ is a polyhedral cone if, for some positive integer k and $A \in C^{n\times k}$,

$$S = \{Ax : x \in R_+^k\} .$$

$$S^* = \{z \in C^n : y \in S \Rightarrow \operatorname{Re}\langle y, z\rangle \ge 0\}$$

is the polar of S. For a real scalar $p > 0$, we denote $z(p) = \{Ax : x_i = p, i = 1, 2, \ldots, k\}$. For any $z^1, z^2 \in S$, $z^1 \geqq_S z^2$ states that $z^1 - z^2 \in S$ while $z^1 >_S z^2$ means that $z^1 - z^2 \in \operatorname{int} S$.

3. Solvability of the convex program

A sufficient condition [4] for $z^0 \in C^n$ to be an optimal point of (1.2) is the existence of an $u^0 \in C^m$ such that

$$(3.1)\qquad \begin{gathered} z^0 \in P,\ u^0 \in L,\ g(z^0) \in L^* , \\ \overline{\overline{\nabla_z f(z^0, \overline{z^0})} + \nabla_{\overline{z}} f(z^0, \overline{z^0})} - J_g^H(z^0)u^0 \in P^* , \\ \operatorname{Re}\left\langle \overline{\overline{\nabla_z f(z^0, \overline{z^0})}+\nabla_{\overline{z}} f(z^0, \overline{z^0})}-J_g^H(z^0)u^0,\ z^0\right\rangle = 0 , \\ \operatorname{Re}\langle g(z^0),\ u^0\rangle = 0 , \end{gathered}$$

where $J_g(z^0)$ denotes the $m \times n$ matrix whose i, j-th element is $\partial g_i(z^0)/\partial z_j$. Let the function $G(z, u)$ be defined by

$$(3.2)\qquad G(z, u) = \begin{bmatrix} \overline{\overline{\nabla_z f(z, \overline{z})}+\nabla_{\overline{z}} f(z, \overline{z})}-J_g^H(z)u \\ g(z) \end{bmatrix}$$

for all $(z, u) \in C^{n+m}$. Now it is easy to see that the point (z^0, u^0) satisfying (3.1) is a solution of the system

$$(3.3)\qquad \begin{gathered} (z, u) \in P \times L , \quad G(z, u) \in P^* \times L^* , \\ \operatorname{Re}\langle G(z, u), (z, u)\rangle = 0 , \end{gathered}$$

which is of the form (1.1). So we have:

REMARK 3.1. If (z^0, u^0) is a solution to the nonlinear complementarity problem, as given by (3.3), then z^0 solves the convex program (1.2).

THEOREM 3.2. *Let $f : Q \to C$ and $g : C^n \to C^m$ be analytic in Q and C^n respectively. Let f have a convex real part with respect to R_+ on $\{(z, \overline{z}) : z \in P\}$ and g be concave with respect to L^* on P. Suppose that*

(i) the set $K = \{z \in P : g(z) \in L^*\}$ be bounded,

(ii) there be a $\hat{z} \in P$ such that $g(\hat{z}) \in \operatorname{int} L^*$.

Then there exists a z^0 which is optimal for problem (1.2).

Proof. Since P, L are polyhedral cones, there exist matrices B, E and positive integers q, r such that

$$P = \{Bs : s \in R_+^q\} ,$$

$$L = \{Et : t \in R_+^r\} .$$

For each $p \in R_+$, define

$$D_1(p) = \{z = Bs \in P : \|s\|_\infty \le \min(p, \hat{p})\} ,$$

$$D_2(p) = \{u = Et \in L : \|t\|_\infty \le p\} ,$$

$$D(p) = D_1(p) \times D_2(p) ,$$

where $\hat{p}$ is a positive number such that $z(\hat{p}) \overset{P}{\geq} z$ for all $z \in K$. Since $D(p)$ is a nonempty, compact, convex set, from Theorem 3.1 of [5] it follows that there exists a $\zeta = (z, u) \in D(p)$ such that

(3.4) $$\operatorname{Re}\langle G(\zeta), \zeta'-\zeta\rangle \ge 0$$

for all $\zeta' = (z', u') \in D(p)$. If $\zeta = (z, u)$ satisfies (3.4), then z satisfies

(3.5) $$\operatorname{Re}\left\langle \overline{\nabla_z f(z, \bar{z})+\nabla_{\bar{z}} f(z, \bar{z})}-J_g^H(z)u, z'-z\right\rangle \ge 0$$

for all $z' \in D_1(p)$, and u satisfies

(3.6) $$\operatorname{Re}\langle g(z), u'-u\rangle \ge 0 \text{ for all } u' \in D_2(p) .$$

This can be checked by setting $u' = u$ and $z' = z$ in (3.4) separately.

Note that for each $p \in R_+$, we get a point $(z, u) \in D(p)$ satisfying (3.4). Let V denote the set of all such points. We shall show that V is bounded. Assume to the contrary that V is unbounded. This implies that there is a sequence $\{(z^i, u^i)\}$ in V such that

(3.7) $$\|s^i\|_\infty + \|t^i\|_\infty \to \infty \text{ as } i \to \infty .$$

But each z^i is in the compact set $D_1(\hat{p})$, and consequently, from (3.7), we get that $\|t^i\|_\infty \to \infty$ as $i \to \infty$. Hence we can assume that $\|t^i\|_\infty \ge \hat{p}$ for $i = 1, 2, \ldots$. Noting that $(\hat{z}, 0) \in D(p)$ for each $p \ge \hat{p}$, from (3.4)-(3.6) we obtain

$$(3.8)\qquad \begin{aligned} &\mathrm{Re}\left\langle \frac{\overline{\nabla_z f\left(z^i, \overline{z^i}\right)}}{\|t^i\|_\infty} + \frac{\nabla_{\bar z} f\left(z^i, \overline{z^i}\right)}{\|t^i\|_\infty} - J_g^H\left(z^i\right) \frac{u^i}{\|t^i\|_\infty},\ \hat{z}-z^i \right\rangle \geq 0\ , \\ &\mathrm{Re}\left\langle g\left(z^i\right),\ \frac{u^i}{\|t^i\|_\infty} \right\rangle \leq 0\ . \end{aligned}$$

Now the point $\dfrac{u^i}{\|t^i\|_\infty} \in D_2(1)$ for $i = 1, 2, \ldots$. Thus the sequence

$$\left\{\left(z^i,\ \frac{u^i}{\|t^i\|_\infty}\right)\right\}$$

lies in the compact set $D_1(\hat{p}) \times D_2(1)$, and therefore there is a subsequence which converges to some $(\tilde{z}, \tilde{u})$ in the compact set. If we retain the same superscripts to denote the subsequence and go for the limit of (3.8), we get

$$(3.9)\qquad \begin{aligned} &\mathrm{Re}\left\langle J_g^H(\tilde{z})\tilde{u},\ \hat{z}-\tilde{z} \right\rangle \leq 0\ , \\ &\mathrm{Re}\langle g(\tilde{z}),\ \tilde{u}\rangle \leq 0\ . \end{aligned}$$

From the concavity of g with respect to L^* on P and $\tilde{u} \in L$, we have

$$\mathrm{Re}\left[\left\langle J_g^H(\tilde{z})\tilde{u},\ \hat{z}-\tilde{z}\right\rangle + \langle g(\tilde{z}),\ \tilde{u}\rangle\right] \geq \mathrm{Re}\langle g(\hat{z}),\ \tilde{u}\rangle\ ,$$

and by (3.9), $\mathrm{Re}\langle g(\hat{z}), \tilde{u}\rangle \leq 0$. This contradicts the assumption that $g(z) \in \operatorname{int} L^*$. Thus we have shown that V is bounded.

The boundedness of V implies that there exists a $p^0 > \hat{p}$ such that $u\left(p^0\right) \overset{L}{>} u$ if $(z, u) \in V$. Let $\left(z^0, u^0\right)$ satisfy (3.4) for all $(z', u') \in D\left(p^0\right)$. Thus, for $u\left(p^0\right) \overset{L}{>} u$,

$$\mathrm{Re}\langle g\left(z^0\right),\ u'-u^0\rangle \geq 0 \quad \text{for all} \quad u' \in D_2\left(p^0\right)\ ,$$

and by Lemmas 3.2 and 4.1 of [5],

$$u^0 \in L,\ g\left(z^0\right) \in L^*,\ \mathrm{Re}\langle g\left(z^0\right),\ u^0\rangle = 0\ .$$

Hence $z^0 \in K$. This implies that for $z\left(p^0\right) \overset{P}{>} z^0$,

$$\operatorname{Re}\left\langle \overline{\nabla_z f\left(z^0, \overline{z^0}\right)} + \nabla_{\bar{z}} f\left(z^0, \overline{z^0}\right) - J_g^H\left(z^0\right)u^0,\ z' - z^0 \right\rangle \geq 0$$

for all $z' \in D_1\left(p^0\right)$. Again using the lemmas quoted just above, we obtain

$$z^0 \in P\ ,\quad \overline{\nabla_z f\left(z^0, \overline{z^0}\right)} + \nabla_{\bar{z}} f\left(z^0, \overline{z^0}\right) - J_g^H\left(z^0\right)u^0 \in P^*\ ,$$

$$\operatorname{Re}\left\langle \overline{\nabla_z f\left(z^0, \overline{z^0}\right)} + \nabla_{\bar{z}} f\left(z^0, \overline{z^0}\right) - J_g^H\left(z^0\right)u^0,\ z^0 \right\rangle = 0\ .$$

Thus we prove that $\left(z^0, u^0\right)$ is a solution to the nonlinear complementarity problem, as given by (3.3). Now, from Remark 3.1, it follows that z^0 is an optimal solution to the convex program (1.2).

In the above theorem, the set K of the feasible solutions to (1.2) is assumed to be bounded. In the next theorem, we shall show that this boundedness of K can be relaxed by imposing stricter conditions on the function f .

LEMMA 3.3. *Let* f *have a convex real part with respect to* R_+ *on* $\{(z, \bar{z}) : z \in P\}$ *and* g *be concave with respect to* L^* *on* P . *Then* $G(z, u)$ *, as given by* (3.2)*, is monotone over* $P \times L$.

Proof. Since g is concave with respect to L^* on P , for $x, y \in P$ and $u, v \in L$,

$$\text{(3.10)}\qquad \operatorname{Re}\left[(u-v)^H\left(g(x)-g(y)\right)-(x-y)^H\left(J_g^H(x)u-J_g^H(y)v\right)\right] \geq 0\ .$$

From the convexity of $\operatorname{Re} f$, we have

$$\text{(3.11)}\qquad \operatorname{Re}(x-y)^H\left(\overline{\nabla_z f(x, \bar{x})}+\nabla_{\bar{z}} f(x, \bar{x})-\overline{\nabla_z f(y, \bar{y})}-\nabla_{\bar{z}} f(y, \bar{y})\right) \geq 0\ .$$

Now adding (3.10) and (3.11), the result of the lemma follows.

THEOREM 3.4. *Let* f *and* g *be defined as in Theorem* 3.2*, and let there be a* $\hat{z} \in P$ *such that*

$$\overline{\nabla_z f(\hat{z}, \bar{\hat{z}})} + \nabla_{\bar{z}} f(\hat{z}, \bar{\hat{z}}) \in \operatorname{int} P^*\ ,$$

$$g(\hat{z}) \in \operatorname{int} L^*\ .$$

Then there exists a solution to the convex minimization problem (1.2).

Proof. From the assumption of this theorem and Lemma 3.3, it follows that $G(z, u)$ is monotone over $P \times L$, and for $(\hat{z}, 0) \in P \times L$, $G(\hat{z}, 0) \in \operatorname{int}(P \times L)^*$. Now, applying Theorem 4.2 of [5], we get a point (z^0, u^0) that solves (3.3). The conclusion of the theorem then follows from Remark 3.1.

REMARKS 3.5. Recently, Kojima [2] has studied the existence of a solution to a convex minimization problem over orthant domains in real space. If $L = R_+^m$, $P = R_+^n$, and f, g are real-valued continuously differentiable functions on R^n, then Theorem 3.2 reduces to the result of Kojima [2, page 269]. Moreover, our Theorem 3.4 suggests an alternate set of hypotheses under which the result of Kojima holds.

If f is defined by $f(z, \bar{z}) = c^H z + \frac{1}{2} z^H M z$ for some positive semi-definite hermitian matrix M and $c \in C^n$, then the assumptions of Theorem 3.4 precisely mean that $\operatorname{Re} f$ is bounded below on the nonempty polyhedral convex set K of the feasible points of (1.2). If this is the case, then the conclusion of Theorem 3.4 follows from the Complex Frank-Wolfe Theorem [3, Theorem 4.3.6], [1].

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