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THE HEAT EQUATION ON THE SPACES OF POSITIVE DEFINITE MATRICES

P. SAWYER

ABSTRACT. The main topic of this paper is the study of the fundamental solution of the heat equation for the symmetric spaces of positive definite matrices, $\mathrm{Pos}(n, R)$.

Our first step is to develop a "False Abel Inverse Transform" $\mathcal{G}$ which transforms functions of compact support on an euclidean space into integrable functions on the symmetric space. The transform $\mathcal{G}$ is shown to satisfy the relation $\Delta\mathcal{G}(f;\cdot) = \mathcal{G}(\Gamma(\Delta)f;\cdot)$ ($\Gamma(\Delta)$ is the usual Laplacian with a constant drift).

Using this transform, we find explicit formulas for the heat kernel in the cases $n = 2$ and $n = 3$. These formulas allow us to give the asymptotic development for the heat kernel as t tends to infinity. Finally, we give an upper and lower bound of the same type for the heat kernel.

Introduction. Our notation will reflect that used by S. Helgason in [7] and in [8] (except in a few instances). To complement this, especially for the spaces of positive definite matrices, we refer to the notation used by C. S. Herz in [9].

In this work, we endeavour to study the heat equation on the symmetric spaces of positive definite matrices. This paper is a shortened and (hopefully) improved presentation of the author's McGill University dissertation [14]. Maurice Chayet, also of McGill University, has investigated similar questions in [3].

It is well known that P_t, the fundamental solution of the heat equation, must be K-invariant. To determine the heat kernel, in view of the Cartan decomposition $G = K\overline{A^+}K$, it is then enough to find $P_t(a \cdot o)$ with $a \in A^+$. For simplicity sake, we will write $P_t(a)$ (or most of the time $P_t(e^H)$) with $a \in A^+$ ($H \in \mathbf{a}^+$).

In the rank one case, one can find P_t explicitly by finding the inverse of the Abel transform F_h (to be denoted by $F(h,\cdot)$ in this paper). If the rank is greater than one, this is not quite so easy.

It is natural, using the usual scalar product on L^2 spaces, to define for Weyl-invariant functions, the adjoint of the Abel transform (also called the *dual Abel transform* in [11]):

$$\langle F(h;\cdot), f\rangle_{L^2(\mathbf{a}/W, dH)} = \langle h, G(f;\cdot)\rangle_{L^2(\mathbf{a}^+, \delta(H)\, dH)}. \tag{1}$$

For all $h \in C_c^\infty(K\backslash X)$ (the K-invariant smooth functions on X of compact support)

$$\Delta G(f;\cdot) = G\big(\Gamma(\Delta)f;\cdot\big). \tag{2}$$

This is a consequence of the property $F(\Delta h;\cdot) = \Gamma(\Delta)F(h;\cdot)$.

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The property shown in (2) is one would expect from the inverse of the Abel transform. However, for $f \in C_c(\mathbf{a}/W)$, $G(f;\cdot)$ is generally not integrable. A good candidate for the inverse of the Abel transform should satisfy (2) and transform functions of compact support into integrable functions.

We will not try to find the inverse of the Abel transform, but rather a transform having these properties calling the result a "False Abel Inverse Transform".

As the title of this paper indicates, our main interest will be $\mathrm{Pos}(n,\mathbf{R})$, the space of positive definite matrices over $\mathbf{R}$. Note that in this case, $\mathbf{a}$ stands for the space of diagonal matrices and $H \in \mathbf{a}^+$ if $H = \mathrm{diag}[H_1,\ldots,H_n]$ with $H_i > H_{i+1}$.

In Section 1, we give the definition of the adjoint of the Abel transform and give an explicit formulation for it. We express $G(n,f;\cdot)$ (the adjoint of the Abel transform for the symmetric space $\mathrm{Pos}(n,\mathbf{R})$) in terms of $G(n-1,f_{|H|};\cdot)$ where $f_{|H|}$ is the function f restricted to a subset of A;

$$G(n,f;H) = \int_{\mathrm{Dom}(H)} G(n-1,f_{|H|};\xi)\psi(H,\xi)\,d\xi. \tag{3}$$

The object of this is, by modifying the domain of integration $\mathrm{Dom}(H)$ in (3), to define the "False Abel Inverse Transform". We want to change the domain in such a way that the property in (2) is preserved, but that the transform of functions of compact support be integrable. This is the program for the following sections.

In Section 2, we define formally the "False Abel Inverse Transform" $\mathcal{G}(n,f;\cdot)$ for the space $\mathrm{Pos}(n,\mathbf{R})$ and prove some of its properties, in particular, that

$$\Delta\mathcal{G}(n,f;\cdot) = \mathcal{G}\big(n,\Gamma(\Delta)f;\cdot\big). \tag{4}$$

We will give a partial proof in Section 2 which is sufficient for the cases $n = 2$, 3 and finish the proof in Section 6.

In the third section, we use the "False Abel Inverse Transform" to define a "candidate" for the heat kernel for $\mathrm{Pos}(n,\mathbf{R})$. The problem of finding the heat kernel for $\mathrm{Pos}(n,\mathbf{R})$ is equivalent to that of finding the heat kernel for $\mathrm{Pos}_1(n,\mathbf{R})$ (the positive definite matrices over the real numbers, with determinant equal to 1): *this is an important consideration; most results in this paper are about the symmetric space* $\mathrm{Pos}(n,\mathbf{R})$ *and not about* $\mathrm{Pos}_1(n,\mathbf{R})$*, the symmetric space associated to* $\mathrm{SL}(n,\mathbf{R})$.

The complexity of our formulas makes it difficult to make explicit computations in the general case. We study the cases $n = 2$ and $n = 3$ in more details.

Using the expression of the heat kernel based on Plancherel's formula, it can be shown that for some positive integer q,

$$P_t(e^H) = Ce^{-\gamma^2 t}t^{-q/2}\exp\big(-r^2(H)/(4t)\big)\phi_0(H)V_t(H) \tag{5}$$

where $r(H) = \|H\|$, $\gamma^2 = \langle \rho,\rho\rangle$, ϕ_0 is the Legendre function and $\lim_{t\to\infty} V_t(H) = 1$ for all H. q is known to be equal to $l + 2|\Sigma_0^+|$ where l is the rank of the symmetric space and Σ_0^+ is the set of positive indivisible roots. We will call q the dimension of the symmetric space at infinity. The (ordinary) dimension of the symmetric space is $l+\sum_{\alpha\in\Sigma_0^+}(m_\alpha+m_{2\alpha})$ where m_α is the multiplicity of the root α.

In the Sections 4 and 5, we will investigate the term $V_t(H)$ for the spaces $X = \mathrm{Pos}(2,\mathbf{R})$ and $X = \mathrm{Pos}(3,\mathbf{R})$ without relying on Plancherel's formula. All the elements of expression (5) will be given explicitly.

We will show that V_t has the asymptotic expansion

$$V_t(H) \asymp 1 + \sum_{m=1}^{\infty}(-1)^m b_m(H)t^{-m} \text{ as } t \longrightarrow \infty.$$

More precisely, for all $M \geq 0$,

$$\begin{aligned} 0 &\leq (-1)^M\Big[1 + \sum_{m=1}^{M}(-1)^m b_m(H)t^{-q} - V_t(H)\Big] \\ &\leq b_{M+1}(H)t^{-(M+1)} \\ &\leq C_M\Big(\frac{1+r(H)}{t}\Big)^{M+1}. \end{aligned}$$

In the paper [2], Jean-Philippe Anker gives an upper bound for the heat kernel P_t for the symmetric spaces $\mathbf{U}(p,q)/\mathbf{U}(p)\times\mathbf{U}(q)$. If we make use of the estimates given by Jean-Philippe Anker in [1] for the Legendre function ϕ_0, his upper bound can be written as

$$V_t(H) \leq C \prod_{\alpha\in\Sigma_0^+}\Big(1 + \frac{1+\alpha}{t}\Big)^{(m_\alpha+m_{2\alpha})/2-1} \tag{6}$$

(for $t > 0$ and $H \in \overline{\mathbf{a}^+}$).

He also conjectures that the upper bound in (6) holds for all symmetric spaces of noncompact type. This is true for the complex case since $V_t(H)$ is identically 1 on such spaces (see [6]) and for all symmetric spaces of rank 1 (as pointed out in [2] by Jean-Philippe Anker).

In Section 5, we show that for the spaces $\mathrm{Pos}(2,\mathbf{R})$ and $\mathrm{Pos}(3,\mathbf{R})$, $V_t(H)$ is bounded above and below by constant multiples of the right hand side of (6). The corresponding results for heat kernel of the real hyperbolic spaces had been obtained by E. B. Davies and N. Mandouvalos (see Theorem 5.7.2 in [4]).

As mentioned above, we conclude the proof of equation (4) in Section 6. We also explain there why our "candidate" for the heat kernel as given in Section 3 is the right one in all cases.

1. **The adjoint of the Abel transform.**

1.1 *Definition of the adjoint of the Abel transform.* We define in this section the adjoint of the Abel transform for any symmetric space of noncompact type.

Let h be a function in $C_c(G)$ bi-invariant under K and f be a function in $C_c(A)$ invariant under the action of the Weyl group W.

The Abel transform of h is

$$F(h;a) = e^{\rho(\log a)} \int_N h(an)\,dn.$$

If we use the integration formulas associated to the Iwasawa and Cartan decompositions, we obtain:

DEFINITION 1.1. The adjoint of the Abel transform is defined by the equation

$$G(f;a) = \int_K e^{-\rho(H(ak))} f(e^{H(ak)})\,dk.$$

1.2 *The adjoint of the Abel transform for* $\mathrm{Pos}(n,\mathbf{R})$. We consider $G(n,f;H)$, the adjoint of the Abel transform for the symmetric space corresponding to $\mathrm{GL}^+(n,\mathbf{R})$.

THEOREM 1.2. *If f is a continuous function on A and $H \in \mathbf{a}^+$ then*

$$G(1,f;H) = f(e^H)$$

and, for $n \geq 2$,

$$G(n,f;H) = A_n \int_{H_n}^{H_{n-1}} \cdots \int_{H_2}^{H_1} G(n-1, f_{\mathrm{tr}\,H}; \xi) \prod_{i=1}^{n-1} \prod_{j=1}^{n} |\sinh(\xi_i - H_j)|^{-1/2} \delta(\xi)\,d\xi$$

where $f_{\mathrm{tr}\,H}(e^\xi) = f\big(\exp(\mathrm{diag}[\xi, \mathrm{tr}\,H - \mathrm{tr}\,\xi])\big)$ *and* $\delta(\xi)\,d\xi = \prod_{\alpha>0} \sinh\big(\alpha(\xi)\big)\,d\xi$.

This result can be proven if we adapt an idea of I. M. Gelfand and M. A. Naimark, namely using induction on the integration over the group $K = \mathrm{SO}(n)$ (see [5]). We will not provide the proof here as we will use this result strictly for "inspiration".

2. **The False Abel Inverse Transform.**

2.1 *Defining the "False Abel Inverse Transform"*. In the Theorem 1.2, the adjoint of the Abel transform is expressed in a form involving induction. This suggests that we define the "False Abel Inverse Transform" the following way:

DEFINITION 2.1 THE "FALSE ABEL INVERSE TRANSFORM". If $H \in \mathbf{a}^+$ then

$$\mathcal{G}(1,f;H) = f(e^H)$$

and, for $n \geq 2$,

$$\mathcal{G}(n,f;H) = \int_{H_{n-1}}^{H_{n-2}} \cdots \int_{H_1}^{\infty} \mathcal{G}(n-1, f_{\mathrm{tr}\,H}; \xi) \prod_{i=1}^{n-1} \prod_{j=1}^{n} |\sinh(\xi_i - H_j)|^{-1/2} \delta(\xi)\,d\xi.$$

We recall that $f_{\mathrm{tr}\,H}(e^\xi) = f\big(\exp(\mathrm{diag}[\xi, \mathrm{tr}\,H - \mathrm{tr}\,\xi])\big)$.

2.2 *Properties of the "False Abel Inverse Transform"*. We will need some information about $\mathcal{G}(n,f;H)$ (from now on, we assume that $H \in \mathbf{a}^+$).

DEFINITION 2.2. Let $D^{(n)}(H)$ be the support of $f \longrightarrow \mathcal{G}(n,f;H)$ seen as a distribution:

$$D^{(1)}(H) = \{H\} \text{ and, for } n \geq 2,$$

$$D^{(n)}(H) = \Big\{F = \begin{pmatrix} \omega & 0 \\ 0 & \operatorname{tr} H - \operatorname{tr} \omega \end{pmatrix} : \omega \in D^{(n-1)}(\xi), H_i \leq \xi_i \leq H_{i-1}, 1 \leq i \leq n-1\Big\}.$$

To simplify notation, we adopt the convention that $H_0 = \infty$.

It is clear that $D^{(n)}(H) \subset C^{(n)}(H) = \{F \in \mathbf{a} : \Sigma_{k=1}^{r} F_k \geq \Sigma_{k=1}^{r} H_k \ (r \leq n-1)$ and $\operatorname{tr} F = \operatorname{tr} H\}$. As a consequence, if $f \in C_c(A)$ then $\{H \in \mathbf{a}^+ : \mathcal{G}(n,f;H) \neq 0\}$ is bounded.

We will now investigate how fast $\mathcal{G}(n,f;\cdot)$ decreases at infinity.

LEMMA 2.3. *Assume $H \in \mathbf{a}^+$ and f is continuous. We write*

$$\|f\|_H = \sup_{F \in D^{(n)}(H)} |f(e^F)|.$$

We have:

(1) $|\mathcal{G}(n,f;H)| \leq \mathcal{G}(n,|f|;H)$.

(2) $|\mathcal{G}(n,f;H)| \leq C_n \|f\|_H [\delta(H)]^{-1/2}$ *where* $0 < C_n < \infty$ *depends uniquely on* n.

PROOF. (1) The statement is clear since $\mathcal{G}(n,f;H)$ corresponds to an integration against a positive kernel.

(2) It is enough to show that $\mathcal{G}(n,1;H) \leq C_n[\delta(H)]^{-1/2}$. This is true for $n = 1$ with $C_1 = 1$. We assume true for $n-1$ $(n \geq 2)$.

$$\begin{aligned} \mathcal{G}(n,1;H) &= \int_{H_{n-1}}^{H_{n-2}} \cdots \int_{H_1}^{\infty} \mathcal{G}(n-1,1;\xi) \prod_{i=1}^{n-1} \prod_{j=1}^{n} |\sinh(\xi_i - H_j)|^{-1/2} \delta(\xi)\, d\xi \\ &\leq C_{n-1} \int_{H_{n-1}}^{H_{n-2}} \cdots \int_{H_1}^{\infty} \prod_{i=1}^{n-1} \prod_{j=1}^{n} |\sinh(\xi_i - H_j)|^{-1/2} \delta^{1/2}(\xi)\, d\xi. \end{aligned} \tag{7}$$

We now use the following: for $x \geq 0$, $\sinh x \sim \frac{x}{1+x} e^x$ and

$$|\xi_i - H_j| = \begin{cases} \xi_i - H_j & \text{if } i \leq j \\ H_j - \xi_i & \text{if } j < i. \end{cases}$$

We have

$$
\begin{aligned}
&\frac{\delta(\xi)\delta(H)}{\prod_{i=1}^{n-1}\prod_{j=1}^{n}|\sinh(\xi_i-H_j)|}\\
&\leq Ce^{-2(\xi_1-H_1)}\frac{1+\xi_1-H_1}{\xi_1-H_1}\prod_{i=2}^{n-1}\frac{(1+H_{i-1}-\xi_i)(1+\xi_i-H_i)e^{-2(\xi_i-H_i)}}{(H_{i-1}-\xi_i)(\xi_i-H_i)}\\
&\quad\cdot\prod_{i<j<n}\frac{1+\xi_i-H_j}{\xi_i-H_j}\prod_{i<j<n}\frac{\xi_i-\xi_j}{1+\xi_i-\xi_j}\prod_{j=1}^{n-3}\prod_{i=j+2}^{n-1}\frac{1+H_j-\xi_i}{H_j-\xi_i}\\
&\quad\cdot\prod_{i=1}^{n-1}\frac{1+\xi_i-H_n}{\xi_i-H_n}\prod_{i<j\leq n}\frac{H_i-H_j}{1+H_i-H_j}\\
&\leq Ce^{-2(\xi_1-H_1)}\frac{1+\xi_1-H_1}{\xi_1-H_1}\prod_{i=2}^{n-1}\frac{(1+H_{i-1}-H_i)(1+\xi_i-H_i)e^{-2(\xi_i-H_i)}}{(H_{i-1}-\xi_i)(\xi_i-H_i)}\\
&\quad\cdot\prod_{i<j<n}\frac{(1+\xi_i-\xi_j)(1+\xi_j-H_j)}{\xi_i-\xi_j}\prod_{i<j<n}\frac{\xi_i-\xi_j}{1+\xi_i-\xi_j}\\
&\quad\cdot\prod_{j=1}^{n-3}\prod_{i=j+2}^{n-1}\frac{1+H_j-H_{i-1}}{H_j-H_{i-1}}\prod_{i=1}^{n-1}\frac{(1+\xi_i-H_i)(1+H_i-H_n)}{H_i-H_n}\\
&\quad\cdot\prod_{i<j\leq n}\frac{H_i-H_j}{1+H_i-H_j}\\
&\leq Ce^{-2(\xi_1-H_1)}\frac{(1+\xi_1-H_1)^{N_1}}{\xi_1-H_1}\\
&\quad\cdot\prod_{i=2}^{n-1}\frac{(1+H_{i-1}-H_i)(1+\xi_i-H_i)^{N_i}e^{-2(\xi_i-H_i)}}{(H_{i-1}-\xi_i)(\xi_i-H_i)}.
\end{aligned}
\tag{8}
$$

Now it is easy to see that for $a < b$, $\int_a^b \frac{(1+x-a)^N e^{-(x-a)}}{\sqrt{(b-x)(x-a)}}\,dx < C_N\frac{1}{\sqrt{1+b-a}}$ where C_N is independent of a and b. This, coupled with (7) and (8), allows us to conclude. ∎

DEFINITION 2.4. We will say that a function f on $\mathbf{a}$ is *odd* (with respect to the Weyl group W) if given $s \in W, f(s\cdot H) = (\det s)f(H)$ for all H in $\mathbf{a}$. We will use the same term for the corresponding property of functions on A.

THEOREM 2.5. *Suppose f is a smooth odd function of compact support. Then $\mathcal{G}(n,f;\cdot)$ is smooth, Weyl invariant and*

$$\Delta\mathcal{G}(n,f;\cdot) = \mathcal{G}\big(n,\Gamma(\Delta)f;\cdot\big).$$

PROOF (BEGINNING). The radial part of the Laplace-Beltrami operator is given by

$$\Delta = \sum_{j=1}^{n}\frac{\partial^2}{\partial H_j^2} + \sum_{j=1}^{n}\sum_{k\neq j}\coth(H_j-H_k)\frac{\partial}{\partial H_j}.$$

To say that $\mathcal{G}(n,f;\cdot)$ is smooth and Weyl invariant means that it can be extended to a smooth Weyl invariant function on A.

We prove the result by induction. It is clearly valid for $n=1$. If we assume the result true for $n-1$, use the linearity of the transform $\mathcal{G}$ and integration by parts, we obtain

$$\Delta \int_{[\xi:H]} \mathcal{G}(n-1, f_{\mathrm{tr}\,H}; \xi) \prod_{i=1}^{n-1}\prod_{j=1}^{n} |\sinh(\xi_i - H_j)|^z \delta(\xi)\, d\xi$$

$$= \int_{[\xi:H]} \mathcal{G}\big(n-1 (\Gamma(\Delta) f)_{\mathrm{tr}\,H} \xi\big) \prod_{i=1}^{n-1}\prod_{j=1}^{n} |\sinh(\xi_i - H_j)|^z \delta(\xi)\, d\xi$$

$$- (z^2 - 1/4) n(n-1) \int_{[\xi:H]} \mathcal{G}(n-1, f_{\mathrm{tr}\,H}; \xi) \prod_{i=1}^{n-1}\prod_{j=1}^{n} |\sinh(\xi_i - H_j)|^z \delta(\xi)\, d\xi$$

$$+ 2(z + 1/2)$$

$$\cdot \Big[\sum_{p<q} \int_{[\xi:H]} \Big(\frac{\partial}{\partial \xi_p} - \frac{\partial}{\partial \xi_q} \Big) \big(\mathcal{G}(n-1, f_{\mathrm{tr}\,H}; \xi) \coth(\xi_p - \xi_q) \delta(\xi) \big)$$

$$\cdot \prod_{i=1}^{n-1}\prod_{j=1}^{n} |\sinh(\xi_i - H_j)|^z\, d\xi$$

$$- \sum_{k<l} \coth(H_l - H_k) \Big(\frac{\partial}{\partial H_k} - \frac{\partial}{\partial H_l} \Big) \Big(\int_{[\xi:H]} \mathcal{G}(n-1, f_{\mathrm{tr}\,H}; \xi)$$

$$\cdot \prod_{i=1}^{n-1}\prod_{j=1}^{n} |\sinh(\xi_i - H_j)|^z \delta(\xi)\, d\xi \Big) \Big]$$

as long as $\Re z$ is large enough, say greater than 2 ($[\xi\,; H]$ stands for the domain of integration in the definition of $\mathcal{G}$). If we assume that both sides are analytic functions of z provided $\Re z > -1$ then, by analytic continuation, the equality is valid in that domain. Taking $z = -1/2$ gives us the desired result.

However, the right hand side may not be analytic for $-1 < \Re z < 0$ if $\mathcal{G}(n-1, f_{\mathrm{tr}\,H}; \xi)$ does not behave well near $\xi_i = H_j$. For $n=2$ and $n=3$ (if f is odd), it is easy to see that $\mathcal{G}(n-1, f_{\mathrm{tr}\,H}; \xi)$ remains bounded and that the proof above applies. This is enough to justify the development in Sections 3, 4 and 5.

To finish the proof (for $n > 3$) we need tools we want to avoid for now. We will conclude it in Section 6. ∎

We can extend this result to functions that decreases suitably fast.

The results of Sections 1 and 2 can be extended by similar methods to the spaces $\mathrm{Pos}(n, \mathbf{C})$ and $\mathrm{Pos}(n, \mathbf{H})$ (the spaces of positive definite matrices over the complex numbers and over the quaternionic numbers).

3. **The heat kernel for** $\mathrm{Pos}(n, \mathbf{R})$.

3.1 $\mathrm{Pos}(n, \mathbf{R})$. In this section we will use the "False Abel Inverse Transform" to define a "candidate" for the fundamental solution to the heat equation in $\mathrm{Pos}(n, \mathbf{R})$. Any solution to the system $L_A - \gamma^2 = \frac{\partial}{\partial t}$ will be transformed into a solution of the heat equation for the symmetric space provided it is odd and decreases rapidly enough. $\tilde{Q}_t(H) = e^{-\gamma^2 t} t^{-n/2} \exp\big(-r^2(H)/(4t)\big)$ is a solution of $L_A - \gamma^2 = \frac{\partial}{\partial t}$ as will be $D\tilde{Q}_t$ for any differential operator D with constant coefficients.

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The next step is then to try with $\mathcal{G}(n, DQ_t; \cdot)$ where D is a differential operator with constant coefficients such that $D\tilde{Q}_t$ is odd.

PROPOSITION 3.1. *Let* $\partial(\pi) = \prod_{i<j}\left(\frac{\partial}{\partial H_i} - \frac{\partial}{\partial H_j}\right)$ *and* $\pi(H) = \prod_{i<j}(H_i - H_j)$*;*

$$\partial(\pi)\exp\left(-r^2(H)/(4t)\right) = (-2t)^{-n(n-1)/2}\pi(H)\exp\left(-r^2(H)/(4t)\right).$$

PROOF. See for instance [8], Corollary 3.8 in Chapter III. ∎

PROPOSITION 3.2. *If we write* $W_t(H) = Ce^{-\gamma^2 t}t^{-n/2}\partial(\pi)\exp\left(-r^2(H)/(4t)\right)$ *(C a constant) then* $P_t(e^H) = \mathcal{G}(n, W_t; H)$ *is a solution to the heat equation for the symmetric space* $\mathrm{Pos}(n, \mathbf{R})$ *for* $n = 2$ *and* $n = 3$ *(see Section 6 for* $n > 3$*).*

PROOF. First note that $Q_t(H) = t^{-n/2}\exp(-r^2(H)/(4t))$ is the standard solution (modulo a constant) to the euclidean heat equation on A ($L_A Q_t = \frac{\partial}{\partial t}Q_t$). The factor $e^{-\gamma^2 t}$ takes care of the drift so that $\Gamma(\Delta)e^{-\gamma^2 t}Q_t = \frac{\partial}{\partial t}e^{-\gamma^2 t}Q_t$ (we recall that $\Gamma(\Delta) = L_A - \gamma^2$). The differential operator $\partial(\pi)$ commutes with $\Gamma(\Delta)$ and $\frac{\partial}{\partial t}$. Finally, W_t is an odd solution of the equation $\Gamma(\Delta)W_t = \frac{\partial}{\partial t}W_t$. $\Delta P_t(e^H) = \Delta\mathcal{G}(n, W_t; H) = \mathcal{G}\left(n, \Gamma(\Delta)W_t; H\right) = \mathcal{G}(n, \frac{\partial}{\partial t}W_t; H) = \frac{\partial}{\partial t}\mathcal{G}(n, W_t; H) = \frac{\partial}{\partial t}P_t(e^H)$. ∎

It will more practical to consider

$$W_t(e^H) = Ce^{-\gamma^2 t}t^{-n/2}t^{-n(n-1)/2}\prod_{i<j}(H_i - H_j)\exp\left(-r^2(H)/(4t)\right)$$

with $C > 0$. The differential operator $\partial(\pi)$ has played its role in the proof of the result above and is no longer needed.

The function P_t, as we will see in the cases $n = 2$ and $n = 3$, can be written as

$$P_t(e^H) = Ce^{-\gamma^2 t}t^{-(\dim X)/2}\exp\left(-r^2(H)/(4t)\right)E(t, H).$$

$E(t, H)$ is of course the term of interest. In order to prove that $P_t(e^H)$ is indeed the fundamental solution of the heat equation we will have to study the behaviour of $E(t, H)$ when t is close to 0 and, more specifically, the behaviour of $E(t, \sqrt{t}H)$ when t is close to 0. To prove that P_t is the heat kernel, it is enough to show that $0 < E(t, H) < C$ and that $\lim_{t\to 0^+} E(t, \sqrt{t}H) = C$ ($C > 0$).

3.2 $\mathrm{Pos}(2, \mathbf{R})$. Since this can be found in the literature (for instance [12]) we will give the heat kernel for $\mathrm{Pos}(2, \mathbf{R})$ without proof.

We will use the notation $\alpha = \alpha(H) = H_1 - H_2$.

$$\begin{aligned} P_t(e^H) &= \frac{1}{8\pi}e^{-\gamma^2 t}t^{-3/2}\exp\left(-r^2(H)/(4t)\right)t^{-1/2}\int_{H_1}^{\infty}(2F_1 - H_1 - H_2) \\ &\quad \cdot \exp\left(-\left(r^2(F) - r^2(H)\right)/(4t)\right)[\sinh(F_1 - H_1)\sinh(F_1 - H_2)]^{-1/2}\,dF_1 \\ &= \frac{1}{8\pi}e^{-\gamma^2 t}t^{-3/2}\exp\left(-r^2(H)/(4t)\right)t^{-1/2}\int_0^{\infty}(2x + \alpha) \qquad (9) \\ &\quad \cdot \exp\left(-x(x + \alpha)/(2t)\right)[\sinh x\sinh(x + \alpha)]^{-1/2}\,dx. \end{aligned}$$

Note that $F_1 + F_2 = H_1 + H_2$ and $r^2(F) - r^2(H) = 2(F_1 - H_1)(F_1 - H_2)$.

3.3 Pos$(3, \mathbf{R})$. Even with Pos$(3, \mathbf{R})$, the function W_t in the definition of $P_t(e^H) = \mathcal{G}(3, W_t; H)$ is not positive on all the domain of integration (which we called $D^{(3)}(H)$ in Section 2). This is the main difficulty for $n \geq 3$. However, in the case $n = 3$, explicit computations are still possible.

We will use the notation $\alpha = \alpha(H) = H_1 - H_2$ and $\beta = \beta(H) = H_2 - H_3$.

We recall here that $W_t(H) = Ce^{-\gamma^2 t}t^{-n/2}t^{-n(n-1)/2}\prod_{i<j}(H_i - H_j)\exp\bigl(-r^2(H)/(4t)\bigr)$ is an odd function. Indeed, it is easy to show that if h is a Weyl invariant function then $\partial(\pi)h$ becomes an odd function. The reason we stress this here is Proposition 3.5 where it is shown that if h is an odd function then $\mathcal{G}(3, h; \cdot)$ corresponds to the integration of h on a portion of $\mathbf{a}^+$ against a positive kernel. This ensures that $\mathcal{G}(3, W_t; \cdot) \geq 0$.

$$\begin{aligned}
\mathcal{G}(3,f;H) = &\int_{H_2}^{H_1}\int_{H_1}^{\infty}\int_{\xi_1}^{\infty} f\bigl(\exp(\operatorname{diag}[\xi_0, \xi_1 + \xi_2 - \xi_0, \operatorname{tr} H - \xi_1 - \xi_2])\bigr)\\
&\cdot [\sinh(\xi_0 - \xi_1)\sinh(\xi_0 - \xi_2)]^{-1/2}\\
&\cdot \prod_{j=1}^{3} |\sinh(\xi_1 - H_j)\sinh(\xi_2 - H_j)|^{-1/2}\sinh(\xi_1 - \xi_2)\,d\xi_0\,d\xi_1\,d\xi_2\\
= 4&\int_{H_2}^{H_1}\int_{H_1}^{\infty}\int_{\xi_1}^{\infty} f\bigl(\exp(\operatorname{diag}[\xi_0, \xi_1 + \xi_2 - \xi_0, \operatorname{tr} H - \xi_1 - \xi_2])\bigr)\\
&\cdot [\cosh(2\xi_0 - \xi_1 - \xi_2) - \cosh(\xi_1 - \xi_2)]^{-1/2}\,d\xi_0\\
&\cdot \prod_{j=1}^{3} |\cosh(2H_j - \xi_1 - \xi_2) - \cosh(\xi_1 - \xi_2)|^{-1/2}\sinh(\xi_1 - \xi_2)\,d\xi_1\,d\xi_2.
\end{aligned}$$

If we write $F_1 = \xi_0$, $F_2 = \xi_1 + \xi_2 - \xi_0$ ($F_3 = \operatorname{tr} H - F_1 - F_2$) and $z = \xi_1 - \xi_2$ then

$$\mathcal{G}(3,f;H) = 2\int_{\mathcal{D}=\mathcal{C}\cup\mathcal{B}\cup s\cdot\mathcal{B}}\int_{|2H_1-F_1-F_2|}^{\min\{F_1-F_2, F_1+F_2-2H_2\}} f(e^F)[\cosh(F_1 - F_2) - \cosh z]^{-1/2}$$

$$\cdot \sinh z\prod_{j=1}^{3} |\cosh(2H_j - F_1 - F_2) - \cosh z|^{-1/2}\,dz\,dF \tag{10}$$

$$= \int_{\mathcal{D}=\mathcal{C}\cup\mathcal{B}\cup s\cdot\mathcal{B}} f(e^F)\Theta(H,F)\,dF. \tag{11}$$

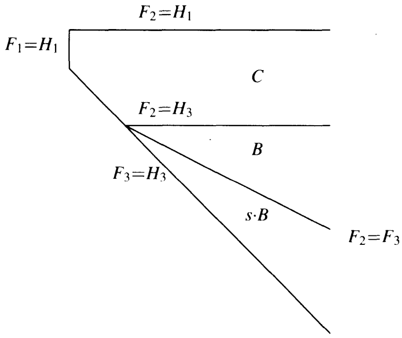
We recall that $H_1 > H_2 > H_3$ ($H \in \mathbf{a}^+$). Since $\operatorname{tr} F = \operatorname{tr} H$, the diagonal matrix F can be described by the coordinates (F_1, F_2).

$$\mathcal{D} = \{F : F_2 \leq H_1,\ H_1 \leq F_1,\ \overbrace{H_1 + H_2 \leq F_1 + F_2}^{F_3 \leq H_3}\}$$

$$C = \{F : H_3 \leq F_2 \leq H_1,\ H_1 \leq F_1,\ \overbrace{H_1 + H_2 \leq F_1 + F_2}^{F_3 \leq H_3}\}$$

$$B = \{F : F_2 \leq H_3,\ \overbrace{H_1 + H_2 + H_3 - F_1 - F_2 \leq F_2}^{F_3 \leq F_2}\}$$

$$s \cdot B = \{F : \overbrace{F_2 \leq H_1 + H_2 + H_3 - F_1 - F_2}^{F_2 \leq F_3},\ \overbrace{H_1 + H_2 \leq F_1 + F_2}^{F_3 \leq H_3}\}.$$



Domain of integration $\mathcal{D}$: coordinates (F_1, F_2).

In order to achieve some symmetry in our results we introduce the following modified elliptic integral:

DEFINITION 3.3.

$$\mathcal{K}(a,b) = \begin{cases} b^{-1/2}K(\sqrt{a/b}) & \text{if } 0 \leq a < b, \\ a^{-1/2}K(\sqrt{b/a}) & \text{if } 0 \leq b < a. \end{cases}$$

where $K(m) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-m^2\sin^2\theta}}$ is the complete elliptic integral of the first kind.

$\mathcal{K}$ is smooth for $a \geq 0$, $b \geq 0$ except on the diagonal.

LEMMA 3.4. *Let (a,b) be in the domain of $\mathcal{K}$.*

(1) $\mathcal{K}(a,b) = \mathcal{K}(b,a)$.

(2) If $c > 0$ *then* $\mathcal{K}(ca, cb) = c^{-1/2}\mathcal{K}(a,b)$.

(3) $\mathcal{K}(a,b) = \frac{\pi}{2}\int_0^\infty \frac{du}{\sqrt{(u^2+|a-b|)(u^2+\max\{a,b\})}}$.

(4) $\mathcal{K}(a,b) = \int_{\max\{\sqrt{a},\sqrt{b}\}}^\infty \frac{dt}{\sqrt{(t^2-a)(t^2-b)}}$ *if* $a \neq b$.

(5) If $\max\{a,b\} \leq \max\{A,B\}$ *and* $|a-b| \leq |A-B|$ *then* $\mathcal{K}(A,B) \leq \mathcal{K}(a,b)$ *($\leq$ can be replaced by $<$). We assume here that (a,b) and (A,B) are in the domain of $\mathcal{K}$.*

(6) $\frac{\pi}{2}[\max\{a,b\}]^{-1/2} \leq \mathcal{K}(a,b) \leq \frac{\pi}{2}|a-b|^{-1/2}$.

(7) If $b > 0$ *then* $\mathcal{K}(0,b) = \frac{\pi}{2}b^{-1/2}$.

PROOF. Either obvious or a consequence of the other properties. ∎

Using (10), the change of variable $x = \cosh z$ and the standard properties of the elliptic integral, $\Theta(H,F)$ is given by the following equations:

If $H_2 < F_2 < H_1$ (*i.e.* $\cosh(2H_1 - F_1 - F_2) < \cosh(F_1 - F_2) < \cosh(2H_2 - F_1 - F_2)$) then

$$\Theta(H,F) = 4\mathcal{K}\Big(\sinh\beta\prod_{j=1}^{3}\sinh(F_j - H_1), -\sinh\alpha\prod_{j=1}^{3}\sinh(F_j - H_3)\Big),$$

if $H_3 < F_2 < H_2$ (*i.e.* $\cosh(2H_2 - F_1 - F_2) < \cosh(F_1 - F_2) < \cosh(2H_3 - F_1 - F_2)$) then

$$\Theta(H,F) = 4\mathcal{K}\Big(-\sinh\alpha \prod_{j=1}^{3}\sinh(F_j - H_3), \sinh\beta \prod_{j=1}^{3}\sinh(F_j - H_1)\Big)$$

and if $F_2 < H_3$ (*i.e.* $\cosh(F_1 - F_2) < \cosh(2H_3 - F_1 - F_2)$) then

$$\Theta(H,F) = 4\mathcal{K}\Big(-\sinh\alpha \prod_{j=1}^{3}\sinh(F_j - H_3), \sinh(\alpha+\beta) \prod_{j=1}^{3}\sinh(F_j - H_2)\Big). \tag{12}$$

Now,

PROPOSITION 3.5. *The notation being as above, if f is odd then*

$$\mathcal{G}(3,f;H) = \int_C f(e^F)\Theta(H,F)\,dF.$$

PROOF. This is clear from (11) and (12). ∎

In particular, we have

$$P_t(e^H) = \mathcal{G}(3, W_t; H) = \int_C W_t(e^F)\Theta(H,F)\,dF.$$

We know that P_t satisfies the heat equation and that $P_t(e^H) \geq 0$ for all $H \in \mathbf{a}^+$. We must now prove that P_t is the fundamental solution.

LEMMA 3.6.

(1) $-\sinh\alpha \prod_{j=1}^3 \sinh(F_j - H_3)$
$- \sinh\beta \prod_{j=1}^3 \sinh(F_j - H_1) = -\sinh(\alpha+\beta)\prod_{j=1}^3 \sinh(F_j - H_2)$.

(2) $-\alpha \prod_{j=1}^3 (F_j - H_3) - \beta \prod_{j=1}^3 (F_j - H_1) = -(\alpha+\beta)\prod_{j=1}^3 (F_j - H_2)$.

(3) *If* $p \neq q$, $r^2(F) - r^2(H) = 2\,\frac{\prod_{k=1}^3 (F_k - H_p) - \prod_{k=1}^3 (F_k - H_q)}{H_p - H_q}$ *if* $p \neq q$ $= (F_1 - H_1)^2 + 2(\alpha + \beta + F_2 - H_2)(F_1 - H_1) + 2(F_2 - H_2)(F_2 - H_3)$ *($\operatorname{tr} F = \operatorname{tr} H$).*

PROOF.

(1) One can multiply both sides by $e^{2\operatorname{tr} H}$ and express the result as sums of exponential terms.

(2) Replacing, in the previous result, H_j by tH_j, F_j by tF_j, dividing both sides by t^4 and letting t tends to 0 we obtain the result.

(3) Can be proven, as in the first part, by direct computations. ∎

We now will write $P_t(e^H) = Ct^{-3}e^{-\gamma^2 t}\exp\big(-r^2(H)/(4t)\big)E(t,H)$ (the constant $C > 0$ absorbing any previous constants) where

$$\begin{aligned} E(t,H) \\ = t^{-3/2} \int_{H_1}^{\infty}\int_{\max\{H_3,H_1+H_2-F_1\}}^{H_1} & \prod_{i<j}(F_i - F_j)\exp\Big(-\big(r^2(F) - r^2(H)\big)/(4t)\Big) \\ & \cdot \prod_{i<j}(F_i - F_j)\exp\Big(-\big(r^2(F) - r^2(H)\big)/(4t)\Big) \\ & \cdot \mathcal{K}\Big(\sinh\beta \prod_{j=1}^3 \sinh(F_j - H_1), -\sinh\alpha \prod_{j=1}^3 \sinh(F_j - H_3)\Big)\,dF_2\,dF_1. \end{aligned}$$

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LEMMA 3.7.

$$0 < E(t,H) \leq \frac{(\sqrt{2\pi})^3}{2}\Big[\frac{\pi(H)}{\delta(H)}\Big]^{1/2} \leq \frac{(\sqrt{2\pi})^3}{2}$$

where, as before, $\pi(H) = \prod_{i<j}(H_i - H_j)$.

PROOF. Only the second inequality requires a proof.

$$\begin{aligned} E(t,H) \leq t^{-3/2} & \int_{H_1}^{\infty}\int_{\max\{H_3,H_1+H_2-F_1\}}^{H_1} \\ & \cdot \prod_{i<j}(F_i - F_j)\exp\big(-\big(r^2(F) - r^2(H)\big)/(4t)\big) \\ & \cdot \mathcal{K}\Big(\frac{\delta(H)}{\alpha(\alpha+\beta)}\prod_{j=1}^{3}(F_j - H_1), -\frac{\delta(H)}{\beta(\alpha+\beta)}\prod_{j=1}^{3}(F_j - H_3)\Big)\, dF_2\, dF_1. \end{aligned}$$

This is a consequence of the fact that for $x \geq 0$, $a \geq 0$, $\sinh(x+a) \geq (x+a)\frac{\sinh a}{a}$ and of the Lemmas 3.4 and 3.6 (the first and second parts). Hence,

$$\begin{aligned} & E(t,H) \\ & \leq t^{-3/2}\Big[\frac{\pi(H)}{\delta(H)}\Big]^{1/2}\int_{H_1}^{\infty}\int_{\max\{H_3,H_1+H_2-F_1\}}^{H_1}\prod_{i<j}(F_i - F_j) \\ & \qquad \cdot \exp\big(-\big(r^2(F) - r^2(H)\big)/(4t)\big) \\ & \qquad \cdot \mathcal{K}\Big(\beta\prod_{j=1}^{3}(F_j - H_1), -\alpha\prod_{j=1}^{3}(F_j - H_3)\Big)\, dF_2\, dF_1 \\ & = t^{-3/2}\Big[\frac{\pi(H)}{\delta(H)}\Big]^{1/2}\big(\pi(H)\big)^{-1} \\ & \qquad \cdot \int_0^{\infty}\int_0^{\infty}\exp\Big(-\Big(\prod_{i<j}(H_i - H_j)\Big)^{-1}[\alpha x_1 + \beta x_2]/(2t)\Big)\mathcal{K}(x_1,x_2)\, dx_2\, dx_1 \\ & = \frac{(\sqrt{2\pi})^3}{2}\Big[\frac{\pi(H)}{\delta(H)}\Big]^{1/2}. \end{aligned}$$

The change of variables used above was $x_1 = \beta\prod_{j=1}^{3}(F_i - H_1)$, $x_2 = -\alpha\prod_{j=1}^{3}(F_i - H_3)$. ∎

This bound is going to be used with the Lebesgue dominated convergence theorem.

COROLLARY 3.8. *If* $H \neq 0$ *then* $\lim_{t\to 0^+} P_t(e^H) = 0$.

PROOF. An immediate consequence of Lemma 3.7. ∎

LEMMA 3.9.

$$\lim_{t\to 0^+} E(t,\sqrt{t}H) = \frac{(\sqrt{2\pi})^3}{2}.$$

PROOF.

$$
\begin{aligned}
&\lim_{t\to 0^+} E(t,\sqrt{t}H)\\
&\quad = \lim_{t\to 0^+} t^{-3/2}\int_{\sqrt{t}H_1}^{\infty}\int_{\max\{\sqrt{t}H_3,\sqrt{t}H_1+\sqrt{t}H_2-F_1\}}^{\sqrt{t}H_1}\prod_{i<j}(F_i-F_j)\\
&\qquad \cdot \exp\Big(-\big(r^2(F)-r^2(\sqrt{t}H)\big)/(4t)\Big)\\
&\qquad \cdot \mathcal{K}\Big(\sinh(\sqrt{t}\beta)\prod_{j=1}^{3}\sinh(F_j-\sqrt{t}H_1),-\sinh(\sqrt{t}\alpha)\\
&\qquad \cdot \prod_{j=1}^{3}\sinh(F_j-\sqrt{t}H_3)\Big)\,dF_2\,dF_1\\
&\quad = \lim_{t\to 0^+}\int_{H_1}^{\infty}\int_{\max\{H_3,H_1+H_2-L_1\}}^{H_1}\prod_{i<j}(L_i-L_j)\exp\Big(-\big(r^2(L)-r^2(H)\big)/4\Big)\\
&\qquad \mathcal{K}\Big(\sinh(\sqrt{t}\beta)/\sqrt{t}\prod_{j=1}^{3}\sinh\big(\sqrt{t}(L_j-H_1)\big)/\sqrt{t},\\
&\qquad -\sinh(\sqrt{t}\alpha)/\sqrt{t}\prod_{j=1}^{3}\sinh\big(\sqrt{t}(L_j-H_3)\big)/\sqrt{t}\Big)\,dL_2\,dL_1\\
&\quad = \int_{H_1}^{\infty}\int_{\max\{H_3,H_1+H_2-L_1\}}^{H_1}\prod_{i<j}(L_i-L_j)\exp\Big(-\big(r^2(L)-r^2(H)\big)/4\Big)\\
&\qquad \cdot \mathcal{K}\Big(\beta\prod_{j=1}^{3}(L_j-H_1),-\alpha\prod_{j=1}^{3}(L_j-H_3)\Big)\,dL_2\,dL_1\\
&\quad = \big(\pi(H)\big)^{-1}\int_0^{\infty}\int_0^{\infty}\exp\Big(-\big(\prod_{i<j}(H_i-H_j)\big)^{-1}[\alpha z_1+\beta z_2]/2\Big)\mathcal{K}(z_1,z_2)\,dz_2\,dz_1\\
&\quad = \frac{(\sqrt{2\pi})^3}{2}.
\end{aligned}
$$

The change of variables used above was $z_1=\beta\prod_{j=1}^{3}(L_i-H_1)$ and $z_2=-\alpha\prod_{j=1}^{3}(L_i-H_3)$. The limit is justified by Lebesgue monotone convergence theorem. ∎

THEOREM 3.10. *P_t is the fundamental solution of the heat equation for* $\mathrm{Pos}(3,\mathbf{R})$ *(with the appropriate constant).*

PROOF. All that remains to prove is that $\lim_{t\to 0^+}\int_{\mathbf{a}^+}P_t(e^H)f(e^H)\delta(H)\,dH=f(I)$.

$$
\begin{aligned}
&\lim_{t\to 0^+}\int_{\mathbf{a}^+}P_t(e^H)f(e^H)\delta(H)\,dH\\
&\qquad = C\lim_{t\to 0^+}e^{-\gamma^2 t}t^{-3}\int_{\mathbf{a}^+}\exp\big(-r^2(H)/(4t)\big)E(t,H)f(e^H)\delta(H)\,dH\\
&\qquad = C\lim_{t\to 0^+}\int_{\mathbf{a}^+}\exp\big(-r^2(\sqrt{t}H)/(4t)\big)E(t,\sqrt{t}H)f(e^{\sqrt{t}H})\frac{\delta(\sqrt{t}H)}{t^{3/2}}\,dH\\
&\qquad = C\int_{\mathbf{a}^+}\exp\big(-r^2(H)/4\big)\frac{(\sqrt{2\pi})^3}{2}f(I)\prod_{i<j}(H_i-H_j)\,dH=16\pi^{5/2}Cf(I).
\end{aligned}
$$

We used the Lebesgue dominated convergence theorem. The only assumptions on f used here are that $f \in L^\infty(A^+)$ and that f is continuous at the identity. ∎

With the previous computations in mind, we choose the constant C to obtain

$$
\begin{aligned}
P_t(e^H) &= \frac{1}{16\pi^{5/2}} t^{-3} e^{-\gamma^2 t} \exp\bigl(-r^2(H)/(4t)\bigr) t^{-3/2} \\
&\quad \cdot \int_{H_1}^{\infty} \int_{\max\{H_3, H_1+H_2-F_1\}}^{H_1} \prod_{i<j} (F_i - F_j) \exp\Bigl(-\bigl(r^2(F) - r^2(H)\bigr)/(4t)\Bigr) \\
&\quad \cdot \mathcal{K}\Bigl(\sinh\beta \prod_{j=1}^{3} \sinh(F_j - H_1), -\sinh\alpha \prod_{j=1}^{3} \sinh(F_j - H_3)\Bigr)\, dF_2\, dF_1.
\end{aligned} \tag{13}
$$

We point out that

$$
\begin{aligned}
&\int_{H_1}^{\infty} \int_{\max\{H_3, H_1+H_2-F_1\}}^{H_1} [\cdots]\, dF_2\, dF_1 \\
&\qquad = \int_{H_1}^{\infty} \int_{H_2}^{H_1} [\cdots]\, dF_2\, dF_1 + \int_{-\infty}^{H_3} \int_{H_3}^{H_2} [\cdots]\, dF_2\, dF_3.
\end{aligned} \tag{14}
$$

If we use the change of variable $x_1 = F_1 - H_1$, $x_2 = F_2 - H_2$ $(H_3 - F_3 = x_1 + x_2)$ in the first integral of (14) and the change of variable $x_1 = H_3 - F_3$, $x_2 = H_2 - F_2$ $(F_1 - H_1 = x_1 + x_2)$ in the second, we find that P_t is a symmetric function of the roots α and β.

4. **The asymptotic expansion of the heat kernel of** $\mathrm{Pos}(n, \mathbf{R})$. We have seen in Section 3 that, for t near 0 and H close to the origin, the behaviour of the heat kernels of $\mathrm{Pos}(2, \mathbf{R})$ and $\mathrm{Pos}(3, \mathbf{R})$ is quite similar to that of the heat kernels of the euclidean spaces of corresponding dimensions, as we should have expected. We will now investigate what happens when $t \longrightarrow \infty$.

As mentioned in the Introduction, we now write

$$
P_t(e^H) = C e^{-\gamma^2 t} t^{-q/2} \exp\bigl(-r^2(H)/(4t)\bigr) \phi_0(H)\, V_t(H) \tag{15}
$$

where q is the dimension at infinity (the constant C will be determined later). We will investigate the asymptotic expansion of $V_t(H)$ in powers of t^{-1}, as t tends to infinity.

The terminology and results concerning asymptotic expansions are standard.

4.1 $\mathrm{Pos}(2, \mathbf{R})$. The dimension at infinity in this case is 4.

We have (see (9))

$$
\begin{aligned}
C\phi_0(H)\, V_t(H) &= \int_{H_1}^{\infty} (2F_1 - H_1 - H_2) \exp\Bigl(-\bigl(r^2(F) - r^2(H)\bigr)/(4t)\Bigr) \\
&\quad \cdot [\sinh(F_1 - H_1) \sinh(F_1 - H_2)]^{-1/2}\, dF_1.
\end{aligned}
$$

We will state, without proofs, the results in this case. The proofs are in the same spirit as those in the case $n = 3$, albeit much simpler.

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THEOREM 4.1. *For $m \geq 0$, let*

$$C\phi_0(H)\, b_m(H) = \frac{2^{-m}}{m!} \int_{H_1}^{\infty} (2F_1 - H_1 - H_2)\Big((F_1 - H_1)(F_1 - H_2)\Big)^m \cdot [\sinh(F_1 - H_1)\sinh(F_1 - H_2)]^{-1/2}\, dF_1.$$

Then,

$$V_t(H) \asymp b_0(H) + \sum_{m=1}^{\infty} (-1)^m b_m(H) t^{-m} \text{ as } t \longrightarrow \infty,$$

uniformly over compact subsets of $\mathbf{a}^+$.

Furthermore, we have $0 \leq C\phi_0(H) b_m(H) \leq C_m\big(1 + r(H)\big)^m (1+\alpha)e^{-\alpha/2}$. *Finally, with an appropriate choice for the constant C, we have* $b_0(H) = 1$ *for all H.*

COROLLARY 4.2. *For all $M \geq 0$,*

$$0 \leq (-1)^M\Big[1 + \sum_{m=1}^{M} (-1)^m b_m(H) t^{-q} - V_t(H)\Big]\phi_0(H)$$
$$\leq b_{M+1}(H)\phi_0(H) \leq C_M\Big(\frac{1 + r(H)}{t}\Big)^{M+1}(1+\alpha)e^{-\alpha/2}.$$

In view of Anker's estimates for the Legendre function ϕ_0, this can be improved. However, we will return to that later as these estimates can be obtained directly from our computations.

4.2 Pos(3, $\mathbf{R}$). The dimension at infinity in this case is 9.

We have (see (13))

$$C\phi_0(H)V_t(H)$$
$$= \int_{H_1}^{\infty} \int_{\max\{H_3, H_1+H_2-F_1\}}^{H_1} \prod_{i<j}(F_i - F_j)\exp\Big(-\big(r^2(F) - r^2(H)\big)/(4t)\Big)$$
$$\cdot\, \mathcal{K}\Big(\sinh\beta \prod_{j=1}^{3}\sinh(F_j - H_1), -\sinh\alpha \prod_{j=1}^{3}\sinh(F_j - H_3)\Big)\, dF_2\, dF_1.$$

We first need some estimates.

LEMMA 4.3. *There exists a polynomial p such that*

$$r^2(F) - r^2(H) \leq \big(1 + r(H)\big)\Big|p\big((F_i - H_i)_{i\leq n}\big)\Big|.$$

PROOF. Using the third part of Lemma 3.6, we write:

$$\big(r^2(F) - r^2(H)\big)/2 = (F_1 - H_1)^2 + 2(\alpha + \beta + F_2 - H_2)(F_1 - H_1)$$
$$+ 2(F_2 - H_2)(F_2 - H_2 + \beta).$$

The result is then an easy consequence of the fact that $\alpha \leq r(H)$ and $\beta \leq r(H)$ (actually $r(H)$ and the largest of α, β and $\operatorname{tr} H$ are of the same order).

LEMMA 4.4. *There exists a constant* $0 < C_m < \infty$ *depending only on m such that*

$$\int_{H_1}^{\infty}\int_{\max\{H_3,H_1+H_2-F_1\}}^{H_1} \prod_{i<j}(F_i - F_j)[r^2(F) - r^2(H)]^m$$
$$\cdot \mathcal{K}\Big(\sinh\beta \prod_{j=1}^{3}\sinh(F_j - H_1), -\sinh\alpha \prod_{j=1}^{3}\sinh(F_j - H_3)\Big)\, dF_2\, dF_1$$
$$\leq C_m\big(1 + r(H)\big)^m(1+\alpha)(1+\beta)(1+\alpha+\beta)\exp\big(-(\alpha+\beta)\big).$$

PROOF. It will be convenient to use the form given in (14). Also, because of the symmetry involved, it is enough to consider only one of the terms. We will repeatedly use the fact that for $x \geq 0$, $\sinh x \sim \frac{x}{1+x}e^x$.

$$\int_{H_1}^{\infty}\int_{H_2}^{H_1} \prod_{i<j}(F_i - F_j)[r^2(F) - r^2(H)]^m$$
$$\cdot \mathcal{K}\Big(\sinh\beta \prod_{j=1}^{3}\sinh(F_j - H_1), -\sinh\alpha \prod_{j=1}^{3}\sinh(F_j - H_3)\Big)\, dF_2\, dF_1$$
$$\leq \frac{\pi}{2}\int_{H_1}^{\infty}\int_{H_2}^{H_1} \prod_{i<j}(F_i - F_j)[r^2(F) - r^2(H)]^m$$
$$\cdot \Big[-\sinh(\alpha+\beta)\prod_{j=1}^{3}\sinh(F_j - H_2)\Big]^{-1/2} dF_2\, dF_1$$

from Lemmas 3.4 and 3.6

$$\leq \frac{\pi}{2}\big(1 + r(H)\big)^m \int_0^{\infty}\int_0^{\alpha}(x_1 - x_2 + \alpha)(x_1 + 2x_2 + \beta)(2x_1 + x_2 + \alpha + \beta)$$
$$\cdot |p(x)|^m[\sinh(\alpha+\beta)\sinh(x_1+\alpha)\sinh x_2 \sinh(x_1 + x_2 + \beta)]^{-1/2}\, dx_2\, dx_1$$

$x_1 = F_1 - H_1,\ x_2 = F_2 - H_2,\ p$ as in the Lemma 4.3

$$\leq C\big(1 + r(H)\big)^m \exp\big(-(\alpha+\beta)\big)\sqrt{\frac{1+\alpha+\beta}{\alpha+\beta}}$$
$$\cdot \int_0^{\infty}\int_0^{\alpha}(x_1+\alpha)(x_1+2x_2+\beta)(2x_1+x_2+\alpha+\beta)\sqrt{1+x_1+\alpha}$$
$$\cdot |p(x)|^m \exp(-x_1 - x_2/2)\Big[(x_1+\alpha)\sinh x_2 \frac{x_1+x_2+\beta}{1+x_1+x_2+\beta}\Big]^{-1/2} dx_2\, dx_1$$
$$\leq C\big(1 + r(H)\big)^m \exp\big(-(\alpha+\beta)\big)\sqrt{\frac{1+\alpha+\beta}{\alpha+\beta}}$$
$$\cdot \int_0^{\infty}\int_0^{\alpha}\sqrt{x_1+\alpha}(x_1+2x_2+\beta)(2x_1+x_2+\alpha+\beta)$$
$$\cdot \sqrt{1+x_1+\alpha}|p(x)| \exp(-x_1 - x_2/2)\Big[\sinh x_2 \frac{x_1+x_2}{1+x_1+x_2}\Big]^{-1/2} dx_2\, dx_1$$

$$\leq C\big(1+r(H)\big)^m \exp\big(-(\alpha+\beta)\big)(1+\alpha)(1+\beta)(1+\alpha+\beta)\sqrt{\frac{1+\alpha+\beta}{\alpha+\beta}}$$
$$\cdot \int_0^\infty\int_0^\alpha \sqrt{x_1+1}(x_1+2x_2+1)(2x_1+x_2+1)\sqrt{2+x_1}\sqrt{1+x_1+x_2}$$
$$\cdot |p(x)| \exp(-x_1-x_2/2)[\sinh x_2(x_1+x_2)]^{-1/2}\,dx_2\,dx_1.$$

The last integral is like $\int_0^\infty\int_0^\infty \tilde{p}(x)\exp(-x_1-x_2/2)[\sinh x_2(x_1+x_2)]^{-1/2}\,dx_2\,dx_1$ ($\tilde{p}$ a polynomial) if $\alpha > 1$ and like $\int_0^\alpha \frac{dx}{\sqrt{x}}$ if $\alpha \leq 1$. In either case, we can conclude. ∎

THEOREM 4.5. *For $m \geq 0$, let*

$$C\phi_0(H)b_m(H)$$
$$= \frac{1}{4^m m!}\int_{\max\{H_3,H_1+H_2-F_1\}}^{H_1} \prod_{i<j}(F_i-F_j)\big(r^2(F)-r^2(H)\big)^m$$
$$\cdot \mathcal{K}\Big(\sinh\beta\prod_{j=1}^3 \sinh(F_j-H_1), -\sinh\alpha\prod_{j=1}^3\sinh(F_j-H_3)\Big)\,dF_2\,dF_1.$$

Then,

$$V_t(H) \asymp b_0(H) + \sum_{m=1}^\infty (-1)^m b_m(H)t^{-m} \quad as \quad t\longrightarrow\infty,$$

uniformly over compact subsets of $\mathbf{a}^+$.

Furthermore, we have

$$0 \leq C\phi_0(H)b_m(H) \leq C_m\big(1+r(H)\big)^m(1+\alpha)(1+\beta)(1+\alpha+\beta)e^{-(\alpha+\beta)}.$$

Finally, with an appropriate choice for the constant C, we have $b_0(H) = 1$ for all H.

PROOF. Naively, this comes to expanding the term $\exp\big(-\big(r^2(F)-r^2(H)\big)/(4t)\big)$ in powers of t^{-1}.

We have, for each $M \geq 0$,

$$0 \leq (-1)^M\Big[\sum_{m=0}^M \frac{(-1)^m}{4^m m!}\big(r^2(F)-r^2(H)\big)^m t^{-m} - \exp\big(-\big(r^2(F)-r^2(H)\big)/(4t)\big)\Big]$$
$$\leq \frac{1}{4^{M+1}(M+1)!}\big(r^2(F)-r^2(H)\big)^{M+1}t^{-(M+1)}.$$

This is a simple application of the Lagrange remainder theorem to the function e^{-x}. The asymptotic expansion for $V_t(H)$ and the upper bounds on the coefficients $b_m(H)$ follow immediately from Lemma 4.4.

$V_t(H)$ is analytic in t in the domain $\Re t > 0$. Indeed, if Υ is any closed curve in the domain $\Re t > 0$, $\int_\Upsilon V_t(H)\,dt = \int_C\Big[\int_\Upsilon \exp\big(-\big(r^2(F)-r^2(H)\big)/(4t)\big)\,dt\Big]\pi(F)\Theta(H,F)\,dF = 0$ (whenever $\Re t > 0$, $|\exp\big(-\big(r^2(F)-r^2(H)\big)/(4t)\big)| < 1$: our bounds in Lemma 4.4 allow the use of Lebesgue dominated convergence theorem).

This is enough to ensure that $\frac{\partial}{\partial t}V_t(H)$ has an asymptotic expansion, the expansion of $V_t(H)$ differentiated term by term.

From the uniform convergence of the asymptotic expansion of $V_t(H)$ over compact subsets of $\mathbf{a}^+$, the fact that Δ is self-adjoint and that $\Delta P_t = \frac{\partial}{\partial t}P_t$, we can conclude that $\Delta(\phi_0 b_0) = -\gamma^2(\phi_0 b_0)$. That implies that b_0 is a constant function. ∎

COROLLARY 4.6. *For all $M \geq 0$,*

$$0 \leq (-1)^M \Big[1 + \sum_{m=1}^{M} (-1)^m b_m(H) t^{-q} - V_t(H)\Big] \phi_0(H)$$
$$\leq b_{M+1}(H)\phi_0(H)$$
$$\leq C_M \Big(\frac{1 + r(H)}{t}\Big)^{M+1} e^{-(\alpha+\beta)}(1+\alpha)(1+\beta)(1+\alpha+\beta).$$

PROOF. This is a direct consequence of the proof of the theorem. ∎

5. **Estimates.** We mention in the Introduction our intention to prove that Jean-Philippe Anker's upper bound,

$$V_t(H) \leq C \prod_{\alpha \in \Sigma_0^+} \Big(1 + \frac{1+\alpha}{t}\Big)^{(m_\alpha + m_{2\alpha})/2-1}, \tag{16}$$

is valid for the symmetric spaces $\mathrm{Pos}(2, \mathbf{R})$ and $\mathrm{Pos}(3, \mathbf{R})$ and that a constant multiple of that upper bound will serve as lower bound. We will endeavour to fulfill this program in this section.

5.1 $\mathrm{Pos}(2, \mathbf{R})$. For the space of positive definite matrices, Anker's bound (16) can be written in simpler terms. Indeed, the roots of the symmetric space $\mathrm{Pos}(n, \mathbf{R})$ are all indivisible and of multiplicity one.

In this case, we have

$$P_t(e^H) = Ce^{-\gamma^2 t} t^{-4/2} \exp\big(-r^2(H)/(4t)\big)\phi_0(H)V_t(H).$$

For the space $\mathrm{Pos}(2, \mathbf{R})$, the bound for $V_t(H)$ is given by

$$C\Big(1 + \frac{1+\alpha}{t}\Big)^{-1/2}.$$

As before, we will state the results for the case $n = 2$ without proofs: the case $n = 3$ is again much more difficult and is sufficiently suggestive as to what the proofs would be in the simpler case.

THEOREM 5.1. *There exists a constant $C > 0$ such that*

$$V_t(H) \geq C\Big(1 + \frac{1+\alpha}{t}\Big)^{-1/2}.$$

A consequence of the proof (as can be seen in the case $n = 3$), is that

COROLLARY 5.2. *There exists $C_1 > 0$ and $C_2 > 0$ such that*

$$C_1 e^{-\alpha/2}(1+\alpha) \leq \phi_0(e^H) \leq C_2 e^{-\alpha/2}(1+\alpha)$$

for all $H \in \overline{\mathbf{a}^+}$.

This result has been known for a long time.

5.1.1. *Upper bound.* On the other hand, it is even easier to prove that

THEOREM 5.3. *There exists a constant $C < \infty$ such that*

$$V_t(H) \le C\Big(1 + \frac{1+\alpha}{t}\Big)^{-1/2}.$$

5.2 Pos(3, **R**). As in the previous section, we can state Anker's bound (16) in simpler terms.

In this case, we have

$$P_t(e^H) = Ce^{-\gamma^2 t}t^{-9/2}\exp\big(-r^2(H)/(4t)\big)\phi_0(H)V_t(H).$$

For the space Pos(3, **R**), the bound for $V_t(H)$ is given by

$$C\Big(1 + \frac{1+\alpha}{t}\Big)^{-1/2}\Big(1 + \frac{1+\beta}{t}\Big)^{-1/2}\Big(1 + \frac{1+\alpha+\beta}{t}\Big)^{-1/2}.$$

As before, $\alpha = \alpha(H) = H_1 - H_2$ and $\beta = \beta(H) = H_2 - H_3$.

We will first prove that with the appropriate choice of positive constant C, the above expression constitutes a lower bound for $V_t(H)$. We will then show that, with a different choice of positive constant C, the above expression can also serve as an upper bound for $V_t(H)$.

5.2.2. *Lower bound.*

THEOREM 5.4.

$$V_t(H) \ge C\Big(1 + \frac{1+\alpha}{t}\Big)^{-1/2}\Big(1 + \frac{1+\beta}{t}\Big)^{-1/2}\Big(1 + \frac{1+\alpha+\beta}{t}\Big)^{-1/2}.$$

PROOF. We will first show that

$$\begin{aligned}\phi_0(H)\, V_t(H) &\ge Ce^{-(\alpha+\beta)}(1+\alpha)(1+\beta)(1+\alpha+\beta)\\ &\quad\cdot\Big(1 + \frac{1+\alpha}{t}\Big)^{-1/2}\Big(1 + \frac{1+\beta}{t}\Big)^{-1/2}\Big(1 + \frac{1+\alpha+\beta}{t}\Big)^{-1/2}.\end{aligned} \tag{17}$$

The proof of that equation is rather cumbersome. The strategy is to break it down into several cases, proving each of them in turn. Going here through all these cases would be rather tedious. We will instead indicate what these cases are, provide some pointers and give an example; the rest will be left as "exercises".

We first note that since the problem is symmetric in α and β, we may assume without loss of generality that $\beta \le \alpha$. We also can make use of the fact that α and $\alpha+\beta$ are then of the same order. We also recall that for $x \ge 0$, $\sinh x \sim \frac{x}{1+x}e^x$.

We also have $\mathcal{K}(a,b) \ge \frac{\pi}{2}\frac{1}{\sqrt{\max\{a,b\}}}$; if we combine that observation with Lemma 3.6 and the first term of (14) along with the change of variable suggested there, we have

$$\begin{aligned}&\phi_0(H)\, V_t(H)\\ &\quad\ge C\int_0^\infty\int_0^\alpha (x_1 - x_2 + \alpha)(x_1 + 2x_2 + \beta)(2x_1 + x_2 + \alpha + \beta)\\ &\qquad\cdot\exp\big(-\big(x_1^2 + 2x_1(x_2 + \alpha + \beta) + 2x_2(x_2 + \beta)\big)/(2t)\big)\\ &\qquad\cdot[\sinh\alpha\,\sinh(x_1 + \alpha + \beta)\sinh(x_2 + \beta)\sinh(x_1 + x_2)]^{-1/2}\, dx_2\, dx_1\end{aligned}$$

where C is a generic constant.

We break down the proof of (17) into the following cases:

$$1 \le \beta \le \alpha : \begin{cases} t \ge \alpha \\ \beta \le t \le \alpha \\ t \le \beta \end{cases}$$

$$\beta \le 1 \le \alpha : \begin{cases} t \ge \alpha \\ 1 \le t \le \alpha \\ \beta \le \sqrt{t} \le 1 \\ \sqrt{t} \le \beta \end{cases} \quad \text{and}$$

$$\beta \le \alpha \le 1 : \begin{cases} t \ge 1 \\ \alpha \le \sqrt{t} \le 1 \\ \beta \le \sqrt{t} \le \alpha \\ \sqrt{t} \le \beta . \end{cases}$$

We prove the case $\beta \le 1 \le \alpha, \beta \le \sqrt{t} \le 1$ as an example:

$$\begin{aligned}
&\phi_0(H)V_t(H) \\
&\quad \ge \int_0^\infty \int_0^\alpha (x_1 - x_2 + \alpha)(x_1 + 2x_2 + \beta)(2x_1 + x_2 + \alpha + \beta) \\
&\qquad \cdot \exp\Big(-\big(x_1^2 + 2x_1(x_2 + \alpha + \beta) + 2x_2(x_2 + \beta)\big)/(2t)\Big) \\
&\qquad \cdot [\sinh\alpha \sinh(x_1 + \alpha + \beta)\sinh(x_2 + \beta)\sinh(x_1 + x_2)]^{-1/2}\, dx_2\, dx_1 \\
&\quad \ge Ce^{-\alpha}(\alpha/2)(\alpha + \beta) \int_0^\infty \int_0^{1/2} (x_1 + 2x_2 + \beta) \\
&\qquad \cdot \exp\Big(-\big(x_1^2 + 2x_1(x_2 + \alpha + \beta) + 2x_2(x_2 + \beta)\big)/(2t)\Big) e^{-x_1/2} \\
&\qquad \cdot [\sinh(x_2 + \beta)\sinh(x_1 + x_2)]^{-1}\, dx_2\, dx_1 \\
&\quad \ge Ce^{-\alpha}(\alpha/2)(\alpha + \beta) \int_0^\infty \int_0^{1/2} (x_1 + 2x_2 + \beta) \\
&\qquad \cdot \exp\Big(-\big(x_1^2 + 2x_1(x_2 + \alpha + \beta) + 2x_2(x_2 + \beta)\big)/(2t)\Big) e^{-x_1/2} \\
&\qquad \cdot [\sinh(x_2 + \beta)\sinh(x_1 + x_2)]^{-1/2}\, dx_2\, dx_1 .
\end{aligned}$$

Now,

$$\begin{aligned}
&\int_0^\infty \int_0^{1/2} (x_1 + 2x_2 + \beta)\exp\Big(-\big(x_1^2 + 2x_1(x_2 + \alpha + \beta) + 2x_2(x_2 + \beta)\big)/(2t)\Big) \\
&\qquad \cdot e^{-x_1/2}[\sinh(x_2 + \beta)\sinh(x_1 + x_2)]^{-1/2}\, dx_2 dx_1 \\
&\quad \ge \frac{t}{\alpha} \int_0^\infty \int_0^{1/2} (2y_2 + \beta)\exp\Big(-\big(y_1^2 + 2y_1(y_2 + 1 + 1)\big)/2\Big) e^{-y_2^2/t - \beta y_2/t} e^{-y_1/2} \\
&\qquad \cdot \Big[\sinh(y_2 + \beta)\sinh(\frac{t}{\alpha} y_1 + y_2)\Big]^{-1/2} dy_2\, dy_1 \qquad y_1 = \frac{\alpha}{t} x_1,\ y_2 = x_2
\end{aligned}$$

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$$\geq C\frac{t}{\alpha}\int_0^1\int_0^{1/2}(y_2+\beta)e^{-y_2^2/t-\beta y_2/t}[(y_2+\beta)(ty_1+y_2)]^{-1/2}\,dy_2\,dy_1$$
$$\geq C\frac{t^{3/2}}{\alpha}\int_0^1\int_0^{1/(2\sqrt{t})}(\sqrt{t}z_2+\beta)^{1/2}e^{-z_2^2-\beta z_2/\sqrt{t}}[tz_1+\sqrt{t}z_2]^{-1/2}\,dz_2\,dz_1$$
$$z_1=y_1,\ z_2=y_2/\sqrt{t}$$
$$\geq C\frac{t^{3/2}}{\alpha}\int_0^1\int_0^{1/2}(\sqrt{t}z_2)^{1/2}[\sqrt{t}z_1+\sqrt{t}z_2]^{-1/2}\,dz_2\,dz_1$$
$$= C\frac{t^{3/2}}{\alpha}\int_0^1\int_0^{1/2}z_2^{1/2}[z_1+z_2]^{-1/2}\,dz_2\,dz_1.$$

The other cases require similar methods.

If we refer to Theorem 4.5 together with (17), we have:

$$Ce^{-(\alpha+\beta)}(1+\alpha)(1+\beta)\Big(1+\frac{1+\alpha}{t}\Big)^{-1/2}\Big(1+\frac{1+\beta}{t}\Big)^{-1/2}\Big(1+\frac{1+\alpha+\beta}{t}\Big)^{-1/2}$$
$$\leq \phi_0(H)\,V_t(H)\leq C'e^{-(\alpha+\beta)}(1+\alpha)(1+\beta)(1+\alpha+\beta).$$

We take the limit as t tends to infinity; the result is $Ce^{-(\alpha+\beta)}(1+\alpha)(1+\beta)(1+\alpha+\beta)\leq \phi_0(H)\leq C'e^{-(\alpha+\beta)}(1+\alpha)(1+\beta)(1+\alpha+\beta)$.

This, together with (17), allows us to conclude. ∎

We have proven the following corollary:

COROLLARY 5.5. *There exists* $C_1>0$ *and* $C_2>0$ *such that*

$$C_1e^{-(\alpha+\beta)}(1+\alpha)(1+\beta)(1+\alpha+\beta)\leq\phi_0(H)$$
$$\leq C_2e^{-(\alpha+\beta)}(1+\alpha)(1+\beta)(1+\alpha+\beta)$$

for all $H\in\overline{\mathbf{a}^+}$.

This result was obtained first by Carl Herz in [10].

5.2.2. *Upper bound*

THEOREM 5.6.

$$V_t(H)\leq C\Big(1+\frac{1+\alpha}{t}\Big)^{-1/2}\Big(1+\frac{1+\beta}{t}\Big)^{-1/2}\Big(1+\frac{1+\alpha+\beta}{t}\Big)^{-1/2}.$$

PROOF. As a consequence of Corollary 5.5, it is enough to show that

$$\phi_0(H)\,V_t(H)\leq Ce^{-(\alpha+\beta)}(1+\alpha)(1+\beta)(1+\alpha+\beta)$$
$$\cdot\Big(1+\frac{1+\alpha}{t}\Big)^{-1/2}\Big(1+\frac{1+\beta}{t}\Big)^{-1/2}\Big(1+\frac{1+\alpha+\beta}{t}\Big)^{-1/2}.$$

The proof of that equation can also be broken into cases, each of them to be proven in turn. We will indicate what these cases are, give some pointers and other indications; the rest will be left as "exercises".

We again will assume that $\beta\leq\alpha$.

We also have $\mathcal{K}(a,b) \leq \frac{\pi}{2}\frac{1}{\sqrt{|a-b|}}$. If we refer to (14), we can write $\phi_0(H)\, V_t(H) = \int_0^\infty\int_0^\alpha \cdots + \int_0^\infty\int_0^\beta \cdots = I_1 + I_2$. We then have

$$I_1 \leq C\int_0^\infty\int_0^\alpha (x_1 - x_2 + \alpha)(x_1 + 2x_2 + \beta)(2x_1 + x_2 + \alpha + \beta)$$
$$\cdot \exp\Big(-\big(x_1^2 + 2x_1(x_2 + \alpha + \beta) + 2x_2(x_2 + \beta)\big)/(2t)\Big)$$
$$\cdot [\sinh(\alpha+\beta)\sinh(x_1+\alpha)\sinh x_2 \sinh(x_1 + x_2 + \beta)]^{-1/2}\, dx_2\, dx_1$$

where C is a generic constant. A similar bound can be found for I_2. It is important to keep in mind that we have two terms to consider.

The cases $t \leq 1+\beta$ and $0 < \beta \leq \alpha \leq 1$ are consequences of Lemma 3.7. The case $t \geq 1+\alpha$ is easily derived from Corollary 4.6. The other "cases" are $1 \leq \beta \leq \alpha$, and $\beta \leq 1 \leq \alpha$ with $1+\beta \leq t \leq 1+\alpha$.

These cases are solved using the same techniques as in the previous section and in Section 4. ∎

Finally, we state the results of Sections 4 and 5 in the most general form possible:

THEOREM 5.7. *Consider the symmetric spaces* Pos(2, **R**) *and* Pos(3, **R**). *Suppose* P_t *is the fundamental solution of the heat equation and write*

$$P_t(e^H) = Ce^{-\gamma^2 t}t^{-q/2}\exp\big(-r^2(H)/(4t)\big)\phi_0(H)V_t(H)$$

where q *is the dimension at infinity.*

Then,

$$0 \leq (-1)^M\Big[1 + \sum_{m=1}^{M} b_m(H)t^{-m} - V_t(H)\Big]$$
$$\leq b_{M+1}(H)t^{-(M+1)} \leq C_M\Big(\frac{1+r(H)}{t}\Big)^{M+1}$$

for $M \geq 0$, *and*

$$A\prod_{\alpha\in\Sigma_0^+}\Big(1 + \frac{1+\alpha}{t}\Big)^{(m_\alpha+m_{2\alpha})/2-1} \leq V_t(H)$$
$$\leq B\prod_{\alpha\in\Sigma_0^+}\Big(1+\frac{1+\alpha}{t}\Big)^{(m_\alpha+m_{2\alpha})/2-1}$$

(with $0 < A < B$*).*

6. **The False Abel Inverse Transform for** Pos(n, **R**). The main goal of this section is to conclude the proof of Theorem 2.5 started in Section 2. Perhaps we should stress here the difficulty we encountered in the proof as begun in Section 2. If we assume that f is odd, smooth and has compact support, we can use the induction hypothesis to show that $\Delta\mathcal{G}(n,f;\cdot) = \mathcal{G}\big(n,\Gamma(\Delta)f;\cdot\big)$. The problem is to show that $\mathcal{G}(n,f;\cdot)$ is "nice" too. This is a necessary ingredient in the proof by induction as shown earlier.

LEMMA 6.1. *Let $i\nu(F) = \sum_{k=1}^n b_k F_k$ be such that $\Re(b_p - b_q) < 1$ whenever $p < q$. Then, there exists a nonzero constant C which depends only on n such that*

$$\lim_{t\to\infty} e^{(-i\nu+\rho)(H(t))} \mathcal{G}(n, e^{i\nu}; e^{H(t)}) = \prod_{p<q\leq n} B((1-(b_p-b_q))/2, 1/2)$$

provided $\lim_{t\to\infty}\big(H_i(t) - H_{i+1}(t)\big) = \infty$ *(*$1 \leq i \leq n-1$*).*

PROOF. When we use this lemma, $H(t)$ will be tH with $H \in \mathbf{a}^+$. We introduce $H(t)$ in order to resolve a technical difficulty in the proof (see equation (18)).

We assume first that $\Re(b_p - b_q) < 0$ whenever $p < q$. Note that

$$e^{-i\nu(H(t))} \mathcal{G}\big(n - 1(e^{i\nu})_{\mathrm{tr}\, H(t)}; e^{\xi}\big) = e^{-i\nu_0(\tilde H(t))} \mathcal{G}(n-1, e^{i\nu_0}; e^{\xi})$$

where $\tilde H(t) = \mathrm{diag}[H_i(t)]_{1\leq i\leq n-1}$ and $i\nu_0(\xi) = \sum_{k=1}^{n-1}(b_k - b_n)\xi_k$.

This opens the way to a proof by induction. The lemma is clearly true for $n = 1$ (by convention, an empty product is equal to 1). Assume it is true for $n-1$ ($n \geq 2$).

We want to compute

$$\lim_{t\to\infty} \int_{H_{n-1}(t)}^{H_{n-2}(t)} \cdots \int_{H_1(t)}^{\infty} e^{-i\nu_0(\tilde H(t))} \mathcal{G}(n-1, e^{i\nu_0}; e^{\xi}) e^{\rho(H(t))} \delta(\xi) \\ \cdot \prod_{i=1}^{n-1}\prod_{j=1}^{n} |\sinh\big(\xi_i - H_j(t)\big)|^{-1/2}\, d\xi.$$

The idea is to make the change of variable $x_i = \xi_i - H_i(t)$, $1 \leq i \leq n-1$ and use Lebesgue dominated convergence theorem to evaluate the limit. Unfortunately, if x_i is near $H_{i-1}(t) - H_i(t)$ the integrand is not bounded as a function of t since $|\sinh\big(x_i + \big(H_i(t) - H_{i-1}(t)\big)\big)|^{-1/2}$ is then like $\big(\big(H_{i-1}(t) - H_i(t)\big) - x_i\big)^{-1/2}$ (elsewhere it is like $\sqrt{2}\exp\big(-\big(\big(H_{i-1}(t) - H_i(t)\big) - x_i\big)/2\big)$).

We consider the following sets: for $2 \leq k \leq n-1$, if D is the domain of integration, $D_k = \big\{\xi \in D : \xi_k \geq H_k(t) + \big(H_{k-1}(t) - H_k(t)\big)/2\big\}$ and $D' = D - \cup_{2\leq k\leq n-1} D_k$. Using Lemma 2.3,

$$\begin{aligned} |e^{-i\nu_0(\tilde H(t))} \mathcal{G}(n-1, e^{i\nu_0}; e^{\xi})| &= |\mathcal{G}(n-1, e^{-i\nu_0(\tilde H(t))} e^{i\nu_0}; e^{\xi})| \\ &\leq \|e^{-i\nu_0(\tilde H(t))} e^{i\nu_0}\|_{\xi}\, \mathcal{G}(n-1, 1; e^{\xi}). \end{aligned}$$

If $\omega \in D^{(n-1)}(\xi)$, then $\sum_{r=1}^p \omega_r \geq \sum_{r=1}^p \xi_r$ ($1 \leq p \leq n-1$). If we combine this with the inequalities $\Re(b_i - b_j) < 0$ whenever $i < j$ and $\xi_k \geq H_k(t)$ $(1 \leq k \leq n-1)$, then

$$\begin{aligned}
&\Re\Big(-i\nu_0\big(\tilde{H}(t)\big) + i\nu_0(\omega)\Big) \\
&\qquad = \sum_{p=1}^{n-1} \Re(b_p - b_{p+1})\Big(\sum_{r=1}^{p} \omega_r - \sum_{r=1}^{p} H_r(t)\Big) \\
&\qquad \leq \Re(b_k - b_{k+1})\Big(\sum_{r=1}^{k} \omega_r - \sum_{r=1}^{k} H_r(t)\Big) \\
&\qquad \leq \Re(b_k - b_{k+1})\Big(\sum_{r=1}^{k} \xi_r - \sum_{r=1}^{k} H_r(t)\Big) \\
&\qquad \leq \Re(b_k - b_{k+1})\big(\xi_k - H_k(t)\big) \\
&\qquad \leq \Re(b_k - b_{k+1})\big(H_{k-1}(t) - H_k(t)\big)/2 \text{ (whenever } \xi \in D_k) .
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big|\int_{D_k} e^{-i\nu_0(\tilde{H}(t))} \mathcal{G}(n-1, e^{i\nu_0}; e^{\xi})\delta(\xi) e^{\rho(H(t))} \prod_{i=1}^{n-1}\prod_{j=1}^{n} \big|\sinh\big(\xi_i - H_j(t)\big)\big|^{-1/2} d\xi\Big| \\
&\qquad \leq e^{\Re(b_k - b_{k+1})(H_{k-1}(t) - H_k(t))/2} e^{\rho(H(t))} \int_D \mathcal{G}(n-1, 1; e^{\xi})\delta(\xi) \\
&\qquad\qquad \cdot \prod_{i=1}^{n-1}\prod_{j=1}^{n} \big|\sinh\big(\xi_i - H_j(t)\big)\big|^{-1/2} d\xi \\
&\qquad = e^{\Re(b_k - b_{k+1})(H_{k-1}(t) - H_k(t))/2} e^{\rho(H(t))} \mathcal{G}(n, 1; e^{H(t)})
\end{aligned}$$

which tends to 0 as t tends to ∞ $\Big(\mathcal{G}\big(n, 1; H(t)\big) \leq C\big[\delta\big(H(t)\big)\big]^{-1/2} \sim e^{-\rho(H(t))}\Big)$.

The same reasoning shows that $\|e^{-i\nu_0(\tilde{H}(t))} e^{i\nu_0}\|_{\xi}$ is uniformly bounded for $\xi \in D$.

To compute the limit, we can now replace the domain of integration D by D'. We now use the change of variable mentioned above:

$$\begin{aligned}
&\lim_{t \to \infty} \int_0^{(H_{n-2}(t) - H_{n-1}(t))/2} \cdots \int_0^{\infty} e^{(-i\nu_0 + \rho_{n-1})(\tilde{H}(t))} \mathcal{G}(n-1, e^{i\nu_0}; e^{x + \tilde{H}(t)}) \\
&\qquad \cdot \prod_{i<j\leq n-1} \sinh\Big(x_i - x_j + \big(H_i(t) - H_j(t)\big)\Big) e^{\sum_{i=1}^{n-1}(H_i(t) - H_n(t))/2} \\
&\qquad \cdot \prod_{i=1}^{n-1}\prod_{j=1}^{n} \big|\sinh\big(x_i + H_i(t) - H_j(t)\big)\big|^{-1/2} dx.
\end{aligned}$$

By the induction hypothesis,

$$\lim_{t \to \infty} e^{(-i\nu_0 + \rho_{n-1})(\tilde{H}(t))} \mathcal{G}(n-1, e^{i\nu_0}; e^{x + \tilde{H}(t)}) = C_{n-1}(\nu_0) e^{(i\nu_0 - \rho_{n-1})(x)} \tag{18}$$

where $C_{n-1}(\nu_0) = \prod_{p<q\leq n-1} B((1 - (b_p - b_q))/2, 1/2)$.

We use again Lemma 2.3 to show that the integrand is bounded by $Ce^{-\sum_{i=1}^{n-1} x_i/2} \prod_{i=1}^{n-1} \sinh^{-1/2} x_i$.

Finally, the limit becomes

$$\begin{aligned}
&2^{(n-1)/2}\int_0^\infty\cdots\int_0^\infty e^{i\nu_0(x)-\rho_{n-1}(x)}C_{n-1}(\nu_0)e^{\sum_{i=1}^{n-1}(n-1-2i)x_i/2}\prod_{i=1}^{n-1}\sinh^{-1/2}x_i\,dx\\
&\quad= 2^{(n-1)/2}C_{n-1}(\nu_0)\int_0^\infty\cdots\int_0^\infty e^{i\nu_0(x)}e^{-\sum_{i=1}^{n-1}x_i/2}\prod_{i=1}^{n-1}\sinh^{-1/2}x_i\,dx\\
&\quad= 2^{(n-1)/2}C_{n-1}(\nu_0)\prod_{i=1}^{n-1}\int_0^\infty e^{((b_i-b_n)-1/2)x_i}\sinh^{-1/2}x_i\,dx_i\\
&\quad= 2^{(n-1)/2}C_{n-1}(\nu_0)\prod_{i=1}^{n-1}\frac{1}{\sqrt{2}}B\big(1/2-(b_i-b_n)/2,1/2\big)
\end{aligned}$$

which gives the desired expression.

By imitating the proof of Lemma 2.3, one finds that if $\Re(b_p-b_q)<K<1$ whenever $p<q$, there exits $0<C_K<\infty$ such that $|\mathcal{G}(n,e^{i\nu};e^H)|\le C_K e^{\Re(i\nu(H))}[\delta(H)]^{-1/2}$ ($\lim_{K\to 1^-}C_K=\infty$). It follows, by analytic continuation, that the lemma is valid if $\Re(b_p-b_q)<1$ whenever $p<q$. ∎

The notation and results that can be found in chapter IV (§ 5, § 6 et § 7) of Helgason's [8] are particularly pertinent to the rest of the section.

PROOF OF THEOREM 2.5 (END). The result is true for $n=1$. Assume it is true for $n-1$ ($n\ge 2$).

Let Λ be the set of all linear combinations of the positive roots having non-negative integer coefficients and let $\tilde{\Lambda}$ be the set of all linear combinations of the positive roots having integer coefficients.

We define $'\mathbf{a}_{\mathbf{C}}^*$ to be the set of complex-valued linear functionals on $\mathbf{a}$ such that $i(s\lambda-t\lambda)\notin\tilde{\Lambda}$ for $s\ne t$ and $\langle\mu,\mu\rangle\ne 2i\langle\mu,\lambda\rangle$ for all $\mu\in\Lambda-\{0\}$ and $s\in W$. $'\mathbf{a}_{\mathbf{C}}^*$ is a dense open connected subset of the set of complex-valued linear functionals on $\mathbf{a}$.

CLAIM. Suppose $i\nu(F)=\sum_{k=1}^n b_kF_k$ is such that $|\Re(b_p-b_q)|<1$ for all p and q and $\nu\in{}'\mathbf{a}_{\mathbf{C}}^*$. Then, there exists a positive constant independent of ν such that $\mathcal{G}\big(n,\sum_{s\in W}(\det s)e^{is\nu};e^H\big)=C\big(\pi(\nu)\big)^{-1}|c(\nu)|^{-2}\phi_\nu(e^H)$.

Recall that if λ is a linear functional on $\mathbf{a}$ and $s\in W$, then $s\lambda(H)=\lambda(s^{-1}H)$. Note that $\phi_\nu(e^H)=\int_K e^{(i\nu-\rho)(H(e^Hk))}\,dk$ and c is Harish-Chandra c-function.

We verify the claim. A consequence of the beginning of the proof and of the induction hypothesis is that if f is odd and decreases quickly enough, then $\Delta\mathcal{G}(n,f;\cdot)=\mathcal{G}\big(n,\Gamma(\Delta)f;\cdot\big)$. We can extend the result further by the following device: Suppose $f=\sum_{s\in W}(\det s)e^{is\nu}$ with ν as in the claim: $e^{-r^2/t}f$ is odd, smooth and decreases quickly enough. If one computes the asymptotic expansion of $\Delta\mathcal{G}(n,e^{-r^2/t}f;\cdot)=\mathcal{G}\big(n,\Gamma(\Delta)(e^{-r^2/t}f);\cdot\big)$, one finds that $\Delta\mathcal{G}(n,f;\cdot)=\mathcal{G}\big(n,\Gamma(\Delta)f;\cdot\big)$.

$\mathcal{G}\big(n,\sum_{s\in W}(\det s)e^{is\nu};e^H\big)$ is then an eigenvector of Δ for the eigenvalue $-(\gamma^2+\nu^2)$ ($\nu^2=\langle\nu,\nu\rangle=-\sum_{k=1}^n b_k^2$).

Assume for now that $0<\Re(b_p-b_q)<1$ whenever $p<q$.

Using the notation preceding the claim, we write

$$\Phi_\nu(H) = e^{(i\nu-\rho)(H)} \sum_{\mu\in\Lambda} \Gamma_\mu(\nu) e^{-\mu(H)}$$

whenever $\nu \in {}'\mathbf{a}_{\mathbf{C}}^*$. We set $\Gamma_0(\nu) = 1$; with the proper recursive definition on the $\Gamma_\mu(\nu)$, the conditions on ${}'\mathbf{a}_{\mathbf{C}}^*$ ensure that functions $\Phi_{s\nu}$, with s running in the Weyl group, form a basis of the eigenspace of Δ for the eigenvalue $-(\gamma^2+\nu^2)$.

Hence, $\mathcal{G}\big(n, \sum_{s\in W}(\det s)e^{is\nu}; e^H\big) = \sum_{s\in W} a_s(\nu)\Phi_{s\nu}(H)$.

Since $H \in \mathbf{a}^+$, $a_I(\nu) = \lim_{t\to\infty} e^{(-i\nu+\rho)(tH)} \mathcal{G}(n, e^{i\nu}; e^{tH}) = \prod_{p<q\leq n} B\Big(\big(1-(b_p - b_q)\big)/2, 1/2\Big)$: the first equality follows from the fact that the series defining Φ_ν converges absolutely and uniformly on each subchamber $\{H \in \mathbf{a}^+ : \alpha_i(H) > C > 0\ (1 \leq i \leq l)\}$ (again see [8]) and the second equality is a consequence of Lemma 6.1.

This gives $a_I(\nu) = C\big(\pi(\nu)\big)^{-1} c(\nu)|c(\nu)|^{-2}$: this is a consequence of the relation $B(1/2 - x, 1/2) = \tan(\pi x) B(x, 1/2)$ and the expression for the c-function as given by T. S. Bhanu Murti in [13].

The functions $\mathcal{G}\big(n, \sum_{s\in W}(\det s)e^{is\nu}; e^H\big)$ are analytic in ν in the domain $|\Re(b_p - b_q)| < 1$. As a consequence, one finds that the coefficients $a_s(\nu)$ are also analytic in the domain ${}'\mathbf{a}_{\mathbf{C}}^* \cap |\Re(b_p - b_q)| < 1$: see for instance exercise B17 page 486 in [8]. The formula $a_I(\nu) = C\big(\pi(\nu)\big)^{-1} c(\nu)|c(\nu)|^{-2}$ is then valid in the same domain. Finally, if we replace ν by $s\nu'$ in $\mathcal{G}\big(n, \sum_{s\in W}(\det s)e^{is\nu}; e^H\big) = \sum_{s\in W} a_s(\nu)\Phi_{s\nu}(H)$, we find that $a_{s'}(\nu) = (\det s')a_I(s\nu') = C\big(\pi(\nu)\big)^{-1} c(s\nu')|c(\nu)|^{-2}$.

This proves the claim since $\phi_\nu = \sum_{s\in W} c(s\nu)\Phi_{s\nu}$.

Suppose that f is an odd function. We write $\hat{f}$ for the Euclidean Fourier transform of f.

$$\begin{aligned} f(H) &= C\int_{\mathbf{a}^*} \hat{f}(\nu)e^{i\nu(H)}\, d\nu \\ &= \frac{C}{|W|}\sum_{s\in W}\int_{\mathbf{a}^*} \hat{f}(s\nu)e^{is\nu(H)}\, d\nu \\ &= \frac{C}{|W|}\int_{\mathbf{a}^*} \hat{f}(\nu)\sum_{s\in W}(\det s)e^{is\nu(H)}\, d\nu. \end{aligned}$$

Hence,

$$\mathcal{G}(n,f;e^H) = C\int_{\mathbf{a}^*} \frac{\hat{f}(\nu)}{\pi(\nu)} \phi_\nu(e^H)|c(\nu)|^{-2}\, d\nu. \tag{19}$$

The equation on the right hand side satisfies the conclusions of the theorem. ■

COROLLARY 6.2. *If f is a K-invariant function on* $\mathrm{Pos}(n, \mathbf{R})$ *of compact support then*

$$f = C\mathcal{G}\big(n, \partial(\pi)F(f;\cdot);\cdot\big)$$

(we recall that $F(f;\cdot)$ is the Abel transform of f).

PROOF. $F(f;\cdot)$ is Weyl invariant so $\partial(\pi)F(f;\cdot)$ is odd, decreases quickly enough and its Euclidean Fourier transform is $C\pi(\nu)\tilde{f}(\nu)$ ($\tilde{f}$ is the spherical Fourier transform of f). The rest follows from equation (19) and the inversion formula (see [8], Theorem 7.5 page 454). ■

COROLLARY 6.3. *Let* $W_t(H) = Ce^{-\gamma^2 t}t^{-n/2}\partial(\pi)\exp\bigl(-r^2(H)/(4t)\bigr)$. $\mathcal{G}(n, W_t; \cdot)$ *is the fundamental solution of the heat equation for* Pos$(n, \mathbf{R})$ *(modulo a constant).*

PROOF. The Abel transform of the heat kernel is known to be

$$Ce^{-\gamma^2 t}t^{-n/2}\exp\bigl(-r^2(H)/(4t)\bigr).$$

The result follows from the previous corollary. ■

7. **Conclusion.** In Section 6, we show that the "candidate" we gave in Section 3 for the heat kernel of Pos$(n, \mathbf{R})$, $P_t(e^H) = \mathcal{G}(n, W_t; H)$ where $W_t(H) = Ce^{-\gamma^2 t}t^{-n/2}\partial(\pi)\exp\bigl(-r^2(H)/(4t)\bigr)$, is actually the right one even for $n > 3$. However, to prove this, we had to rely on the heavy machinery involving Plancherel's formula and the eigenfunctions of the Laplace-Beltrami operator. The methods we have used in Sections 3, 4 and 5, we believe, are more likely to be open to generalization to other type of Riemannian manifolds.

Another point is that to prove similar bounds for Pos$(n, \mathbf{R})$ ($n > 3$) will be quite difficult using our transform, especially the lower bound. On the other hand, a slight improvement of Lemma 2.3 would allow us to compute an asymptotic expansion for the heat kernel, something the expression of P_t based on Plancherel's formula does not provide.

We think it is better to leave this development to a further paper.

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Department of Mathematics
University of Ottawa
Ottawa, Ontario K1N 6N5