Zeros, chaotic ratios and the computational complexity of approximating the independence polynomial

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Abstract

The independence polynomial originates in statistical physics as the partition function of the hard-core model. The location of the complex zeros of the polynomial is related to phase transitions, and plays an important role in the design of efficient algorithms to approximately compute evaluations of the polynomial.

In this paper we directly relate the location of the complex zeros of the independence polynomial to computational hardness of approximating evaluations of the independence polynomial. We do this by moreover relating the location of zeros to chaotic behaviour of a naturally associated family of rational functions; the occupation ratios.

1. *Introduction*

The independence polynomial of a graph $G = (V, E)$ is defined by

$$Z_G(\lambda) = \sum_{I \subseteq V} \lambda^{|I|},$$

where the sum is taken over all *independent* subsets I of the vertex set V. Recall that I is said to be independent if no two vertices in I are connected by an edge. Note that $Z_G(1)$ equals the number of independent subsets of V.

In statistical physics the independence polynomial is known as the partition function of the hard-core model. Of particular interest from a physics perspective is the location of the zeros of the partition function for certain classes of graphs. Away from these zeros the free energy is analytic, i.e. there are no phase transitions in the Lee–Yang sense cf. [**YL52**].

It turns out that exact computation of the independence polynomial for large graphs is not feasible for most values of λ; it is a #P-Hard problem[1], cf. [**Gre00, Rot96, Vad01**]. A

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[1] The complexity class #P may be seen as the counting version of the complexity class NP. For example, the problem of deciding whether a graph on n vertices contains an independent set of size k is a problem in
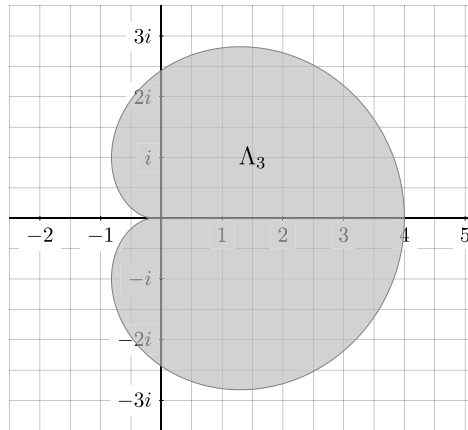
Fig. 1. The cardioid Λ_3.

question that has received significant interest is for which $\lambda \in \mathbb{C}$ there exist polynomial time algorithms that approximate $Z_G(\lambda)$, up to small multiplicative constants. See e.g. [**ALG20, Bar16, BGGŠ20, PR17, SS14, Wei06**] and the references therein.

Surprisingly, much like absence of zeros implies absence of phase transitions (in the Lee-Yang sense), absence of zeros implies the existence of efficient algorithms for this computational problem. More formally, on the maximal simply connected open set containing the origin on which the independence polynomial does not vanish for all graphs of a given maximum degree Δ there exists an efficient algorithm for approximating the independence polynomial [**Bar16, PR17**]. Let us denote this maximal 'zero-free' set by $\mathcal{U}_\Delta$. For real values of λ in the complement of the closure of $\mathcal{U}_\Delta$, approximating the partition function is computationally hard [**BGGŠ20, PR19, SS05, SS14**]. In other words, the absence/presence of complex zeros near the real axis marks a transition in the computational complexity of approximating the independence polynomial of graphs of bounded degree Δ for real values of λ. The transition point for positive λ coincides with the phase transition for the hard-core model on the Cayley tree of degree Δ.

A natural question is whether a similar phenomenon manifests itself for non-real λ. Bezaková, Galanis, Goldberg and Štefankovič [**BGGŠ20**] made an important contribution towards solving this question, by showing that for any integer $\Delta \geq 3$ and non-real λ outside a certain cardioid[2], Λ_Δ, approximation of the independence polynomial for graphs of bounded degree at most Δ is computationally hard. (In fact #P-hard.) See Figure 1 for a picture of Λ_3 and Definition 2·5 for the definition of Λ_Δ. Earlier it was shown by two of the authors of the present paper [**PR19**] that zeros of the independence polynomial of graphs of maximum degree at most Δ accumulate on the entire boundary of Λ_Δ. In particular the 'zero-free' set $\mathcal{U}_\Delta$ is contained in the cardioid; their intersections with the real axis in fact coincide [**PR19, SS05**]. Buys [**Buy21**] however showed that Λ_Δ does contain zeros of the

NP, while the problem of determining the number of independent sets of size k is in #P. See [**AB09, Val79**] for further background.

[2] Although the domain Λ_Δ resembles a cardioid, it is formally not a cardioid. However, as discussed in section 7, it plays an analogous role as the Main Cardioid of the Mandelbrot set, justifying our use of the term cardioid.

independence polynomial of graphs of bounded degree Δ. This in particular indicates that the result of [**BGGŠ20**] does not fully answer the question how zeros relate to computational hardness for non-real λ.

The goal of this paper is to solve this question by directly relating, for any fixed integer $\Delta \geq 3$, the zeros for the family of graphs of maximum degree at most Δ to the parameters where approximating evaluations of the independence polynomial is computationally hard. Our result is obtained by studying a natural family of rational maps associated to this family of graphs, using techniques and ideas from complex dynamics. We show that 'chaotic behaviour' of this family is equivalent to the presence of zeros, and implies computational hardness.

1·1. *Occupation ratios*

Given $\Delta \in \mathbb{Z}_{\geq 2}$, we define $\mathcal{G}_\Delta$ as the collection of finite simple rooted graphs (G, v) such that the maximum degree of G is at most Δ. For $i \in \{1, \ldots, \Delta\}$ we define $\mathcal{G}_\Delta^i = \{(G, v) \in \mathcal{G}_\Delta : \deg(v) \leq i\}$. The *occupation ratio*, or *ratio* for short, of a rooted graph (G, v) is defined by the rational function

$$R_{G,v}(\lambda) := \frac{Z_G^{\text{in}}(\lambda)}{Z_G^{\text{out}}(\lambda)},$$

where "in" means that in the definition of $Z_G(\lambda)$ the sum is taken only over independent sets I that contain the marked point v, while "out" means that the independent sets do not contain v. The ratio is a very useful tool in studying the zeros of the independence polynomial, see Lemma 2·1 below, and has been key in several of the aforementioned works. The ratio is also relevant from a statistical physics perspective as it is closely related to the *free energy*.

When (G, v) is a rooted Cayley tree of depth $n - 1$ and down-degree $d = \Delta - 1$, the ratio satisfies

$$R_{G,v}(\lambda) = f_{\lambda,d}^n(0),$$

where

$$f_{\lambda,d}(z) := \frac{\lambda}{(1 + z)^d}$$

and throughout the paper we write f^n for the nth iterate of the map f.

In this context it is therefore natural to consider λ as the parameter which determines the orbit of the marked point 0. This type of setting is often studied in complex dynamical systems, where one is interested in the sets where the parameter λ is *active* or *passive*. A parameter λ_0 is said to be passive if the family of rational functions $\{\lambda \mapsto f_{\lambda,d}^{\circ n}(0)\}$ is normal at λ_0, i.e. there exists a neighbourhood such that every sequence in this family has a subsequence that converges uniformly. A parameter is active if it is not passive. The most well-known activity-locus is undoubtedly the boundary of the Mandelbrot set, where the iterates of the functions $z^2 + c$ are considered. Following this terminology we define the *activity-locus*, $\mathcal{A}_\Delta$, by

$$\mathcal{A}_\Delta := \{\lambda_0 \in \mathbb{C} \mid \{\lambda \mapsto R_{G,v}(\lambda) \mid (G, v) \in \mathcal{G}_\Delta\} \text{ is not locally normal at } \lambda_0\}.$$

Another notion of chaotic behaviour of the ratios appears in the proof of the result of Bezáková, Galanis, Goldberg and Šefankovič [**BGGŠ20**]. An important step towards

proving #P-hardness is showing that for every non-real λ outside of the closed cardioid $\overline{\Lambda_\Delta}$ the set of values $\{R_{G,v}(\lambda) \mid (G, v) \in \mathcal{G}_\Delta^1\}$ is dense in $\hat{\mathbb{C}}$. Motivated by this we define

$$\mathcal{D}_\Delta := \{\lambda \in \mathbb{C} \mid \{R_{G,v}(\lambda) \mid (G, v) \in \mathcal{G}_\Delta^1\} \text{ is dense in } \hat{\mathbb{C}}\}$$

and refer to the closure of $\mathcal{D}_\Delta$ as the *density-locus*. We will prove it is equal to the activity-locus, thereby showing that these two notions of chaotic behaviour of the ratios are essentially equivalent.

1·2. *Main result*

To state our main result connecting the presence of zeros to computational hardness, we define the *zero-locus* as the closure of

$$\mathcal{Z}_\Delta = \{\lambda \in \mathbb{C} : Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}.$$

We informally define the #$\mathcal{P}$-*locus* as the closure of the collection of λ for which approximating $Z_G(\lambda)$ is #P-hard for $G \in \mathcal{G}_\Delta$. See subsection 1·3 below for a formal definition.

The main results of this paper can now be stated succinctly as follows.

THEOREM 1·1 (Main Theorem). *For any integer* $\Delta \geq 3$ *the zero-locus, the activity-locus and the density-locus are equal and contained in the* #$\mathcal{P}$-*locus. In other words*:

$$\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta = \overline{\mathcal{D}_\Delta} \subseteq \overline{\#\mathcal{P}_\Delta}.$$

We remark that the topological structure of the complement of the zero-locus is not yet understood. We have the following conjecture.

CONJECTURE 1·2. *For each integer* $\Delta \geq 3$, *the set* $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$ *is connected*.

Should this conjecture be true, then by Proposition 4·3 below, we know that the maximal 'zero-free' set containing 0, $\mathcal{U}_\Delta$, equals the complement of the zero-locus. Since there exists a polynomial time algorithm [**Bar16, PR17**] for approximating the independence polynomial on $\mathcal{U}_\Delta$, this would imply with our main theorem a complete understanding of the computational complexity of approximating the independence polynomial in terms of the location of the zeros as well as in terms of chaotic behaviour of the ratios.

Remark 1·3. We note that [**PR20**] and [**BGPR22**] combined implicitly contain similar equivalent characterisations for the Lee–Yang zeros of the partition function of the ferromagnetic Ising model on bounded degree graphs. In that setting the complement of the zero-locus is in fact connected when the edge interaction parameter is sub-critical.

1·3. *Computational complexity*

We formally state here the computational problems we are interested in. We denote by $\mathbb{Q}[i]$ the collection of complex numbers with rational real and imaginary part. Let $\lambda \in \mathbb{Q}[i]$, $\Delta \in \mathbb{N}$ and consider the following computational problems.

Name #Hard-CoreNorm(λ, Δ)

Input A graph G of maximum degree at most Δ.

Output If $Z_G(\lambda) \neq 0$ the algorithm must output a rational number N such that $N/1.001 \leq |Z_G(\lambda)| \leq 1.001N$. If $Z_G(\lambda) = 0$ the algorithm may output any rational number.

Name #Hard-CoreArg(λ, Δ)

Input A graph G of maximum degree at most Δ.

Output If $Z_G(\lambda) \neq 0$ the algorithm must output a rational number A such that $|A - a| \leq \pi/3$ for some $a \in \arg(Z_G(\lambda))$. If $Z_G(\lambda) = 0$ the algorithm may output any rational number.

We can now formally define the #$\mathcal{P}$-locus as the closure of the set

$$\#\mathcal{P}_\Delta := \{\lambda \in \mathbb{Q}[i] : \text{the problem } \#\text{Hard-CoreNorm}(\lambda, \Delta) \text{ is } \#\text{P-hard}\}.$$

We remark that in the definition of #$\mathcal{P}_\Delta$ we could also replace #Hard-CoreNorm(λ, Δ) by #Hard-CoreArg(λ, Δ) without altering the validity of Theorem 1·1.

We moreover note that the constant 1.001 is rather arbitrary. It originates from [**BGGŠ20**]. As remarked there the constant can be replaced by any other constant. Let us quickly explain the idea. If say #Hard-CoreNorm(λ, Δ) is #P-hard, but there would be a polynomial time algorithm for the problem with 1.001 replaced by 1.001^2, then we could run this algorithm on the disjoint union of two copies of the same graph G obtaining an a 1.001^2 approximation to the norm of $Z_{G \cup G}(\lambda) = Z_G(\lambda)^2$. This would immediately gives us a 1.001-approximation to the norm of $Z_G(\lambda)$. Since the number of vertices of $G \cup G$ is polynomial in the number of vertices of G, we would thus also get a polynomial time algorithm for the problem with constant 1.001.

Organisation. After introducing preliminary definitions and results in section 2, we treat the degree $\Delta = 2$ case in section 3. While the equalities between different loci are different when $\Delta = 2$, the explicit descriptions of the zero- and activity-locus will be used in the higher degree cases.

In Section 4 we prove the equality of the zero-locus and the activity-locus. The inclusion of the latter in the former is actually an immediate consequence of Montel's Theorem, and proved earlier in Corollary 2·12. We end that section by showing that connected components of the complement of the zero-locus are simply connected.

In Section 5 we prove the equality of the activity- and the density-locus, and in Section 6 we prove that the density-locus is contained in the #$\mathcal{P}$-locus.

We end our paper by discussing a special subclass of graphs: the finite Cayley trees of fixed down-degree $\Delta - 1$. In this setting classical results from complex dynamical systems can be used to obtain a precise description of the zero- and activity-locus. While there zeros do not lie in the activity-locus, the activity-locus equals the accumulation set of the zeros.

2. *Preliminaries*

In this section we collect some preliminary results and conventions that will be used in the remainder of the paper. The results in this section are not necessarily new, but often cannot be found in the literature in the exact way they are stated here. For convenience of the reader we include proofs, especially when the methods are similar to those used later in the paper.

2·1. *Ratios of graphs and trees*

Recall that for a rooted graph (G, v) the occupation ratio is defined as the following rational function in λ

$$R_{G,v}(\lambda) = \frac{Z_G^{\text{in}}(\lambda)}{Z_G^{\text{out}}(\lambda)}.$$

We note that $Z_G(\lambda) = Z_G^{\text{in}}(\lambda) + Z_G^{\text{out}}(\lambda)$, which implies that $Z_G(\lambda) = 0$ if and only if $R_{G,v}(\lambda) = -1$, unless $Z_{G,v}^{\text{in}}(\lambda)$ and $Z_{G,v}^{\text{out}}(\lambda)$ both vanish, in which case the value of the rational function $R_{G,v}(\lambda)$ may not equal -1. The next lemma will show that we can often ignore this difficulty.

We will write $G - v$ for the graph G with vertex v removed, and $G - N[v]$ for the graph with $N[v]$ removed, where $N[v] = \{u \in V(G) : \{u, v\} \in E(G)\} \cup \{v\}$ is the closed neighbourhood of v. We observe that $Z_{G,v}^{\text{out}}(\lambda) = Z_{G-v}(\lambda)$, and similarly $Z_{G,v}^{\text{in}}(\lambda) = \lambda \cdot Z_{G-N[v]}(\lambda)$.

LEMMA 2·1. *Let $\lambda \in \mathbb{C}^*$. The following three statements are equivalent.*

   (i) *There exists a graph G of maximum degree at most Δ for which $Z_G(\lambda) = 0$.*
   (ii) *There exists a rooted graph $(G, v) \in \mathcal{G}_\Delta$ for which $R_{G,v}(\lambda) = -1$.*
   (iii) *There exists a rooted graph $(G, v) \in \mathcal{G}_\Delta$ for which $R_{G,v}(\lambda) \in \{-1, 0, \infty\}$.*

Proof. Assume that (i) holds, then there is a graph G of maximum degree at most Δ for which $Z_G(\lambda) = 0$. Without loss of generality we can assume $G \in \mathcal{G}_\Delta$ satisfies $Z_G(\lambda) = 0$ and has a minimal number of vertices, i.e. for any graph $H \in \mathcal{G}_\Delta$ with $Z_H(\lambda) = 0$ we have $|V(G)| \leq |V(H)|$. For any vertex $v \in V(G)$ we have

$$0 = Z_G(\lambda) = Z_{G,v}^{\text{in}}(\lambda) + Z_{G,v}^{\text{out}}(\lambda).$$

As $|V(G - v)| < V(G)$ we have $Z_{G,v}^{\text{out}}(\lambda) = Z_{G-v}(\lambda) \neq 0$, which implies $R_{G,v}(\lambda) = -1$. Thus (ii) holds. Trivially, if (ii) holds then also (iii) holds. To complete the proof we will assume (iii) holds and show that (i) follows.

Assume there is a rooted graph $(G, v) \in \mathcal{G}_\Delta$ for which $R_{G,v}(\lambda) \in \{-1, 0, \infty\}$. If $R_{G,v}(\lambda) = -1$, we either have $Z_{G,v}^{\text{out}}(\lambda) = 0$, in which case (i) follows, or $Z_{G,v}^{\text{out}}(\lambda) \neq 0$, in which case $Z_{G,v}^{\text{in}}(\lambda) = -Z_{G,v}^{\text{out}}(\lambda)$ and (i) follows as well. If $R_{G,v}(\lambda) = \infty$ we have $Z_{G,v}^{\text{out}}(\lambda) = 0$. As $Z_{G,v}^{\text{out}}(\lambda) = Z_{G-v}(\lambda)$ we see (i) holds. The final case is $R_{G,v}(\lambda) = 0$, in which case we have $0 = Z_{G,v}^{\text{in}}(\lambda) = \lambda \cdot Z_{G-N[v]}(\lambda)$. Now as $\lambda \neq 0$, we must have $Z_{G-N[v]}(\lambda) = 0$, which concludes the proof.

Note that for $\lambda = 0$ we have $R_{G,v}(\lambda) = 0$ and $Z_G(\lambda) = 1$ for any graph G and any vertex $v \in V(G)$. Hence for $\lambda = 0$, statements (i) and (ii) in Lemma 2·1 are still equivalent, while statement (iii) is not equivalent to (i) or (ii).

The following result due to Bencs [**Ben18**] will play an important role in this paper.

THEOREM 2·2. *Let $(G, v) \in \mathcal{G}_\Delta^i$ be a rooted connected graph. Then there is a rooted tree $(T, u) \in \mathcal{G}_\Delta^i$ and induced graphs $G_1, \ldots, G_k$ of G such that*:

   (i) $Z_T = Z_G \prod_{i=1}^k Z_{G_i}$;
   (ii) $R_{G,v} = R_{T,u}$.

The following result is a consequence.

LEMMA 2·3. *Let $\lambda \in \mathbb{C}$ and $(G, v) \in \mathcal{G}_\Delta$ with $Z_G(\lambda) = 0$. Then there is a rooted tree $(T, u) \in \mathcal{G}_\Delta^1$ such that $Z_T(\lambda) = 0$ and $R_{T,u}(\lambda) = -1$.*

Proof. Note that for any graph G we have $Z_G(0) = 1$, so we can assume $\lambda \neq 0$. By Lemma 2·1 there exists a rooted graph $(G, v) \in \mathcal{G}_\Delta$ such that $R_{G,v}(\lambda) = -1$. By Theorem 2·2(i) we see there is a rooted tree $(T, u) \in \mathcal{G}_\Delta$ with $Z_T(\lambda) = 0$. It follows there is a tree $\widetilde{T}$ of maximum degree Δ with a minimal number of vertices such that $Z_{\widetilde{T}}(\lambda) = 0$. For $\widetilde{T}$ and any vertex $v \in V(\widetilde{T})$ we have $R_{\widetilde{T},v}(\lambda) = -1$. The lemma follows by choosing v a leaf of $\widetilde{T}$.

At a later stage we will need to worry about the degree of the root vertex in our definition of the activity- and density-locus. We therefore introduce some definitions to facilitate their discussion.

Fix an integer $\Delta \geq 2$ throughout. For $i = 1, \ldots, \Delta$ we denote the family of ratios with root degree at most i by

$$\mathcal{R}_\Delta^i := \{R_{G,v)} \mid (G, v) \in \mathcal{G}_\Delta^i\}.$$

We just write $\mathcal{R}_\Delta$ instead of $\mathcal{R}_\Delta^\Delta$. For a given $\lambda \in \mathbb{C}$, we denote the set of values of these ratios by

$$\mathcal{R}_\Delta^i(\lambda) := \{R_{G,v)}(\lambda) \mid (G, v) \in \mathcal{G}_\Delta^i\}.$$

Then we define $\mathcal{A}_\Delta^i$ to be the collection of λ_0 at which the family $\mathcal{R}_\Delta^i$ is not normal. We just write $\mathcal{A}_\Delta$ instead of $\mathcal{A}_\Delta^\Delta$. Finally, we introduce $\mathcal{D}_\Delta^i$ to be the collection of λ for which the set $\mathcal{R}_\Delta^i(\lambda)$ is dense in $\mathbb{C}$. Note that we denote $\mathcal{D}_\Delta^1$ by $\mathcal{D}_\Delta$ (as opposed to the above convention).

2·2. *Graph manipulations and definition of the cardioid*

The recursion formula given in the following lemma is well known.

LEMMA 2·4. *Let $T = (V, E)$ be a tree and v a vertex of T. Suppose v is connected to $d \geq 1$ other vertices $u_1, \ldots, u_d$. Denote T_s for the tree that is the connected component of $T - v$ containing u_s. Then we have*

$$R_{T,v}(\lambda) = \frac{\lambda}{\prod_{s=1}^d (1 + R_{T_s,u_s}(\lambda))}. \tag{2·1}$$

Proof. We have

$$\frac{Z_{T,v}^{in}(\lambda)}{Z_{T,v}^{out}(\lambda)} = \lambda \frac{Z_{T-N[v]}(\lambda)}{Z_{T-v}(\lambda)} = \lambda \prod_{s=1}^d \frac{Z_{T_s,u_s}^{out}(\lambda)}{Z_{T_s}(\lambda)} = \lambda \prod_{s=1}^d \frac{Z_{T_s,u_s}^{out}(\lambda)}{Z_{T_s,u_s}^{out}(\lambda) + Z_{T_s,u_s}^{in}(\lambda)}, \tag{2·2}$$

where in the second equality we use that the partition function of a graph factors into the partition functions of its connected components. By dividing for each $s \in \{1, \ldots, d\}$ the denominator and enumerator of the right-hand side of equation (2·2) by $Z_{T_s,u_s}^{out}(\lambda)$ we obtain the desired formula.

This lemma implies the claim from the introduction that the ratios of Cayley trees are given by iterating $f_{\lambda,d}(z) = \lambda/(1 + z)^d$. We refer to Section 7 for an in-depth discussion of Cayley trees and their associated dynamics.

Definition 2·5. Define the cardioid Λ_Δ as the closure of the set of parameters λ for which $f_{\lambda,d}$ has an attracting fixed point.
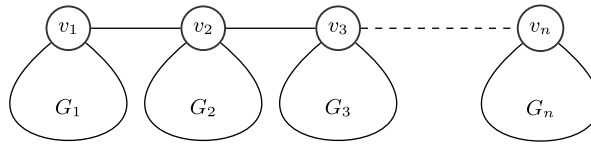
Fig. 2. The graph $\widetilde{P}_n$ in Lemma 2·6.

Note that $0 \in \Lambda_\Delta$. One can show, see [**PR19**, section 2·1], that

$$\Lambda_\Delta = \left\{ \frac{z}{(1-z)^\Delta} \mid |z| \le \frac{1}{\Delta - 1} \right\}.$$

Taking $z = \frac{-1}{\Delta - 1}$, we observe that

$$\lambda^*(\Delta) = \frac{-(\Delta - 1)^{\Delta - 1}}{\Delta^\Delta}$$

is the intersection point of Λ_Δ with the negative real line.

Let $G = (V, E)$ be a graph and let (G_i, v_i) be rooted graphs for $i \in V$. We refer to the graph obtained from G and the G_i by identifying each vertex $i \in V$ with v_i as *implementing* the G_i in G, see Figure 2. The next lemmas describe the effect on the ratios for various choices of G and G_i.

LEMMA 2·6. *Let P_n denote the path on n vertices. Let (G_i, v_i) be rooted graphs for $i \in \{1, \ldots, n\}$ and denote $\mu_i(\lambda) = R_{G_i, v_i}(\lambda)$. Let $\widetilde{P}_n$ be the graph obtained by implementing the G_i in P_n. Then*

$$R_{\widetilde{P}_n, v_n}(\lambda) = (f_{\mu_n(\lambda)} \circ \cdots \circ f_{\mu_1(\lambda)})(0),$$

where $f_\mu(z) = \mu/(1 + z)$.

Proof. We use induction on n. For $n = 1$, by definition we have $R_{G_1, v_1}(\lambda) = \mu_1(\lambda) = f_{\mu_1(\lambda)}(0)$. As $\widetilde{P}_1 = G_1$, we have $R_{\widetilde{P}_1, v_1}(\lambda) = R_{G_1, v_1}(\lambda)$. The base case follows.

Suppose the statement holds for some $n \ge 1$. The vertex v_{n+1} has 1 neighbour that is part of the path P_n. Let us denote that neighbour as v_n. It follows that

$$R_{\widetilde{P}_{n+1}, v_{n+1}}(\lambda) = \frac{Z^{in}_{\widetilde{P}_{n+1}, v_{n+1}}(\lambda)}{Z^{out}_{\widetilde{P}_{n+1}, v_{n+1}}(\lambda)} = \frac{Z^{in}_{G_{n+1}, v_{n+1}}(\lambda)}{Z^{out}_{G_{n+1}, v_{n+1}}(\lambda)} \cdot \frac{Z^{out}_{\widetilde{P}_n, v_n}(\lambda)}{Z_{\widetilde{P}_n, v_n}(\lambda)}$$

$$= R_{G_{n+1}, v_{n+1}}(\lambda) \cdot \frac{Z^{out}_{\widetilde{P}_n, v_n}(\lambda)}{Z^{out}_{\widetilde{P}_n, v_n}(\lambda) + Z^{in}_{\widetilde{P}_n, v_n}(\lambda)} = \frac{R_{G_{n+1}, v_{n+1}}(\lambda)}{1 + R_{\widetilde{P}_n, v_n}(\lambda)}$$

$$= f_{\mu_{n+1}(\lambda)}(R_{\widetilde{P}_n, v_n}(\lambda)),$$

where in the second equality we use that the partition function of a graph factors into the partition functions of its connected components.

By the induction hypothesis we have

$$R_{\widetilde{P}_n, v_n}(\lambda) = (f_{\mu_n(\lambda)} \circ \cdots \circ f_{\mu_1(\lambda)})(0),$$

from which it follows that

$$R_{\widetilde{P}_{n+1},v_{n+1}}(\lambda) = (f_{\mu_{n+1}(\lambda)} \circ f_{\mu_n(\lambda)} \circ \cdots \circ f_{\mu_1(\lambda)})(0),$$

completing the proof.

Remark 2·7. Note that if the graphs G_i in Lemma 2·6 are all of maximum degree Δ and the roots v_i have degree at most $\Delta - 2$ for $i \in \{2, \ldots, n-1\}$ and at most degree $\Delta - 1$ for $i \in \{1, n\}$, then the graph $\widetilde{P}_n$ is also of maximum degree Δ.

LEMMA 2·8. *Let $G = (V, E)$ be a graph and denote $n = |V|$. Let (H, v) be a rooted graph. Let $\widetilde{G} = (\widetilde{V}, \widetilde{E})$ be obtained from G by implementing n copies of (H, v) in G. Then for any $w \in V$ we have*

$$\frac{Z_{\widetilde{G},w}(\lambda)}{(Z_{H,v}^{out}(\lambda))^n} = Z_{G,w}(R_{H,u}(\lambda)) \tag{2·3}$$

and

$$R_{\widetilde{G},w}(\lambda) = R_{G,w}(R_{H,v}(\lambda)). \tag{2·4}$$

Proof. We have

$$\frac{Z_{\widetilde{G},w}^{in}(\lambda)}{(Z_{H,v}^{out}(\lambda))^n} = \frac{\sum_{\substack{I \in \mathcal{I}(G) \\ w \in I}} Z_{H,v}^{in}(\lambda)^{|I|} Z_{H,v}^{out}(\lambda)^{n-|I|}}{(Z_{H,v}^{out}(\lambda))^n} = Z_{G,w}^{in}(R_{H,v}(\lambda)) \tag{2·5}$$

and

$$\frac{Z_{\widetilde{G},w}^{out}(\lambda)}{(Z_{H,v}^{out}(\lambda))^n} = \frac{\sum_{\substack{I \in \mathcal{I}(G) \\ w \notin I}} Z_{H,v}^{in}(\lambda)^{|I|} Z_{H,v}^{out}(\lambda)^{n-|I|}}{(Z_{H,v}^{out}(\lambda))^n} = Z_{G,w}^{out}(R_{H,v}(\lambda)). \tag{2·6}$$

Equality (2·3) follows from equalities (2·5) and (2·6) noting that for any graph W and any vertex u of W we have $Z_W(\lambda) = Z_{W,u}^{in}(\lambda) + Z_{W,u}^{out}(\lambda)$. Equality (2·4) follows from equalities (2·5) and (2·6) and the definition of the ratio.

We will also need the following slight variation on Lemma 2·4.

LEMMA 2·9. *Let (G_1, v_1) and (G_2, v_2) be rooted graphs, and define the rooted graph $(\widetilde{G}, \widetilde{v})$ by identifying the roots v_1 and v_2. Then*

$$R_{\widetilde{G},\widetilde{v}}(\lambda) = \lambda^{-1} \cdot R_{G_1,v_1}(\lambda) \cdot R_{G_2,v_2}(\lambda).$$

Proof. We compute

$$R_{\widetilde{G},\widetilde{v}}(\lambda) = \frac{Z_{\widetilde{G},\widetilde{v}}^{in}(\lambda)}{Z_{\widetilde{G},\widetilde{v}}^{out}(\lambda)} = \frac{Z_{G_1,v_1}^{in}(\lambda) \cdot Z_{G_2,v_2}^{in}(\lambda) \cdot \lambda^{-1}}{Z_{G_1,v_1}^{out}(\lambda) \cdot Z_{G_2,v_2}^{out}(\lambda)} = \lambda^{-1} \cdot R_{G_1,v_1}(\lambda) \cdot R_{G_2,v_2}(\lambda).$$

2·3. *The Shearer region*

Denote the open disk around 0 with radius $(\Delta - 1)^{\Delta-1}/\Delta^\Delta$ by B_Δ. This region, also known as the Shearer region, is the maximal open disk centered around 0 that is zero free

for the independence polynomial of graphs of maximum degree Δ [**She85, SS05**]. We will show the Shearer region is also disjoint from the activity-locus and the density-locus, which will later be used to deal with the $\lambda = 0$ case in the proof of our main theorem.

LEMMA 2·10. *Let* $\Delta \geq 2$ *be an integer. Then* B_Δ *is disjoint from the activity-locus, the zero-locus and the density-locus, i.e., we have* $B_\Delta \cap \overline{\mathcal{D}_\Delta} = B_\Delta \cap \mathcal{A}_\Delta = B_\Delta \cap \overline{\mathcal{Z}_\Delta} = \emptyset$.

Proof. We claim that for any rooted graph $(G, v) \in \mathcal{G}_\Delta$ and any $\lambda \in B_\Delta$ we have

$$|R_{G,v}(\lambda)| < \begin{cases} \frac{1}{\Delta} & \text{if } \deg(v) \leq \Delta - 1, \\ \frac{1}{\Delta-1} & \text{otherwise.} \end{cases}$$

By Theorem 2·2 we can equivalently work with rooted trees $(T, v) \in \mathcal{G}_\Delta$ instead of rooted graphs.

We will proof the claim by induction on the number of vertices of T. If $|V(T)| = 1$, we have $\deg(v) = 0$ and therefore $R_{T,v}(\lambda) = \lambda$. The claim then follows as $(\Delta - 1)^{(\Delta-1)}/\Delta^\Delta < 1/\Delta$ for all $\Delta \geq 2$. Suppose the claim holds for all rooted trees $(T, v) \in \mathcal{G}_\Delta$ with $|V(T)| \leq n$ for some $n \geq 1$. Let $(\widetilde{T}, \widetilde{v}) \in \mathcal{G}_\Delta$ be a rooted tree with $n + 1$ vertices. Denote the d children of $\widetilde{v}$ as $u_1, \ldots, u_d$ and denote (T_i, u_i) for the rooted subtree of $\widetilde{T}$ with root u_i. By Lemma 2·4 we have

$$R_{\widetilde{T},\widetilde{v}}(\lambda) = \frac{\lambda}{\prod_{i=1}^{d}(1 + R_{T_i,u_i}(\lambda))}.$$

We note that each (T_i, u_i) has at most n vertices, hence the induction hypotheses applies. Furthermore in T_i we have $\deg(u_i) \leq \Delta - 1$ as $\widetilde{T}$ has maximum degree at most Δ. Thus we see

$$|R_{\widetilde{T},\widetilde{v}}(\lambda)| = \frac{|\lambda|}{\prod_{i=1}^{d}|1 + R_{T_i,u_i}(\lambda)|} \leq \frac{|\lambda|}{\prod_{i=1}^{d}(1 - |R_{T_i,u_i}(\lambda)|)}$$

$$< \frac{|\lambda|}{(1 - \frac{1}{\Delta})^d} = \frac{\Delta^d|\lambda|}{(1 - \Delta)^d} < \frac{(\Delta - 1)^{\Delta-1-d}}{\Delta^{\Delta-d}}.$$

Now if $d \leq \Delta - 1$, we see $(\Delta - 1)^{\Delta-1-d}/\Delta^{\Delta-d} < 1/\Delta$ hence the claim follows for that case. If $d = \Delta$ we have $(\Delta - 1)^{\Delta-1-d}/\Delta^{\Delta-d} = 1/(\Delta - 1)$, which proves the claim.

It follows from the claim above that the family of ratios $\mathcal{R}_\Delta$ maps B_Δ into the open unit disk, for all $\Delta \geq 2$. So clearly $B_\Delta \cap \mathcal{D}_\Delta = \emptyset$. As B_Δ is open, we have $B_\Delta \cap \overline{\mathcal{D}_\Delta} = \emptyset$.

By Montel's Theorem the family $\mathcal{R}_\Delta$ is normal on B_Δ, so $B_\Delta \cap \mathcal{A}_\Delta = \emptyset$. We showed for all rooted graphs $(G, v) \in \mathcal{G}_\Delta$ that $|R_{G,v}(\lambda)| < 1/\Delta - 1 \leq 1$, hence the ratio will never equal -1. For $\lambda \neq 0$, we see by Lemma 2·1 that $\lambda \notin \mathcal{Z}_\Delta$. For $\lambda = 0$ we note that $Z_G(0) = 1$ for any graph $G \in \mathcal{G}_\Delta$. It follows that $B_\Delta \cap \mathcal{Z}_\Delta = \emptyset$. Again, as B_Δ is open, we have $B_\Delta \cap \overline{\mathcal{Z}_\Delta} = \emptyset$. This completes the proof.

Remark 2·11. We note that on the negative real line the Shearer region agrees with the interior of the cardioid, i.e we have $\mathbb{R}_{\leq 0} \cap B_\Delta = \mathbb{R}_{\leq 0} \cap int\Lambda_\Delta$ for all integers $\Delta \geq 3$.

Lemmas 2·10 and 2·1 together imply one of the inclusions in our main result.

COROLLARY 2·12. *For all $\Delta \geq 2$ the activity-locus is contained in the zero-locus, i.e.* $\mathcal{A}_\Delta \subseteq \overline{\mathcal{Z}_\Delta}$.

Proof. Equivalently, we want to show that for any $\lambda \in \mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$ the family $\mathcal{R}_\Delta$ is normal at λ. By Lemma 2·10 this is the case for $\lambda = 0$ and thus we assume that $\lambda \neq 0$. Take a sufficiently small neighbourhood U around λ such that $0 \notin U$ and $U \cap \overline{\mathcal{Z}_\Delta} = \emptyset$. It follows from Lemma 2·1 that the family $\mathcal{R}_\Delta$ avoids $\{-1, 0, \infty\}$ for all $\lambda' \in U$. Hence by Montel's Theorem the family is normal on U.

3. *Graphs with maximum degree at most two*

In this section we will deal with graphs of maximum degree at most two, in other words graphs for which each component is a path or a cycle. We will show that

$$\overline{\mathcal{Z}_2} = \mathcal{A}_2 = (-\infty, -1/4] \quad \text{and} \quad \mathcal{D}_2^1 = \mathcal{D}_2^2 = \emptyset.$$

An explicit description of $\mathcal{Z}_2$ was already known [**HL72, SS05**]; we provide a new proof for the sake of completeness.

Note that this is in contrast to the situation for $\Delta \geq 3$ as stated in Theorem 1·1. It follows from Lemma 2·3 that $\mathcal{Z}_2$ is equal to the set of λ for which there is a $(T, v) \in \mathcal{G}_2^1$, with T a tree, such that $R_{T,v}(\lambda) = -1$. The collection $\mathcal{G}_2^1$ consists of rooted graphs where the component containing the root is a path rooted at an endpoint. Let (P_n, v_n) denote a path on n vertices rooted at an endpoint v_n. If we let $f_\lambda(z) = \lambda/(1 + z)$ then it follows from Lemma 2·6 that $R_{P_n, v_n}(\lambda) = f_\lambda^n(0)$. For fixed λ the map f_λ is a Möbius transformation and therefore we first review some properties of Möbius transformation.

3·1. *Möbius transformations*

Everything that is done in this section can for example be found in [**Bea95**, section 4·3]. Let $\mathcal{M}$ denote the group of Möbius transformations with composition as group operation and let $\mathrm{GL}_2(\mathbb{C})$ denote the group of 2×2 invertible matrices with complex entries. The following map is a surjective group homomorphism.

$$\Phi : \mathrm{GL}_2(\mathbb{C}) \longrightarrow \mathcal{M}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto \frac{az + b}{cz + d} \right).$$

For any $g \in \mathcal{M}$ take an element $A \in \Phi^{-1}(\{g\})$ and define $\mathrm{tr}^2(g) = \mathrm{tr}(A)^2 / \det(A)$. This quantity does not depend on the choice of A and thus tr^2 is a well defined function on $\mathcal{M}$. We say that elements $f, g \in \mathcal{M}$ are conjugate if there exists an $h \in \mathcal{M}$ such that $f = h \circ g \circ h^{-1}$.

LEMMA 3·1 ([**Bea95**, theorem 4·3·4]). *Let $f, g \in \mathcal{M}$ not equal to the identity. The maps f, g are conjugate if and only if $\mathrm{tr}^2(f) = \mathrm{tr}^2(g)$. It follows that g is conjugate to*:

  (i) *a rotation $z \mapsto e^{i\theta} \cdot z$ for some $\theta \in (0, \pi]$ if and only if $\mathrm{tr}^2(g) \in [0, 4)$;*
  (ii) *the translation $z \mapsto z + 1$ if and only if $\mathrm{tr}^2(g) = 4$;*
  (iii) *a multiplication $z \mapsto \xi \cdot z$ for some $\xi \in \mathbb{C}^*$ with $|\xi| < 1$ if and only if $\mathrm{tr}^2(g) \in \mathbb{C} \setminus [0, 4]$.*

  The map g is said to be elliptic, parabolic or loxodromic in these three cases respectively.

Observe that if $f = h \circ g \circ h^{-1}$ then $f^n = h \circ g^n \circ h^{-1}$. It follows that, to understand the dynamical behaviour of a Möbius transformation g, it is enough to understand the dynamical

behaviour of any element in the conjugacy class of g. If g is loxodromic then it has two distinct fixed points in $\widehat{\mathbb{C}}$, one of which is attracting and the other is repelling. Under iteration of g the orbit of every initial value except for the repelling fixed point converges to the attracting fixed point. If g is parabolic then g has a unique fixed point, and under iteration of g all orbits converge to this fixed point. If g is elliptic then g is conjugate to a rotation $z \mapsto e^{i\theta} \cdot z$. We say that g is conjugate to a *rational rotation* if θ is a rational multiple of π and otherwise we say that g is conjugate to an *irrational rotation*. If g is conjugate to a rational rotation there is a positive integer n such that g^n is equal to the identity. If g is conjugate to an irrational rotation it has two fixed points, say p, q, and $\widehat{\mathbb{C}} \setminus \{p, q\}$ is foliated by generalized circles on which g acts conjugately to an irrational rotation.

We end this subsection by classifying f_λ in terms of its parameter.

LEMMA 3·2. *The Möbius transformation f_λ is*:

(i) *elliptic if $\lambda \in (-\infty, -1/4)$;*
(ii) *parabolic if $\lambda = -1/4$;*
(iii) *loxodromic if $\lambda \in \mathbb{C}^* \setminus (-\infty, -1/4]$.*

Proof. This follows from Lemma 3·1 and the fact that $\mathrm{tr}^2(f_\lambda) = -1/\lambda$.

3·2. *Determining the zero and activity-locus*

In this subsection we we will show that both $\overline{\mathcal{Z}_2}$ and $\mathcal{A}_2^1$ are equal to $(-\infty, -1/4]$. By definition we have $\mathcal{A}_2^1 \subseteq \mathcal{A}_2$ and by Corollary 2·12 we have $\mathcal{A}_2 \subseteq \overline{\mathcal{Z}_2}$, hence it will follow that $\mathcal{A}_2$ is equal to $(-\infty, -1/4]$ as well.

LEMMA 3·3. *Zeros of Z_G for graphs $G \in \mathcal{G}_2$ form a dense subset of the interval $(-\infty, -1/4)$, hence $\overline{\mathcal{Z}_2} = (-\infty, -1/4]$.*

Proof. We claim that $\lambda \in \mathcal{Z}_2$ if and only if f_λ is conjugate to a rational rotation.

First suppose that $\lambda \in \mathcal{Z}_2$. Then, by Lemma 2·3, there is an $n \geq 1$ such that for the path on n vertices P_n rooted at the endpoint v_n we have $R_{P_n, v_n}(\lambda) = -1$ and thus $f_\lambda^n(0) = -1$. Because $f_\lambda^2(-1) = 0$ regardless of the value of λ we obtain that $f_\lambda^{n+2}(0) = 0$. This means that 0 is a periodic point of f_λ of period strictly larger than 1. This can only occur if f_λ is conjugate to a rational rotation, as is explained in Section 3.1.

Suppose that f_λ is conjugate to a rational rotation. Note that this implies that λ is not equal to zero. Take the smallest positive integer n such that f_λ^n is equal to the identity and thus specifically $f_\lambda^n(0) = 0$. Note that $f_\lambda(0) = \lambda$ and $f_\lambda^2(0) = \lambda/(1+\lambda)$ and thus $n \geq 3$. Since $f_\lambda^{-2}(0) = -1$ we obtain that $R_{P_{n-2}, v_{n-2}}(\lambda) = f_\lambda^{n-2}(0) = -1$. It follows from the proof of Lemma 2·1 that λ is a root of $Z_{P_{n-2}}$.

Parameters λ for which f_λ is conjugate to a rational rotation lie dense in the set of parameters for which f_λ is conjugate to any rotation. It follows from Lemma 3·2 that $\overline{\mathcal{Z}_2} = (-\infty, -1/4]$.

We remark that tr^2 of the map that sends z to $e^{i\theta} \cdot z$ is equal to $2(1 + \cos(\theta))$. By comparing this to the value of $\mathrm{tr}^2(f_\lambda)$ it follows from the previous proof that

$$\mathcal{Z}_2 = \left\{ \frac{-1}{2(1 + \cos{(t\pi)})} : t \in (0, 1) \cap \mathbb{Q} \right\}.$$

We will now prove the final lemma needed to determine $\mathcal{A}_2$.

LEMMA 3·4. *The family $\mathcal{R}_2^1$ is not normal around any $\lambda \in (-\infty, -1/4]$, in other words* $(-\infty, -1/4] \subseteq \mathcal{A}_2^1$.

Proof. Recall that

$$\mathcal{R}_2^1 = \{ R_{P_n, v_n} : n \geq 1 \} = \{ \lambda \mapsto f_\lambda^n(0) : n \geq 1 \}.$$

Take a $\lambda_0 \in (-\infty, -1/4]$ and suppose for the sake of contradiction that there exists a neighbourhood U of λ_0 on which $\mathcal{R}_2^1$ is normal. We take U connected and sufficiently small so that it does not contain 0, and 0 is not a fixed point of f_λ for any $\lambda \in U$. Because $\mathcal{R}_2^1$ is normal on U there exists a subsequence of $\{R_{P_n, v_n}\}_{n \geq 1}$ that converges locally uniformly to a holomorphic function $F : U \to \widehat{\mathbb{C}}$. For $\lambda \in U \setminus (-\infty, -1/4]$ the map f_λ is loxodromic, hence $f_\lambda^n(0)$ converges to the attracting fixed point of f_λ as n goes to infinity. This means that for $\lambda \in U \setminus (-\infty, -1/4]$ we have $f_\lambda(F(\lambda)) = F(\lambda)$. Because $U \setminus (-\infty, -1/4]$ is non-empty and open in U it follows from the identity theorem for holomorphic functions that the equality $f_\lambda(F(\lambda)) = F(\lambda)$ must hold for all $\lambda \in U$. The set U contains a parameter λ_1 for which f_{λ_1} is elliptic. The value 0 is not a fixed point of f_{λ_1} and thus the distance of $f_{\lambda_1}^n(0)$ to both of the fixed points of f_{λ_1} is uniformly bounded below for all n by a positive constant. This means that no subsequence of $\{R_{P_n, v_n}(\lambda_1)\}_{n \geq 1}$ can converge to the fixed point $F(\lambda_1)$. We conclude that $\mathcal{R}_2^1$ is not normal at λ_0.

It follows from the previous two lemmas and Corollary 2·12 that

$$(-\infty, -1/4] \subseteq \mathcal{A}_2^1 \subseteq \mathcal{A}_2^2 \subseteq \overline{\mathcal{Z}_2} = (-\infty, -1/4].$$

Therefore we can conclude that both $\mathcal{A}_2$ and $\overline{\mathcal{Z}_2}$ are equal to $(-\infty, -1/4]$.

3·3. *Determining the density-locus*

Recall that for $\lambda \in \mathbb{C}$ we defined $\mathcal{R}_\Delta^i(\lambda) = \{ R_{G,v}(\lambda) : (G, v) \in \mathcal{G}_\Delta^i \}$. Subsequently we defined $\mathcal{D}_\Delta^i$ as the set consisting of those λ for which $\mathcal{R}_\Delta^i(\lambda)$ is dense in $\widehat{\mathbb{C}}$. It is thus clear that $\mathcal{D}_2^1 \subseteq \mathcal{D}_2^2$. To conclude the section we show the following.

LEMMA 3·5. *There is no $\lambda_0 \in \mathbb{C}$ for which $\mathcal{R}_2^2(\lambda_0)$ is dense in $\widehat{\mathbb{C}}$, i.e. $\mathcal{D}_2^2 = \emptyset$.*

Proof. It follows from Theorem 2·2 that

$$\mathcal{R}_2^2(\lambda) = \{ R_{T,v}(\lambda) : (T, v) \in \mathcal{G}_2^2 \text{ with } T \text{ a tree} \}.$$

A rooted tree $(T, v) \in \mathcal{G}_2^2$ can be viewed as a vertex v onto which two rooted paths (P_n, v_n) and (P_m, v_m) are attached for $n, m \geq 0$. It follows from Lemma 2·4 and Lemma 2·6 that

$$R_{T,v}(\lambda) = \lambda \cdot \frac{1}{1 + f_\lambda^n(0)} \cdot \frac{1}{1 + f_\lambda^m(0)} = \frac{1}{\lambda} \cdot f_\lambda^{n+1}(0) \cdot f_\lambda^{m+1}(0).$$

For a specific λ_0 the right-hand side of this equality is not defined if $f_{\lambda_0}^{n+1}(0)$ and $f_{\lambda_0}^{m+1}(0)$ take on the values 0 and ∞. Recall that if $f_{\lambda_0}^{n+1}(0) = \infty$, then $f_{\lambda_0}^n(0) = -1$, which implies

that $\lambda_0 \in \mathcal{Z}_2$. If this is the case then Lemma 3·3 implies that λ_0 is real, and thus $\mathcal{R}_2^2(\lambda_0)$ is contained in $\mathbb{R} \cup \{\infty\}$ and is not dense in $\widehat{\mathbb{C}}$.

Assume that λ_0 is not real. In this case we have the equality

$$\mathcal{R}_2^2(\lambda_0) = \left\{ \frac{1}{\lambda_0} \cdot f_{\lambda_0}^{n+1}(0) \cdot f_{\lambda_0}^{m+1}(0) : n, m \geq 0 \right\}.$$

The map f_{λ_0} is loxodromic, hence the orbit of 0 converges to an attracting fixed point without passing through ∞. Note that $f_{\lambda_0}(\infty) = 0$, therefore ∞ is not the attracting fixed point, and thus there is a positive bound $B \in \mathbb{R}_{>0}$ such that $|f_{\lambda_0}^n(0)| < B$ for all n. It follows that

$$\left| \frac{1}{\lambda_0} \cdot f_{\lambda_0}^{n+1}(0) \cdot f_{\lambda_0}^{m+1}(0) \right| < \frac{B^2}{|\lambda_0|}$$

for all n, m, and thus $\mathcal{R}_2^2(\lambda_0)$ is bounded and in particular not dense in $\widehat{\mathbb{C}}$.

4. Equality of the zero-locus and the activity-locus for $\Delta \geq 3$

In this section we prove the equalities $\mathcal{A}_\Delta^1 = \mathcal{A}_\Delta^2 = \cdots = \mathcal{A}_\Delta^\Delta = \overline{\mathcal{Z}_\Delta}$ for $\Delta \geq 3$, thereby proving that the activity-locus is equal to the zero-locus. Our strategy is similar to the $\Delta = 2$ case. By definition we have $\mathcal{A}_\Delta^1 \subseteq \mathcal{A}_\Delta^2 \subseteq \cdots \subseteq \mathcal{A}_\Delta^\Delta$. We will first show that $\mathcal{A}_\Delta^1 = \mathcal{A}_\Delta^2 = \cdots = \mathcal{A}_\Delta^{\Delta-1}$ and subsequently we will show that $\overline{\mathcal{Z}_\Delta} \subseteq \mathcal{A}_\Delta^{\Delta-1}$. Then Corollary 2·12, which states that $\mathcal{A}_\Delta^\Delta \subseteq \overline{\mathcal{Z}_\Delta}$, is enough to arrive at our desired conclusion.

LEMMA 4·1. *The family $\mathcal{R}_\Delta^1$ is normal at $\lambda_0 \in \mathbb{C}$ if and only if $\mathcal{R}_\Delta^{\Delta-1}$ is normal at λ_0, and hence $\mathcal{A}_\Delta^1 = \mathcal{A}_\Delta^{\Delta-1}$.*

Proof. Recall that

$$\mathcal{R}_\Delta^i := \{R_{G,v} : (G, v) \in \mathcal{G}_\Delta^i\}$$

and thus $\mathcal{R}_\Delta^1 \subseteq \mathcal{R}_\Delta^{\Delta-1}$. It follows that if $\mathcal{R}_\Delta^{\Delta-1}$ is normal at λ_0 then the same holds for $\mathcal{R}_\Delta^1$.

To show the other direction, assume that $\mathcal{R}_\Delta^1$ is normal at λ_0. Note that the family $\mathcal{R}_\Delta^\Delta$ is normal at 0 by Lemma 2·10, hence we can assume $\lambda_0 \neq 0$. As $\mathcal{R}_\Delta^1$ is normal at λ_0, there is a neighbourhood U of λ_0 on which $\mathcal{R}_\Delta^1$ is a normal family. We can take U such that $0 \notin U$. We will show that $\mathcal{R}_\Delta^{\Delta-1}$ is also a normal family on U. To that effect take a sequence of rooted graphs $\{(G_n, v_n)\}_{n \geq 1} \subseteq \mathcal{G}_\Delta^{\Delta-1}$. Construct the rooted graphs $(\hat{G}_n, \hat{v}_n) \in \mathcal{G}_\Delta^1$ by attaching a root $\hat{v}_n$ to the root v_n of G_n by a single edge. By assumption the sequence $\{R_{\hat{G}_n, \hat{v}_n}\}_{n \geq 1}$ has a subsequence that converges locally uniformly to a function $H : U \to \widehat{\mathbb{C}}$. Let $I \subseteq \mathbb{N}$ be the indices belonging to this subsequence. By Lemma 2·6 we have $R_{\hat{G}_n, \hat{v}_n}(\lambda) = f_\lambda(R_{G_n, v_n}(\lambda))$ for every $\lambda \in U$. Because U does not contain 0, the Möbius transformation f_λ is invertible for every $\lambda \in U$. Therefore for these λ we have

$$\lim_{\substack{n \to \infty \\ n \in I}} R_{G_n, v_n}(\lambda) = \lim_{\substack{n \to \infty \\ n \in I}} f_\lambda^{-1}(f_\lambda(R_{G_n, v_n}(\lambda))) = \lim_{\substack{n \to \infty \\ n \in I}} f_\lambda^{-1}(R_{\hat{G}_n, \hat{v}_n}(\lambda)) = f_\lambda^{-1}(H(\lambda)).$$

Because the map $U \times \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ that sends (λ, z) to $f_\lambda^{-1}(z)$ is continuous, we can conclude that this limit converges locally uniformly on U. Therefore we have shown that the sequence

$\{R_{G_n,v_n}\}_{n\geq 1}$ has a subsequence that converges locally uniformly to the holomorphic function $\lambda \mapsto f_\lambda^{-1}(H(\lambda))$, and thus $\mathcal{R}_\Delta^{\Delta-1}$ is normal at λ_0.

PROPOSITION 4·2. *Let $\Delta \geq 3$. Then $\overline{\mathcal{Z}_\Delta} \subseteq \mathcal{A}_\Delta^{\Delta-1}$, and hence the zero-locus is contained in the activity-locus.*

Proof. Let us assume $\lambda \in \overline{\mathcal{Z}_\Delta}$. Then for any open neighbourhood V of λ there is a $\lambda_0 \in V$ for which $Z_G(\lambda_0) = 0$ for some $G \in \mathcal{G}_\Delta$. We will prove that the family $\mathcal{R}_\Delta^{\Delta-1}$ cannot be normal on V.

By Lemma 2·3 there is a rooted tree $(T, u) \in \mathcal{G}_\Delta^1$ for which $R_{T,u}(\lambda_0) = -1$. Consider the rooted trees (T_n, v) obtained by implementing a copy of (T, u) in every vertex of the rooted paths (P_n, v). It follows from Lemma 2·8 that

$$R_{T_n,v} = R_{P_n,v} \circ R_{T,u}.$$

We note that in T_n the root v has degree $2 \leq \Delta - 1$. Furthermore $R_{T,u}$ maps a neighbourhood of λ_0 holomorphically to a neighbourhood of -1, since $R_{T,u}$ is not constantly equal to -1. Lemma 3·4 states that the family $\{R_{P_n,v}\}_{n>0}$ is not normal at -1 and thus it follows that $\{R_{T_n,v}\}_{n>0}$ is not normal at λ_0.

Summarising we have the following relations between sets

$$\mathcal{A}_\Delta^1 \stackrel{(1)}{=} \mathcal{A}_\Delta^{\Delta-1} \subseteq \mathcal{A}_\Delta^\Delta \stackrel{(2)}{\subseteq} \overline{\mathcal{Z}_\Delta} \stackrel{(3)}{\subseteq} \mathcal{A}_\Delta^{\Delta-1},$$

where equality (1) is due to Lemma 4·1, inclusion (2) is due to Corollary 2·12 and inclusion (3) is due to Proposition 4·2. It follows that $\mathcal{A}_\Delta^1 = \ldots = \mathcal{A}_\Delta^\Delta$, and $\mathcal{A}_\Delta = \overline{\mathcal{Z}_\Delta}$ for all $\Delta \geq 2$.

4·1. *The complement of the zero-locus*

As an application of the equality of the zero-locus and the activity-locus, we show here that each component of the complement of the zero-locus is simply connected. We recall from the introduction that this implies that if the complement of the zero-locus is connected (as we conjecture in Conjecture 1·2), then our main result gives a complete understanding of the complexity of approximately computing the independence polynomial.

PROPOSITION 4·3. *Let $\Delta \geq 2$ be an integer. Any connected component of the complement of the zero-locus, $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$, is simply connected.*

Proof. For $\Delta = 2$ the statement follows directly by the exact characterization of the closure of the zero-locus. We will therefore assume that $\Delta \geq 3$.

Let γ be a simple closed curve contained in the complement of the zero-locus, $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$. It is sufficient to prove that the interior of γ, which we will denote by V, is zero free. Let us suppose for the sake of a contradiction that this is not the case.

Let T be a minimal tree for which $Z_T(\lambda_0) = 0$ for some $\lambda_0 \in V$. Let v be a leaf of T. Since $|T|$ is chosen minimal it follows that $R_{T,v}(\lambda_0) = -1$. Denote the neighbour of v in T by w. By minimality of T it also follows that $R_{T-v,w}(\lambda) \neq -1$ for any $\lambda \in \overline{V}$.

Note that V is necessarily bounded, as it is a subset of the cardioid, Λ_Δ. Hence by compactness of $\overline{V}$ it follows that $R_{T-v,w}$ is bounded away from -1 on V. Since

$$R_{T,v}(\lambda) = \frac{\lambda}{1 + R_{T-v,w}(\lambda)}$$

it follows that $R_{T,v}$ is bounded on $\overline{V}$. By the Open Mapping Theorem for holomorphic functions it follows that there must be a $\lambda_1 \in \partial V = \gamma$ for which

$$R_{T,v}(\lambda_1) \in (-\infty, -1).$$

Use Lemma 2·6 to implement the rooted tree (T, v) in the paths P_n to obtain a sequence of rooted graphs $\{(G_n, u_n)\}_{n \geq 1}$ with

$$R_{G_n, u_n} = R_{P_n, u_n} \circ R_{T,v}.$$

Since $R_{T,v}(\lambda_1) \in (-\infty, -1)$, which is contained in the half-line where the family $\{\lambda \mapsto R_{P_n, u}(\lambda)\}_{n \in \mathbb{N}}$ is not normal, it follows that the family of ratios $\{R_{G_n, u_n}\}$ is not normal at λ_1. This contradicts the assumption that γ is contained in $\mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$, by the equivalence of the activity-locus and the zero-locus.

5. *Equality of the density-locus and the activity-locus for* $\Delta \geq 3$

We first show the inclusion $\overline{\mathcal{D}_\Delta} \subseteq \mathcal{A}_\Delta$ holds. Note that as $\mathcal{A}_\Delta$ is closed, it suffices to show $\mathcal{D}_\Delta \subseteq \mathcal{A}_\Delta$.

THEOREM 5·1. *The density-locus is contained in the activity-locus. More precisely, we have* $\mathcal{D}_\Delta \subseteq \mathcal{A}_\Delta$ *for all* $\Delta \geq 3$.

Remark 5·2. Recall the remarkable Proposition 6 in [**BGGŠ20**], in which it is shown that non-real $\lambda \in \mathbb{Q}[i]$ outside the cardioid Λ_Δ are contained in the density-locus. As a consequence Theorem 5·1 implies that $\mathcal{Z}_\Delta$ is dense in the complement of the cardioid.

The proof of Theorem 5·1 is by contradiction. So we will assume that there is a $\lambda_0 \in \mathcal{D}_\Delta$ with $\lambda_0 \notin \mathcal{A}_\Delta$ and arrive at a contradiction. In order to do this, we state and prove three helpful lemmas.

LEMMA 5·3. *Let* $\lambda_0 \in \mathbb{C} \setminus \mathcal{A}_\Delta$. *Assume the family* $\mathcal{R}_\Delta$ *is normal on some open neighbourhood* U *of* λ_0 *and that* $\{R_{G_n, v_n}(\lambda_0)\}_{n \geq 1}$ *converges to* -1 *for a sequence* $\{(G_n, v_n)\}_{n \geq 1}$ *of rooted graphs from* $\mathcal{G}_\Delta$. *Then* $\{R_{G_n, v_n}\}_{n \geq 1}$ *converges to* -1 *locally uniformly on* U.

Proof. It follows from the conclusion of Section 4, i.e. $\mathcal{A}_\Delta = \overline{\mathcal{Z}_\Delta}$, that $Z_G(\lambda) \neq 0$ for all $\lambda \in U$ and $G \in \mathcal{G}_\Delta$. Suppose $\{R_{G_n, v_n}\}_{n \geq 1}$ does not converge to -1 locally uniformly on U. Then, after taking a subsequence if necessary, we may assume that $\{R_{G_n, v_n}\}_{n \geq 1}$ converges locally uniformly on U to a non-constant holomorphic function f. Clearly $f(\lambda_0) = -1$. Since zeros of holomorphic functions are isolated there exists $\varepsilon > 0$ so that $\overline{B(\lambda_0, \varepsilon)} \subset U$ and such that

$$\delta := \inf_{\lambda \in \partial B(\lambda_0, \varepsilon)} |f(\lambda) + 1| > 0.$$

Let n be sufficiently large so that $|R_{G_n, v_n} - f| < \delta$ uniformly on $\overline{B(\lambda_0, \varepsilon)}$. Then

$$|(R_{G_n, v_n}(\lambda) + 1) - (f(\lambda) + 1)| < \delta < |f(\lambda) + 1| + |R_{G_n, v_n}(\lambda) + 1|$$

for all $\lambda \in \partial B(\lambda_0, \varepsilon)$. There exists $\lambda_1 \in B(\lambda_0, \varepsilon)$ for which $R_{G_n, v_n}(\lambda_1) = -1$ by Rouché's theorem. By Lemma 2·1 it follows λ_1 is a zero of the independence polynomial Z_G for

some graph G of maximum degree at most Δ, which is a contradiction as we assumed $\lambda_0 \in \mathbb{C} \setminus \mathcal{A}_\Delta = \mathbb{C} \setminus \overline{\mathcal{Z}_\Delta}$.

LEMMA 5·4. *Let $\lambda_0 \in \mathbb{C} \setminus \mathcal{A}_\Delta$. Assume the family $\mathcal{R}_\Delta$ is normal on some open neighbourhood U of λ_0 and that $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ converges to $\mu \leq -1/4$ for a sequence of rooted graphs $\{(G_n, v_n)\}_{n\geq 1}$ in $\mathcal{G}^1_\Delta$. Then $\{R_{G_n,v_n}\}_{n\geq 1}$ converges to μ locally uniformly on U.*

Proof. If this is not the case then, as in the previous lemma, we may assume that $\{R_{G_n,v_n}\}_{n\geq 1}$ converges locally uniformly to a non-constant holomorphic function f with $f(\lambda_0) = \mu$. By Rouché's theorem we can find $\lambda_1 \in U$ and n sufficiently large so that $R_{G_n,v_n}(\lambda_1) = \mu$, by the same argument as in the previous lemma.

Consider the family of rooted graphs $\{(\widetilde{G}_k, w_k)\}$ obtained by implementing (G_n, v_n) in every vertex of the rooted paths (P_k, w_k), where P_k is the path with k vertices and w_k is one of its extreme vertices. Since v_n has degree 1, the graph $\widetilde{G}_k$ has maximum degree at most Δ. Hence by Lemma 2·6 we have

$$R_{\widetilde{G}_k,w_k}(\lambda) = f^k_{R_{G_n,v_n}(\lambda)}(0).$$

By Lemma 3·4 the family $\{R_{P_k,w_k}\} = \{\lambda \mapsto f^k_\lambda(0)\}$ is non-normal at $\lambda = \mu$, and therefore the family $\{R_{\widetilde{G}_k,w_k}\}$ is non-normal at λ_1, contradicting the fact that the family $\mathcal{R}_\Delta$ is normal on U.

LEMMA 5·5. *Assume there is a $\lambda_0 \in \mathcal{D}_\Delta$ with $\lambda_0 \notin \mathcal{A}_\Delta$. Denote U for an open neighbourhood of λ_0 on which the family $\mathcal{R}_\Delta$ is normal. Assume that $\{R_{G_n,v_n}(\lambda_0)\}_{n\geq 1}$ converges to $\mu \in \mathbb{R}$ for a sequence of rooted graphs $\{(G_n, v_n)\}_{n\geq 1}$ in $\mathcal{G}^{\Delta-1}_\Delta$. Then $\{R_{G_n,v_n}\}_{n\geq 1}$ converges to μ locally uniformly on U.*

Proof. If $\mu = -1$ the result follows by Lemma 5·3. We may therefore assume that $\mu \neq -1$. Recall that we denote $f_\lambda(z) = \lambda/(1+z)$. We will show for each $\mu \in \mathbb{R}$ there exists $\mu_1, \mu_2, \mu_3 \leq -1/4$ so that

$$f_{\mu_m} \circ \cdots \circ f_{\mu_1}(\mu) = -1,$$

for some $m \leq 3$. We distinguish between different cases:

(i) $\mu \geq -3/4$. Take $\mu_1 = -1 - \mu \leq -1/4$, one can check $f_{\mu_1}(\mu) = -1$;

(ii) $\mu < -1$. Take $\mu_1 = -1/4$ and $\mu_2 = 1 - f_{\mu_1}(\mu)$, then $f_{\mu_1}(\mu) > 0 > -3/4$ and so $\mu_2 \leq -1/4$. One can check $f_{\mu_2} \circ f_{\mu_1}(\mu) = -1$;

(iii) $-1 < \mu < -3/4$. Take $\mu_1 = \mu_2 = -1/4$ and $\mu_3 = 1 - f_{\mu_2}(f_{\mu_1}(\mu))$, then $f_{\mu_1}(\mu) < -1$ so we see $\mu_3 \leq -1/4$. One can check $f_{\mu_3} \circ f_{\mu_2} \circ f_{\mu_1}(\mu) = -1$.

We may assume that $\{R_{G_n,v_n}\}_{n\geq 1}$ converges locally uniformly on U to a holomorphic function f with $f(\lambda_0) = \mu$. We want to show f is constant on U. Since the set $\{R_{G,v}(\lambda_0) : (G, v) \in \mathcal{G}^1_\Delta\}$ is dense in $\hat{\mathbb{C}}$ by assumption, we can choose sequences of rooted graphs $\{(G^i_n, v^i_n)\}_{n\geq 1}$ in $\mathcal{G}^1_\Delta$ so that $\{R_{G^i_n,v^i_n}(\lambda_0)\}_{n\geq 1}$ converges to μ_i for each $i = 1, \ldots, m$. By Lemma 5·4 every sequence $\{R_{G^i_n,v^i_n}\}_{n\geq 1}$ converges locally uniformly on U to the constant function μ_i for each i.

Consider for each $n \geq 1$, the rooted graph $(\widetilde{G}_n, v^m_n)$ obtained by implementing the rooted graphs $(G_n, v_n), (G^1_n, v^1_n), \ldots, (G^m_n, v^m_n)$ on the vertices of the path P_{m+1} of length m. Note that $\widetilde{G}_n$ has maximum degree at most Δ.
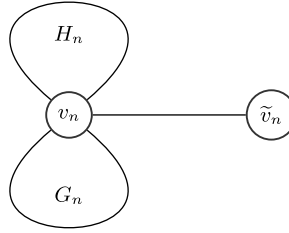
Fig 3. The rooted graph $\widetilde{G}_n$ in the proof of Theorem 5·1.

It follows from Lemma 2·6 that

$$R_{\widetilde{G}_n, v_n^m}(\lambda) = f_{R_{G_n^m, v_n^m}(\lambda)} \circ \cdots \circ f_{R_{G_n^1, v_n^1}(\lambda)} \circ R_{G_n, v_n}(\lambda).$$

By our choice of the μ_i the sequence of ratios $\{R_{\widetilde{G}_n, v_n^m}(\lambda_0)\}_{n \geq 1}$ converges to $f_{\mu_m} \circ \cdots \circ f_{\mu_1}(\mu) = -1$. Hence by Lemma 5·3 the sequence of ratios $\{R_{\widetilde{G}_n, v_n^m}\}_{n \geq 1}$ converges locally uniformly to the constant function -1. Furthermore the sequence of ratios $\{R_{\widetilde{G}_n, v_n^m}\}_{n \geq 1}$ converges to the function $F := f_{\mu_m} \circ \cdots \circ f_{\mu_1} \circ f$. As $f_{\mu_m} \circ \cdots \circ f_{\mu_1}(z)$ is a non-constant holomorphic function and $F = -1$ on U, it follows that f is constant on U, as desired.

We are now ready to prove Theorem 5·1.

Proof of Theorem 5·1. Assume for the purpose of a contradiction that there exists $\lambda_0 \in \mathcal{D}_\Delta^1$ with $\lambda_0 \notin \mathcal{A}_\Delta$. We note that by Lemma 2·10 we know $\lambda_0 \neq 0$. Throughout the proof denote U for an open neighbourhood of λ_0 on which the family $\mathcal{R}_\Delta$ is normal; we may assume $0 \notin U$ by taking U small enough. Assume first that λ_0 is not purely imaginary. Consider the real number $c = -|\lambda_0|^2/2\mathrm{Re}\,\lambda_0$ and notice that

$$\frac{\lambda_0^2}{\lambda_0 + c} = 2\mathrm{Re}\,\lambda_0 \in \mathbb{R}.$$

Choose two sequences of rooted graphs $\{G_n, v_n\}_{n \geq 1}$, $\{(H_n, w_n)\}_{n \geq 1}$ in $\mathcal{G}_\Delta^1$ so that the sequences of ratios $\{R_{G_n, v_n}(\lambda_0)\}_{n \geq 1}$ and $\{R_{H_n, w_n}(\lambda_0)\}_{n \geq 1}$ converge to respectively 1 and c. By Lemma 5·5 we must have that these sequences of ratios converge locally uniformly on U to the respective constants 1 and c.

Consider the sequence of graphs $\widetilde{G}_{n \geq 1}$ constructed by merging v_n and w_n and by then connecting this vertex to a vertex $\widetilde{v}_n$. See Figure 3 for an illustration.

It follows from Lemma 2·6 and Lemma 2·9 for all $\lambda \in U$ that

$$R_{\widetilde{G}_n, \widetilde{v}_n}(\lambda) = \frac{\lambda}{1 + \lambda^{-1} R_{G_n, v_n}(\lambda)\, R_{H_n, w_n}(\lambda)},$$

where we use $0 \notin U$. Therefore the sequence of holomorphic functions $\{R_{\widetilde{G}_n, \widetilde{v}_n}\}_{n \geq 1}$ converges locally uniformly on U to the function $f(\lambda) = \lambda^2/(\lambda + c)$ as $n \to \infty$. Note that f is not a constant function, and that $f(\lambda_0) \in \mathbb{R}$, contradicting Lemma 5·5. This contradiction completes the proof for λ_0 not purely imaginary.

Assume instead that λ_0 is purely imaginary and let $(G, v) \in \mathcal{G}_\Delta^1$ so that $R_{G, v}(\lambda_0)$ is not purely imaginary. For $c \in \mathbb{R}$ to be determined later choose again two sequences of rooted graphs $\{G_n, v_n\}_{n \geq 1}$, $\{(H_n, w_n)\}_{n \geq 1}$ in $\mathcal{G}_\Delta^1$ such that sequences $\{R_{G_n, v_n}(\lambda_0)\}_{n \geq 1}$ and

$\{R_{H_n,w_n}(\lambda_0)\}_{n\geq 1}$ converge to 1 and c respectively. Define for each $n \geq 1$, $(\widetilde{G}_n, \widetilde{v}_n)$ as above and let (K_n, v_n) be the rooted graph obtained from the disjoint union of $(\widetilde{G}_n, \widetilde{v}_n)$ and (G, v) by identifying the vertex $\widetilde{v}_n$ with v. It follows from Lemma 2·6 and Lemma 2·9 for $\lambda \in U$ that

$$R_{K_n,v_n}(\lambda) = \frac{R_{G,v}(\lambda)}{1 + \lambda^{-1}R_{G_n,v_n}(\lambda)\,R_{H_n,w_n}(\lambda)},$$

where we use $0 \notin U$. Thus in order to follow the same argument as before we require $c \in \mathbb{R}$ for which

$$\frac{\lambda_0 \cdot R_{G,v}(\lambda_0)}{\lambda_0 + c} \in \mathbb{R}.$$

It is clear that such real number c exists, hence the identical argument leads to the desired contradiction.

We will now show the other inclusion $\mathcal{A}_\Delta \subseteq \overline{D_\Delta}$ also holds for all $\Delta \geq 3$. We first show the inclusion holds for non-real parameters $\lambda \in \mathcal{A}_\Delta$.

THEOREM 5·6. *Let $\Delta \geq 3$ and suppose that the family $\mathcal{R}_\Delta$ is not normal in any neighbourhood of $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$. Then there exists λ_1 arbitrarily close to λ_0 for which the set $\{R_{G,v}(\lambda_1) : (G, v) \in \mathcal{G}_\Delta^1\}$ is dense in $\hat{\mathbb{C}}$.*

Proof. Because $\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta$ there exists λ_2 arbitrarily close to λ_0 for which there is a graph G of maximum degree at most Δ such that $Z_G(\lambda_2) = 0$. We claim that we can assume $\lambda_2 \notin \mathbb{R}$. This is clear if $\lambda_0 \notin \mathbb{R}$. Moreover, if $\lambda_0 \in \mathbb{R}$ then λ_0 is a strictly positive real number. Because $Z_G(x) > 0$ for any positive real number x, it follows that λ_2 is necessarily not real as long as it is sufficiently close to λ_0.

By Lemma 2·3 there is a rooted tree $(T, v) \in \mathcal{G}_\Delta^1$ such that $R_{T,v}(\lambda_2) = -1$. Since the rational function $\lambda \mapsto R_{T,v}(\lambda)$ is non-constant, it is an open map. The image of a neighbourhood of λ_2 therefore contains a small open real interval around -1. Recall that Lemma 3·2 states that for $\mu \in (-\infty, -1/4)$ the map $f_\mu : z \mapsto \mu/(1 + z)$ is conjugate to a rotation $w \mapsto e^{i\theta} \cdot w$. Furthermore, by comparing tr^2 of both maps, it is not hard to see that those parameters μ for which f_μ is conjugate to an irrational rotation lie dense in $(-\infty, -1/4)$. Therefore we can choose a $\lambda_1 \in \mathbb{C} \setminus \mathbb{R}$ arbitrarily close to λ_2 such that for $\mu := R_{T,v}(\lambda_1)$ the map f_μ is conjugate to an irrational rotation. From now on μ is fixed to be this value.

Let p, q be the two fixed points of the transformation f_μ. In Section 3.1 we explained that $\hat{\mathbb{C}} \setminus \{p, q\}$ is foliated by generalized circles invariant under f_μ, and on which f_μ acts conjugate to an irrational rotation. We denote the generalized circle through z by C_z, and write C_q and C_p for $\{q\}$ and $\{p\}$ respectively. The map $z \mapsto C_z$ is continuous as a map from $\hat{\mathbb{C}}$ to the space $\{K \subseteq \hat{\mathbb{C}} : K \text{ compact}\}$ equipped with the Hausdorff metric.

Our aim is to show that $\mathcal{R}_\Delta^1(\lambda_1)$ is dense in $\hat{\mathbb{C}}$. We first claim that if $w \in \mathcal{R}_\Delta^1(\lambda_1)$, then $\mathcal{R}_\Delta^2(\lambda_1) \cap C_w$ is dense in C_w.

To prove the claim, let $(H, u) \in \mathcal{G}_\Delta^1$ be a rooted graph such that $R_{H,u}(\lambda_1) = w$. Let $\widetilde{G}_n$ as follows be obtained from the path P_{n+1} on $n + 1$ vertices, labeled v_0 up to v_n, by implementing (H, u) at v_0 and the rooted tree (T, v) at the remaining n vertices of P_{n+1}, see Figure 4. Now by Lemma 2·6 we have

$$R_{\widetilde{G}_n,v_n}(\lambda_1) = f_\mu^n(R_{H,u}(\lambda_1)) = f_\mu^n(w).$$
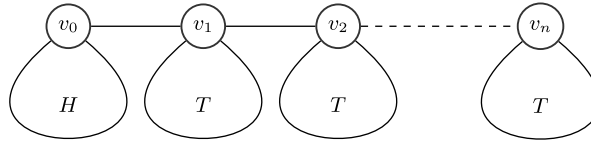
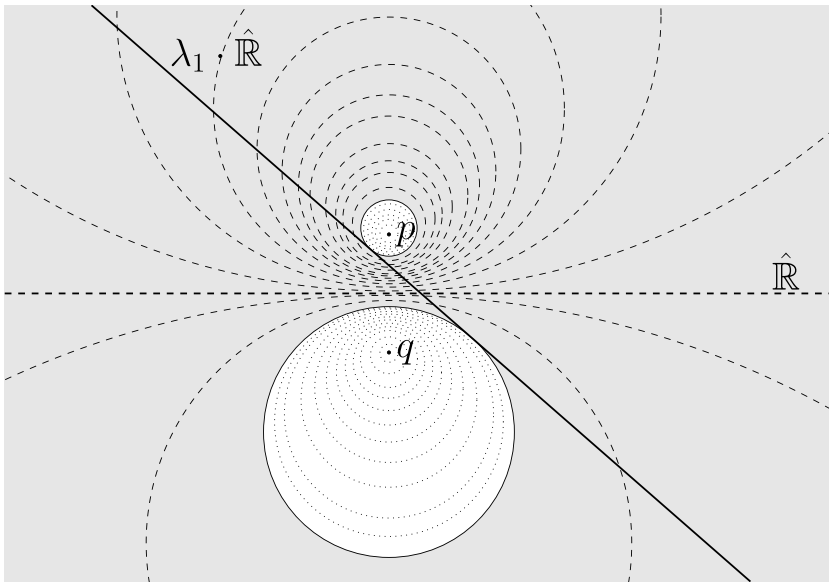Fig 4. The graph $(\widetilde{G}_n, v_n)$ in the proof of the claim.

Fig 5. The generalised circles $\lambda_1 \cdot \hat{\mathbb{R}}$ and $\hat{\mathbb{R}}$ intersect in the points 0 and ∞. A region around 0 is drawn. The open set U is shaded in gray. Examples of generalised circles C_w that intersect $\lambda_1 \cdot \hat{\mathbb{R}}$ transversely are drawn with dashed lines, while examples of circles that do not intersect $\lambda_1 \cdot \hat{\mathbb{R}}$ are drawn with dotted lines.

Observe that for each $n \geq 1$ we have $(\widetilde{G}_n, v_n) \in \mathcal{G}_\Delta^2$. Because f_μ acts conjugately to an irrational rotation on C_w it follows that $\mathcal{R}_\Delta^2(\lambda_1) \cap C_w$ is dense in C_w.

Because $\mu \in \mathcal{R}_\Delta^1(\lambda_1)$ and $C_\mu = \hat{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ it follows from the claim that $\mathcal{R}_\Delta^2(\lambda_1) \cap \hat{\mathbb{R}}$ is dense in $\hat{\mathbb{R}}$. Observe that $f_{\lambda_1}(\hat{\mathbb{R}}) = \lambda_1 \cdot \hat{\mathbb{R}}$. So by attaching a vertex at the root with an edge, we obtain that $\mathcal{R}_\Delta^1(\lambda_1) \cap \lambda_1 \cdot \hat{\mathbb{R}}$ is dense in $\lambda_1 \cdot \hat{\mathbb{R}}$.

The set

$$U = \{z \in \hat{\mathbb{C}} : C_z \text{ intersects } \lambda_1 \cdot \hat{\mathbb{R}} \text{ transversely}\}$$

is an open set in $\hat{\mathbb{C}}$, see Figure 5. Because $\lambda_1 \notin \mathbb{R}$ we see that $C_{-1} = \hat{\mathbb{R}}$ intersects $\lambda_1 \cdot \hat{\mathbb{R}}$ transversely, and thus $-1 \in U$. The set U is contained in $\cup_{w \in \lambda_1 \cdot \hat{\mathbb{R}}} C_w$. Because $\mathcal{R}_\Delta^1(\lambda_1) \cap \lambda_1 \cdot \hat{\mathbb{R}}$ is dense in $\lambda_1 \cdot \hat{\mathbb{R}}$, it follows that $\cup_{w \in \mathcal{R}_\Delta^1(\lambda_1)} C_w$ is dense in U. From the claim we proved earlier, it follows $\mathcal{R}_\Delta^2(\lambda_1)$ is dense in U. Attaching a vertex to the root of a tree in $\mathcal{G}_\Delta^2$ with ratio r yields a rooted tree in $\mathcal{G}_\Delta^1$ with ratio $f_{\lambda_1}(r)$, and thus $\mathcal{R}_\Delta^1(\lambda_1)$ is dense in the neighbourhood $U_\infty := f_{\lambda_1}(U)$ of ∞.

For two rooted trees $(T_1, v_1) \in \mathcal{G}_\Delta^1$ and $(T_2, v_2) \in \mathcal{G}_\Delta^2$ with ratios r_1 and r_2 respectively we can define the rooted tree $(T_3, v_1) \in \mathcal{G}_\Delta^2$ by adding an edge between the roots of T_1 and T_2 and considering v_1 the root of the obtained tree. By Lemma 2·6 the ratio of (T_3, v_1) is given by

$$F(r_1, r_2) := f_{r_1}(r_2) = \frac{r_1}{1 + r_2}$$

under the assumption that this fraction is well defined, i.e., $(r_1, r_2) \notin \{(0, -1), (\infty, \infty)\}$. It is not hard to see that

$$F(U_\infty \times \hat{\mathbb{R}} \setminus \{(0, -1), (\infty, \infty)\}) = \hat{\mathbb{C}}.$$

Because $\mathcal{R}_\Delta^1(\lambda_1)$ is dense in U_∞ and $\mathcal{R}_\Delta^2(\lambda_1)$ is dense in $\hat{\mathbb{R}}$ it follows that $\mathcal{R}_\Delta^2(\lambda_1)$ is dense in $\hat{\mathbb{C}}$. We finally conclude that $\mathcal{R}_\Delta^1(\lambda_1)$ is dense in $f_{\lambda_1}(\hat{\mathbb{C}}) = \hat{\mathbb{C}}$.

We can now finally prove the inclusion $\mathcal{A}_\Delta \subseteq \overline{\mathcal{D}_\Delta}$ building on Proposition 6 of [**BGGŠ20**] to deal with the real parameters $\lambda \in \mathcal{A}_\Delta$.

THEOREM 5·7. *Let $\Delta \geq 3$. Then the activity locus is contained in the density locus, i.e.* $\mathcal{A}_\Delta \subseteq \overline{\mathcal{D}_\Delta}$.

Proof. Let $\lambda_0 \in \mathcal{A}_\Delta$. If $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}_{\leq 0}$, then $\lambda_0 \in \overline{\mathcal{D}_\Delta}$ follows from Theorem 5·6. We know $\overline{\mathcal{Z}_\Delta} = \mathcal{A}_\Delta$. By Remark 2·11 we know that

$$\overline{\mathcal{Z}_\Delta} \cap \mathbb{R}_{\leq 0} = \mathbb{R}_{\leq 0} \setminus int(\Lambda_\Delta)$$

for all $\Delta \geq 3$. Proposition 6 of [**BGGŠ20**] implies that $\mathbb{C} \setminus (\mathbb{R} \cup \Lambda_\Delta) \subseteq \overline{\mathcal{D}_\Delta}$ for $\Delta \geq 3$. Hence it follows that $\mathbb{R}_{\leq 0} \setminus int(\Lambda_\Delta) \subseteq \overline{\mathcal{D}_\Delta}$, which completes the proof.

6. *Density implies #P-hardness*

In this section we will show that the density-locus is contained in the $\#\mathcal{P}$-locus. To prove our result we will need to to show 'exponential' density for ratios of a specific family of trees: we need to get ε-close to a given point $P \in \mathbb{Q}[i]$ with ratios of trees of size at most $O(\log(1/\varepsilon) + \text{size}(P))$. Here $\text{size}(P)$ denotes the sum of the bit sizes of the real and imaginary part of P. Moreover, we denote for rational $\varepsilon > 0$ by $\text{size}(\varepsilon, P)$ the sum of the the bit size of ε and $\text{size}(P)$.

Let $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$. Then the Möbius transformation f_{λ_0} is loxodromic (cf. Section 3.1) and hence has a repelling fixed point, which we denote by z_0.

Let

$$A := \{z \in \mathbb{C} : 2\pi/3 - 0.01 < \arg z < 2\pi/3, 1/17 < |z| < 1/16\}. \tag{6·1}$$

Let $\mathcal{T} = \{(G_1, v_1), \ldots, (G_M, v_M), (\overline{G}_1, \overline{v}_1), \ldots, (\overline{G}_M, \overline{v}_M)\}$ be a family of rooted trees and U an open disk containing z_0. The pair $(\mathcal{T}, U)$ is called a *fast implementer for* λ_0 if the ratios $\mu_i := R_{G_i, v_i}$ and $\chi_i := R_{\overline{G}_i, \overline{v}_i}$ are such that the maps $g_i := f_{\mu_i} \circ f_{\chi_i}$ are loxodromic and satisfy:

  (i) the attracting fixed point z_i of g_i lies in U for all i;
  (ii) $\overline{U} \subseteq \bigcup_{i=1}^M g_i(U)$;
  (iii) $g_i'(z) \in A$ for all i and all $z \in \overline{U}$;

and the disk U is such that:

   (i) $\overline{U} \subset f_{\lambda_0}(U)$;

  (ii) $\overline{U}$ does not contain the attracting fixed point of f_{λ_0};

 (iii) U has three rational points on its boundary.

We have the following results concerning fast implementers.

LEMMA 6·1. *Let $\Delta \geq 3$ be an integer. Let $\lambda_0 \in \mathcal{D}_\Delta$. Then there exists a fast implementer $(\mathcal{T}, U)$ for λ_0.*

LEMMA 6·2. *Let $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ and assume that there exists a fast implementer*

$$(\{(G_1, v_1), \ldots, (G_M, v_M), (\overline{G}_1, \overline{v}_1), \ldots, (\overline{G}_M, \overline{v}_M)\}, U)$$

for λ_0. Then, given $P \in \mathbb{C}$ and $\epsilon > 0$ there exists an algorithm that yields a sequence of ratios

$$w_1, \ldots, w_K \in \{\lambda_0\} \cup \bigcup_{i=1}^{M} \{\mu_i := R_{G_i, v_i}, \chi_i := R_{\overline{G}_i, \overline{v}_i}\}$$

such that $|(f_{w_K} \circ \cdots \circ f_{w_1})(0) - P| < \epsilon$, $w_K = \lambda_0$ and $K = \mathcal{O}(\max(\log(1/\epsilon), \log(|P|/\epsilon)))$. If $\lambda_0 \in \mathbb{Q}[i]$ and the input parameters P, ϵ are also in $\mathbb{Q}[i]$ then the algorithm runs in $\text{poly}(\text{size}(P, \epsilon))$ time.

We provide proofs for these lemmas in the next subsection, but first we collect some consequences.

COROLLARY 6·3 *Let $\Delta \geq 3$ be an integer. The set $\mathcal{D}_\Delta$ is an open set.*

Proof. Let $\lambda_0 \in \mathcal{D}_\Delta$. Let $(\mathcal{T}, U)$ be fast implementer, as guaranteed to exist by Lemma 6·1. For λ nearby λ_0 we still have that the repelling fixed point of f_λ is contained in U, its attracting fixed point does not lie in $\overline{U}$ and $\overline{U} \subset f_\lambda(U)$. In other words $(\mathcal{T}, U)$ is a fast implementer for λ. Therefore applying the algorithm of Lemma 6·2 to λ we obtain that the collection of values $\{R_{G,v}(\lambda) \mid (G, v) \in \mathcal{G}_\Delta^1\}$ is dense in $\hat{\mathbb{C}}$ and hence $\lambda \in \mathcal{D}_\Delta$.

For our next corollary we first need a result about the set

$$\mathcal{E}_\Delta := \{\lambda \in \mathbb{Q}[i] \mid Z_G(\lambda) = 0 \text{ for some } G \in \mathcal{G}_\Delta\}.$$

LEMMA 6·4. *Let $\Delta \geq 3$ be an integer. Then the collection $\mathcal{E}_\Delta$ is contained in the set*

$$\left\{ (a + ib)^{-1} \mid a, b \in \mathbb{Z}, \ 0 < \sqrt{a^2 + b^2} \leq \frac{\Delta^\Delta}{(\Delta - 1)^{\Delta - 1}} \right\}.$$

Proof. Let $\lambda \in \mathcal{E}_\Delta$. Then there exists a graph $G = (V, E)$ such that $1/\lambda$ is a root of $P(z) := z^{|V|} Z_G(1/z)$. Now P is a monic polynomial and therefore $1/\lambda \in \mathbb{Z}[i]$ (since $\mathbb{Z}[i]$ is integrally closed by Gauss's lemma). We also know that $|1/\lambda| \leq \Delta^\Delta/(\Delta - 1)^{\Delta - 1}$ by Lemma 2·10. This proves the lemma.

COROLLARY 6·5. *Let $\Delta \geq 3$ be an integer. Let $\lambda_0 \in (\mathcal{D}_\Delta \cap \mathbb{Q}[i]) \setminus \mathcal{E}_\Delta$. Then given $P \in \mathbb{Q}[i]$ and rational $\varepsilon > 0$ there exists an algorithm that generates a rooted tree (T,v) such that $|R_{T,v}(\lambda_0) - P| \leq \varepsilon$ and $Z_{T,v}^{out}(\lambda_0) \neq 0$, and outputs $Z_{T,v}^{in}(\lambda_0)$ and $Z_{T,v}^{out}(\lambda_0)$ in time bounded by poly(size(ε, P)).*

Proof. We first perform a brute force, but constant time, computation to obtain a fast implementer $(\{(G_1, v_1), \ldots, (G_M, v_M), (\overline{G}_1, \overline{v}_1), \ldots, (\overline{G}_M, \overline{v}_M)\}, U)$ for λ_0. Denote for $i = 1, \ldots, M$, $\mu_i := R_{G_i, v_i}$ and $\chi_i := R_{\overline{G}_i, \overline{v}_i}$ and $g_i := f_{\mu_i} \circ f_{\chi_i}$.

The algorithm of Lemma 6·2 applied to P now returns in time $poly(size(\varepsilon, P))$ a sequence of ratios $\omega_1 \ldots, \omega_K \in \{\lambda_0\} \cup \bigcup_{i=1}^{M} \{\mu_i, \chi_i\}$ that, by Lemma 2·6, correspond to the implementation of the trees G_i and $\overline{G}_i$ on a path with $K = \mathcal{O}(\max(\log(1/\epsilon), \log(|P|/\epsilon)))$ vertices. The resulting rooted tree (T, v) has maximum degree at most Δ and root degree 1 and satisfies $|R_{T,v}(\lambda_0) - P| \leq \varepsilon$. Denote the rooted tree corresponding to the sequence $\omega_1, \ldots \omega_i$ by (T_i, u_i). Then (T_{i+1}, u_{i+1}) is obtained from (T_i, u_i) by adding the edge $\{v_{i+1}, u_i\}$ to T_i and gluing a rooted tree $(H, v) \in \{K_1, (G_j, v_j), (\overline{G}_j, v_j) \mid j = 1, \ldots, M\}$ to u_{i+1} (here K_1 denotes a single vertex.) We then have

$$\left(Z_{T_{i+1}, u_{i+1}}^{in}(\lambda_0), Z_{T_{i+1}, u_{i+1}}^{out}(\lambda_0) \right) = \left(Z_{H,v}^{in}(\lambda_0) Z_{T_i, u_i}^{out}(\lambda_0), Z_{H,v}^{out}(\lambda_0) Z_{T_i}(\lambda_0) \right). \quad (6\cdot2)$$

Note that (6·2) describes a simple recurrence to compute $Z_{T,v}^{in}(\lambda_0)$ and $Z_{T,v}^{out}(\lambda_0)$ in time linear in the number of vertices of T.

Finally, we remark that $Z_{T,v}^{out}(\lambda_0) \neq 0$ since $\lambda_0 \notin \mathcal{E}_\Delta$ by assumption.

We can now prove the desired inclusion of the density-locus in the #$\mathcal{P}$-locus.

THEOREM 6·6. *For any integer $\Delta \geq 3$ the density-locus $\overline{\mathcal{D}_\Delta}$ is contained in the #$\mathcal{P}$-locus $\overline{\#\mathcal{P}_\Delta}$.*

Proof. We will show that for any $\lambda_0 \in (\mathcal{D}_\Delta \cap \mathbb{Q}[i]) \setminus \mathcal{E}_\Delta$ the computational problem #Hard-CoreNorm(λ_0, Δ) is #P-hard. Since $\mathcal{D}_\Delta^1$ is an open set and $\mathcal{E}_\Delta$ is finite, this implies the theorem.

This in fact follows directly from the work of [**BGGŠ20**]. Let us briefly indicate why. In [**BGGŠ20**, section 6] the authors show that a polynomial time algorithm for #Hard-CoreNorm(λ_0, Δ) combined with the statement of Corollary 6·5 for λ_0 yields an algorithm that on input of a graph G of maximum degree at most Δ exactly computes $Z_G(1)$, the number of independent sets of G, in polynomial time in the number of vertices of G. (The algorithm is obtained by cleverly utilising Corollary 6·5 for suitable choices of P and gluing combinations of the obtained trees to G and applying the assumed algorithm for #Hard-CoreNorm(λ_0, Δ) to the resulting graph.) Since determining $Z_G(1)$ is a known #P-complete problem, this implies that #Hard-CoreNorm(λ_0, Δ) is #P-hard.

We note that our result does not allow us to say anything about the complexity of #Hard-CoreNorm(λ_0, Δ) for $\lambda_0 \in \partial(\mathcal{D}_\Delta) \cap \mathbb{Q}[i]$. For example, for $\lambda_0 \in \partial(\mathcal{D}_\Delta) \cap \mathbb{Q}_{\leq 0}$ it follows from [**BGGŠ20**] that the problem #Hard-CoreNorm(λ, Δ) is #P-hard. For $\lambda \in \mathbb{Q}$ such that $\lambda \geq \lambda_c(\Delta) := (\Delta - 1)^{\Delta-1}/(\Delta - 2)^\Delta$ we know from [**BGGŠ20**] that $\lambda \in \partial(\mathcal{D}_\Delta)$, while the complexity of #Hard-CoreNorm($\lambda_c(\Delta)$, Δ) is unknown. For $\lambda > \lambda_c(\Delta)$ the problem Hard-CoreNorm(λ, Δ) is only known to be NP-hard [**SS14**], and unlikely to be #P-hard cf. [**BGGŠ20**].

6·1. *Proofs of Lemma 6·1 and Lemma 6·2*

The next lemma directly implies Lemma 6·1.

LEMMA 6·7. *Given $z_0 \in \mathbb{C} \setminus \{-1, 0\}$, a dense subset D of $\mathbb{C}^*$ and a non-empty open subset A of the unit disk $\mathbb{D}$ then there exists a finite set of tuples $\{(\mu_i, \chi_i)\}_{i=1}^M \subset D \times D$ and an arbitrarily small open disk $U \subseteq \mathbb{C}$ containing z_0 such that the maps $g_i := f_{\mu_i} \circ f_{\chi_i}$ are loxodromic Möbius transformations and*:

  (i) *the attracting fixed point z_i of g_i lies in U for all i;*
 (ii) $\overline{U} \subseteq \bigcup_{i=1}^M g_i(U)$;
(iii) $g_i'(z) \in A$ *for all i and all $z \in \overline{U}$.*

Proof. We denote $g_{\mu,\chi} = f_\mu \circ f_\chi$ throughout this proof. Note $g_{\mu,\chi}$ is a Möbius transformation for $\mu, \chi \neq 0$. Without loss of generality assume that A is bounded away from 0. Take $\alpha \in A$ such that $\alpha \neq z_0/(z_0 + 1)$. Note that $\chi_0 = (z_0 + 1)^2 \alpha/(z_0 - (z_0 + 1)\alpha)$ and $\mu_0 = z_0(z_0 + \chi_0 + 1)/z_0 + 1$ are nonzero and well defined as $z_0 \neq -1, 0$ and $\alpha \neq z_0/z_0 + 1, 0$. Furthermore we have $g_{\mu_0,\chi_0}(z_0) = z_0$ and $g_{\mu_0,\chi_0}'(z_0) = \alpha$.

Define $F: (\mathbb{C}^*)^2 \times \mathbb{C} \to \hat{\mathbb{C}}$ as $F(\mu, \chi, z) = g_{\mu,\chi}(z) - z$. Now as $\frac{\partial F}{\partial z}(\mu_0, \chi_0, z_0) = \alpha - 1 \neq 0$, the implicit function theorem gives an open neighbourhood W of (μ_0, χ_0) and a holomorphic function $h: W \to \mathbb{C}$ with $h(\mu_0, \chi_0) = z_0$ and $F(\mu, \chi, h(\mu, \chi)) = 0$ for all $(\mu, \chi) \in W$. As h is a non-constant holomorphic map, it is an open map and so $h(W)$ is an open neighbourhood of z_0.

Let $B \subseteq A$ be an open set in $\mathbb{C}$ with $\alpha \in B$ and $\overline{B} \subseteq A$. Denote $H(\mu, \chi, z) = (\partial g_{\mu,\chi}/\partial z)(z) = \mu\chi/(1 + z + \chi)^2$, note that H is continuous as a function on $\mathbb{C}^3 \setminus \{(\mu, \chi, z): \chi + z + 1 = 0\}$. It follows there is an open neighbourhood C of z_0 such that we have $H(\mu_0, \xi_0, z) \in B$ for all $z \in C$. We have $\{(\mu_0, \chi_0)\} \times \overline{C} \subseteq H^{-1}(\overline{B}) \subseteq H^{-1}(A)$. As $H^{-1}(A)$ is an open subset of $\mathbb{C}^3 \setminus \{(\mu, \chi, z): \chi + z + 1 = 0\}$ containing the set $\{(\mu_0, \chi_0)\} \times \overline{C}$, by a compactness argument it follows that $H^{-1}(A)$ contains a set of the form $L \times C$, for some open neighbourhood L of the point (μ_0, χ_0). Hence the set $Y := L \cap W \cap h^{-1}(C)$ is an open neighbourhood of (μ_0, χ_0) and so $h(Y)$ is an open neighbourhood of z_0.

Take $U \subset h(Y)$ an open disk containing z_0, such that $\overline{U} \subset h(Y)$. Note that we can take U arbitrarily small. By construction, we have for all $(\mu, \chi) \in Y$ that $g_{\mu,\chi}'(z) \in A$ for all $z \in \overline{U}$. Furthermore, we have $F(\mu, \chi, h(\mu, \chi)) = 0$, so $h(\mu, \chi)$ is the attracting fixed point of $g_{\mu,\chi}$. Note $D \times D$ is dense in $h^{-1}(U)$, hence the fixed points of $g_{\mu,\chi}$ for $(\mu, \chi) \in h^{-1}(U) \cap (D \times D)$ lie dense in U. There is a uniform lower bound on the diameters of the disks $g_{\mu,\chi}(U)$ for $(\mu, \chi) \in h^{-1}(U)$, because $g'_{\mu,\chi}(z) \in A$ for all $z \in U$ and A is bounded away from 0. Therefore

$$\left\{ g_{\mu,\chi}(U): (\mu, \chi) \in h^{-1}(U) \cap (D \times D) \right\}$$

is an open cover of $\overline{U}$. As $\overline{U}$ is compact, there is a finite set of tuples $\{(\mu_i, \chi_i)\}_{i=1}^M \subseteq h^{-1}(U) \cap (D \times D)$ such that $\overline{U} \subseteq \cup_{i=1}^M g_{\mu_i,\chi_i}(U)$. We thus found the desired set of tuples in $D \times D$ and the open disk U containing z_0.

We next focus on proving Lemma 6·2. To this end let $\lambda_0 \in \mathbb{C} \setminus \mathbb{R}$ and let

$$(\{(G_1, v_1), \ldots, (G_M, v_M), (\overline{G_1}, \overline{v_1}), \ldots, (\overline{G_M}, \overline{v_M})\}, U)$$
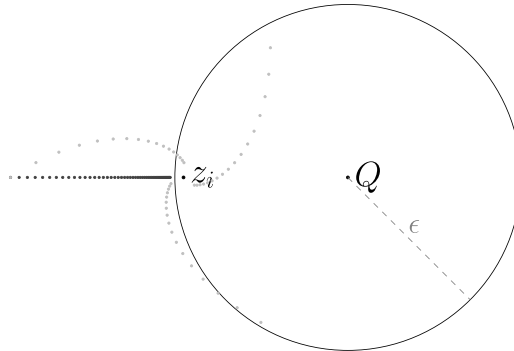
Fig. 6. An example of two orbits with the same initial value converging to z_i under iteration of two different maps. For the dark gray orbit converging horizontally to z_i the derivative at z_i is real. For the light gray orbit the derivative at z_i has the same magnitude, while its argument is a little less than $2\pi/3$.

be a fast implementer for λ_0. We fix these throughout this section. We denote the repelling fixed point of f_{λ_0} by z_0 and we denote for $i = 1, \ldots, M$, $\mu_i := R_{G_i, v_i}$, $\chi_i := R_{\overline{G}_i, \overline{v}_i}$ and $g_i := f_{\mu_i} \circ f_{\chi_i}$. We distinguish between the case that P is close to the attracting fixed point of f_{λ_0} and the case that it is not. In the first case the algorithm is much simpler.

Let a be the attracting fixed point of f_{λ_0}. Because $f_{\lambda_0}(\infty) = 0$ we observe that ∞ is not a fixed point and thus $a \in \mathbb{C}$. Suppose that $|P - a| \leq \epsilon/2$. Choose $\delta > 0$ for which there is a constant $\eta < 1$ such that $|f_{\lambda_0}'(z)| < \eta$ for all $z \in B(a, \delta)$. The point 0 is not a fixed point of f_{λ_0} because $f_{\lambda_0}(0) = \lambda_0 \neq 0$ and thus $f_{\lambda_0}^n(0)$ converges to a as $n \to \infty$. It follows that there is a constant N_0 such that $f_{\lambda_0}^{N_0}(0) \in B(a, \delta)$. Note that the value of N_0 does not depend on the input parameters. Now let $N_\epsilon = \max \{ \lceil \log_\eta (\frac{\epsilon}{2\delta}) \rceil, 0 \} + 1$. Then for any $w \in B(a, \delta)$ we have

$$|f_{\lambda_0}^{N_\epsilon}(w) - a| < \eta^{N_\epsilon} |w - a| < \epsilon/2$$

and thus for $K = N_0 + N_\epsilon$ we have

$$|f_{\lambda_0}^K(0) - P| \leq |f_{\lambda_0}^{N_\epsilon}(f_{\lambda_0}^N(0)) - a| + |a - P| < \epsilon/2 + \epsilon/2 < \epsilon.$$

Because $K = \mathcal{O}(\log(1/\epsilon))$ this describes the algorithm when $|P - a| \leq \epsilon/2$.

The case that $|P - a| > \epsilon/2$ is more involved and we will describe the algorithm as a sequence of simpler subroutines. Just as in Lemma 6·7 let z_i denote the attracting fixed point of g_i. We will show first show that, given a parameter Q that is at most distance ϵ away from some z_i, we only have to apply g_i to the starting value 0 an $\mathcal{O}(\log(1/\epsilon))$ number of times to get ϵ close to Q. Morally, this should be true because after a fixed number of steps the orbit of 0 converges exponentially quickly to z_i and because z_i is close to Q the orbit should also get close to Q. The only way that this reasoning could be incorrect is if z_i and Q are almost ϵ apart and the orbit of 0 converges to z_i from the wrong direction. An example of this is given by the dark gray orbit in Figure 6. This is the reason that we required $g_i'(z_i)$ to have an argument close to $2\pi/3$ in which case the above reasoning is correct as the light gray orbit in Figure 6 demonstrates. In the following proof most time is spent on making this precise.

LEMMA 6·8. *There exists an algorithm that, given $\epsilon > 0$, $Q \in \mathbb{C}$ and $i \in \{1, \ldots, M\}$ such that $|Q - z_i| < \epsilon$, yields an integer K such that $|g_i^K(0) - Q| < \epsilon$, where $K = \mathcal{O}(\log(1/\epsilon))$. If $\lambda_0 \in \mathbb{Q}[i]$ and the input parameters Q, ϵ lie in $\mathbb{Q}[i]$ then the algorithm runs in $\mathrm{poly}(\mathrm{size}(Q, \epsilon))$ time.*

Proof. Let δ be such that $B(z_i, \delta) \subseteq U$ and let $\epsilon' = \min\{\epsilon/2, \delta\}$. Note that $g_i(0) = \mu_i/(1 + \chi_i) \neq 0$ and thus 0 is not a fixed point of g_i. Because z_i is the attracting fixed point of g_i we can find (in a similar way as described above) a positive integer $\widetilde{K}$ that is $\mathcal{O}(\log(1/\epsilon')) = \mathcal{O}(\log(1/\epsilon))$ such that $|g_i^{\widetilde{K}}(0) - z_i| < \epsilon'$. If $|Q - z_i| \leq \epsilon/2$ we are done because then

$$|g_i^{\widetilde{K}}(0) - Q| \leq |g_i^{\widetilde{K}}(0) - z_i| + |Q - z_i| < \epsilon' + \epsilon/2 \leq \epsilon.$$

So from now on we assume that $|Q - z_i| > \epsilon/2$. Define the following sector S of $B(z_i, \epsilon')$

$$S = \left\{ z_i + \xi \cdot \left(\frac{Q - z_i}{|Q - z_i|} \right) : |\xi| < \epsilon', -\pi/3 \leq \arg(\xi) \leq \pi/3 \right\}.$$

We claim that $S \subseteq B(Q, \epsilon)$. To show this note that

$$\left| Q - \left[z_i + \xi \cdot \left(\frac{Q - z_i}{|Q - z_i|} \right) \right] \right| = |Q - z_i| \cdot \left| 1 - \frac{\xi}{|Q - z_i|} \right| < \epsilon \cdot \left| 1 - \frac{\xi}{|Q - z_i|} \right|.$$

If ξ is as in the definition of S the complex number $\xi/|Q - z_i|$ has its argument between $-\pi/3$ and $\pi/3$. Furthermore, because $|\xi| < \epsilon/2$ and $|Q - z_i| > \epsilon/2$, its norm is bounded above by 1. It follows that the norm of $1 - \xi/|Q - z_i|$ is at most 1. Indeed, because $|1 - re^{i\phi}|^2 = 1 + r^2 - 2r\cos(\phi)$, the statement $|1 - re^{i\phi}| \leq 1$ is equivalent to $r = 0$ or $r \leq 2\cos(\phi)$, and the latter is satisfied for all $0 \leq r \leq 1$ and $-\pi/3 \leq \phi \leq \pi/3$. The claim follows.

We now claim that for $w \in B(z_i, \epsilon')$ the intersection of $\{w, g_i(w), g_i^2(w), g_i^3(w)\}$ with S is not empty. Note that because $\epsilon' \leq \delta$ we have that $B(z_i, \epsilon') \subseteq U$ and thus $g'(w) \in A$ for every $w \in B(z_i, \epsilon')$. It follows that applying g_i to w has the effect of rotating around z_i with an angle strictly between $2\pi/3 - 0.01$ and $2\pi/3$ and contracting towards z_i. Therefore applying g_i to w three times has the effect of rotating w a little less than a full circle around z_i, with steps that are strictly less than $2\pi/3$ radians. Because the internal angle of the sector S is $2\pi/3$ the orbit $w, g_i(w), g_i^2(w), g_i^3(w)$ cannot miss S.

To summarize the algorithm, define ϵ' and determine $\widetilde{K}$ such that $g_i^{\widetilde{K}}(0) \in B(z_i, \epsilon')$. Then determine a $j \in \{0, 1, 2, 3\}$ such that $|g_i^{\widetilde{K}+j}(0) - Q| < \epsilon$. We have shown that there exists at least one such j. The output of the algorithm is $K = \widetilde{K} + j$.

We shall now describe an algorithm that does the following. Given a disk D of radius ϵ inside U, it returns an index i, a disk $\widetilde{D}$ of radius at least ϵ containing z_i and a sequence of indices $j_1, \ldots, j_K$ such that $(g_{j_1} \circ \cdots \circ g_{j_{K-1}})(\widetilde{D}) \subseteq D$. To describe the computational complexity of this algorithm, we need a finite way to represent disks in the complex plane. A pleasant way for our purposes is to represent an open disk D by three distinct points P_1, P_2, P_3 on its boundary. This is an unambiguous way to represent a disk because three different points on a circle uniquely determine that circle. If $P_1, P_2, P_3 \in \mathbb{Q}[i]$ we say that the disk D is rational and that $\mathrm{size}(D) = \mathrm{size}(P_1) + \mathrm{size}(P_2) + \mathrm{size}(P_3)$.

Recall that a Möbius transformation maps generalised circles (circles and straight lines) to generalised circles. In what follows, we will apply Möbius transformations to disks in

the complex plane. We shall make sure that the image of the disks involved is always again a disk in the complex plane and not the complement of a disk or a half-plane as it could in general be. Therefore, if D is a disk represented by P_1, P_2 and P_3 and g is one of the Möbius transformations, then $g(D)$ will be a disk represented by $g(P_1), g(P_2)$ and $g(P_3)$. Note that if D is rational and g has rational coefficients then $g(D)$ is a again rational. The Möbius transformations that we will apply come from a fixed finite set and thus there is a fixed constant C for which $\text{size}(g(D)) \le C \cdot \text{size}(D)$.

Let us denote $P_j = x_j + iy_j$ for $j \in \{1, 2, 3\}$. The center $c_D = x + yi$ of the disk D is known as the circumcenter of the triangle with vertices P_1, P_2 and P_3. The coordinates of c_D can be calculated using the well known and easy to derive formulas

$$x = \frac{(x_1^2 + y_1^2)(y_2 - y_3) + (x_2^2 + y_2^2)(y_3 - y_1) + (x_3^2 + y_3^2)(y_1 - y_2)}{2(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2))},$$

$$y = \frac{(x_1^2 + y_1^2)(x_3 - x_2) + (x_2^2 + y_2^2)(x_1 - x_3) + (x_3^2 + y_3^2)(x_2 - x_1)}{2(x_1(y_2 - y_3) + x_2(y_3 - y_1) + x_3(y_1 - y_2))}.$$

We note that if D is rational, then c_D is rational and can be computed in time linear in $\text{size}(D)$. We can also decide whether a given point $Q \in \mathbb{Q}[i]$ lies in a given rational disk D in time linear in $\text{size}(Q)$ and $\text{size}(D)$.

We next need a lemma concerning a geometric construction involving disks.

LEMMA 6·9. *There exists an algorithm that, given two disks A, B in the complex plane for which the center of A is contained in B and B is not contained in A, returns a disk D contained in both A and B, such that the area of D is at least $1/128$ times that of A. Furthermore, if A and B are rational then D is rational and both the running time of the algorithm and $\text{size}(D)$ are bounded by a fixed constant times $\text{size}(A, B)$.*

Proof. For a disk D we denote its center by c_D and its radius by r_D and recall that if D is rational then c_D is rational and can be computed efficiently. For two distinct points P, Q on the boundary of D we denote the closed counterclockwise arc from P to Q by $\text{Arc}_D(P, Q)$ and we denote the sector given by the convex hull of $\text{Arc}_D(P, Q)$ and c_D by $\text{Sec}_D(P, Q)$. We note that the internal angle of both $\text{Arc}_D(P, Q)$ and $\text{Sec}_D(P, Q)$ is given by the arclength of $\text{Arc}_D(P, Q)$ divided by r_D. We claim that either a sector of A whose internal angle is greater than $2\pi/3$ is contained in the closure of B or a sector of B whose internal angle is greater than $2\pi/3$ is contained in the closure of A.

If the boundaries of A and B either do not intersect or intersect in one point then A is contained in B and the claim is obvious. Otherwise let S_1, S_2 be the two intersection points such that $\text{Arc}_A(S_1, S_2)$ is contained in B and thus $\text{Arc}_B(S_2, S_1)$ is contained in A, see Figure 7. Consider the quadrilateral $\square c_B S_1 c_A S_2$ and suppose towards contradiction that the internal angles at both c_A and c_B are at most $2\pi/3$, then the sum of the internal angles at S_1 and S_2 is at least $2\pi/3$ and since they are equal by symmetry the internal angle at S_1 is at least $\pi/3$. By then considering the triangle $\triangle c_B S_1 c_A$ it should follow that $|c_B - c_A| \ge |c_B - S_1| = r_B$, which contradicts the assumption that c_A is contained in B. We therefore find that either the angle $\angle S_2 c_B S_1$ or $\angle S_2 c_A S_1$ is at least $2\pi/3$. If the latter is the case then both $\text{Arc}_A(S_1, S_2)$ and c_A are contained in the closure of B and thus the same is true for $\text{Sec}_A(S_1, S_2)$. If the angle $\angle S_2 c_A S_1$ is less than $2\pi/3$, then $\angle S_2 c_B S_1$ is at least $2\pi/3$. It follows that $\angle c_A c_B S_1$ is the largest internal angle of the triangle $\triangle c_B S_1 c_A$ and thus $|c_A - c_B| < |c_A - S_1| = r_A$, from which it follows that c_B is contained in A. So in this case $\text{Sec}_B(S_2, S_1)$ is contained in A.
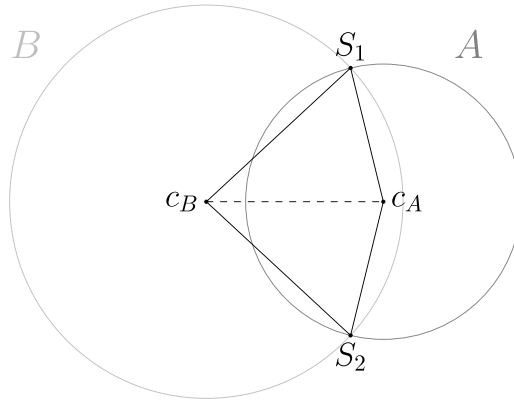
Fig 7. An illustration of the disk A and B such that the arc along B between S_2 and S_1 is contained in A and the arc along A between S_1 and S_2 is contained in B in the proof of Lemma 6·9.
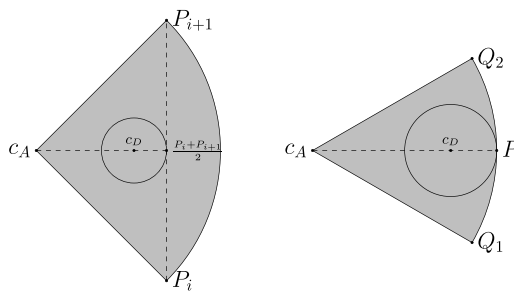

Fig 8. An illustration of how the disk D sits inside $Sec_A(Q_1, Q_2)$ in situation (i), on the left and situation (ii), on the right.

For the algorithm we do not need to know whether a large sector of B is contained in A or vice versa. Assume for simplicity that a sector S of A with internal angle at least $2\pi/3$ is contained in B. Take S to be as large as possible. In the case that A is contained in B we let $S = A$. Let P_0 denote one of the given (rational) points on the boundary of A. Now for $i = 0, 1, 2$ inductively define P_{i+1} as P_i rotated around c_A with an angle of $\pi/2$. Calculating these points is computationally easy because $P_{i+1} = c_A + i(P_i - c_A)$. Now one of the following is guaranteed to be the case:

  (i) two consecutive points P_i and P_{i+1} are contained in S;
  (ii) there is a unique index i such that $P_i \in S$.

Determining which of the two cases is true is easy since checking membership of S is equivalent to checking membership of B. In the first case we note that $Sec_A(P_i, P_{i+1})$ is contained in both the closures of A and B. Now let $R = (P_i + P_{i+1})/2$ and $c_D = (c_A + 3R)/4$ and let D be the disk with center c_D and the point R on the boundary, see Figure 8. It can be checked that D is now contained in $Sec_A(P_i, P_{i+1})$ and its area is $1/32$ that of A.

In the second case note that the arc $Arc_A(Q_1, Q_2)$ containing P_i such that the internal angle of both $Arc_A(Q_1, P_i)$ and $Arc_A(P_i, Q_2)$ is $\pi/6$ contained in S, otherwise, since the internal angle of S is at least $2\pi/3$, S has to contain two consecutive points P_i and P_{i+1}. Now let D be the disk with center $(c_A + 3P_i)/4$ containing P_i on its boundary, see Figure 8. It can be checked that D is contained in $Sec_A(Q_1, Q_2)$ and its area is $1/16$ that of A.

The algorithm above is only guaranteed to successfully return a disk contained in both A and B if a large sector of A is contained in B. Therefore we have to run the algorithm described above (and let it fail if neither of the two described cases is true) and run the same algorithm with the roles of A and B reversed. If both instances of the algorithm return a disk, say D_1 and D_2, then at least one of them is contained in both A and B but the other one might not be. So in this case we have to run one final check to see which one of the two disks is indeed contained in both A and B (which is computationally easy). If they both are we can return either D_1 or D_2.

In conclusion we obtain a disk D contained in both A and B that is either at least $1/32$ of the area of A or $1/32$ of the area of B. Because the area of B is at least $1/4$ that of A (otherwise $r_B < r_A/2$ and $c_A \in B$ would imply $B \subseteq A$), we can conclude that the area of D is at least $(1/4) \cdot (1/32) = 1/128$ that of A.

LEMMA 6·10. *There exists an algorithm that, given a disk $D_1 \subseteq U$ with radius ϵ, returns an index $i \in \{1, \ldots, M\}$, a sequence of indices $j_1, \ldots, j_K \in \{1, \ldots, M\}$ and a disk D_K such that $z_i \in D_K$, the radius of D_K is at least ϵ and $(g_{j_1} \circ \cdots \circ g_{j_{K-1}})(D_K) \subseteq D_1$. Furthermore, $K = \mathcal{O}(\log(1/\epsilon))$. If $\lambda_0 \in \mathbb{Q}[i]$ and D_1 and ϵ are both rational then D_K is rational and the algorithm runs in $\mathrm{poly}(\mathrm{size}(D_1, \epsilon))$ time.*

Proof. For every index i let $U_i = g_i(U)$. Recall that we took U as a rational disk. Because the derivative of g_i is bounded on U, the image U_i is again a disk in the complex plane. If λ_0 is rational, the coefficients of g_i are rational and then U_i is also rational. The point z_i is fixed for g_i and contained in U, therefore, also contained in U_i. We describe a procedure to generate a sequence of disks $\{D_n\}_{n \geq 1}$, starting with the given disk D_1. The sequence is defined in such a way such that $D_n \subseteq U$ for all n, which is, by assumption, the case for D_1.

Suppose we have arrived at disk $D_n \subseteq U$. Check if there is any index $i \in \{1, \ldots, M\}$ such that $z_i \in D_n$, if there is stop the procedure and let $K = n$. Otherwise, let m_n be the center of D_n and determine an index $j_n \in \{1, \ldots, M\}$ such that $m_n \in U_{j_n}$. Such an index must exist because $m_n \in U$ and the disks $U_1, \ldots, U_M$ cover U. Because the center of D_n lies in U_{j_n} but U_{j_n} is not contained in D_n (z_{j_n} does not lie in D_n) we can use Lemma 6·9 to generate a disk $\widetilde{D}_n$ that is contained in both D_n and U_{j_n} whose area is at least $1/128$ times that of D_n and which can be assumed to be rational if D_n is. Now we define $D_{n+1} = g_{j_n}^{-1}(\widetilde{D}_n)$. Because $\widetilde{D}_n \subseteq U_{j_n}$ the disk D_{n+1} lies in U and because $\widetilde{D}_n \subseteq D_n$ the disk $g_{j_n}(D_{n+1})$ lies in D_n. Furthermore, by the properties of the fast implementer, $g_{j_n}^{-1}$ is expanding the norm on U_{j_n} with a factor at least 16, the area of D_{n+1} is at least $16^2 \cdot (1/128) = 2$ times that of D_n. This means that the area of D_n grows exponentially with n and thus, because the area of U is fixed, the procedure will terminate after $K = \mathcal{O}(\log(1/\epsilon))$ steps. Note that indeed $z_i \in D_K$ for some i, the radius of D_K is at least that of D_1 and $(g_{j_1} \circ \cdots \circ j_{K-1})(D_K) \subseteq D_1$.

Recall that we had defined a and z_0 to be the attracting and repelling fixed point of f_{λ_0} respectively. We have already described the algorithm in Lemma 6·2 when P is near a. What follows is the final lemma needed to describe the algorithm when P is not near a.

LEMMA 6·11. *There exists a fixed positive constant c and an algorithm that, given $P \in \mathbb{C}$ and $\epsilon > 0$ such that $|P - a| \geq \epsilon/2$, yields a disk $D \subseteq U$ and a positive integer K with $f_{\lambda_0}^K(D) \subseteq B(P, \epsilon)$, such that the radius of D is at least $c \cdot \min(\epsilon, \epsilon/|P|^2)$ and $K = \mathcal{O}(\log(1/\epsilon))$. If both*

λ_0 and the input parameters are in $\mathbb{Q}[i]$ then D is also rational and both $\mathrm{size}(D)$ and the running time of the algorithm is polynomial in $\mathrm{size}(P, \epsilon)$.

Proof. Let V be a compact neighbourhood of a such that $|f_{\lambda_0}{}'(z)| < \xi < 1$ for some constant ξ for all $z \in V$. We first claim that there is an integer N such that the complement of $f_{\lambda_0}^N(U)$ is contained in V. To show this let $U_n = f_{\lambda_0}^n(U)$. Recall that we assumed that $\overline{U} = \overline{U_0} \subseteq U_1$ and thus inductively $\overline{U_n} \subseteq U_{n+1}$. Under iteration of $f_{\lambda_0}^{-1}$ every initial point that is not a converges to z_0 and thus eventually lands in U. Therefore,

$$\bigcup_{n=1}^{\infty} U_n = \widehat{\mathbb{C}} \setminus \{a\}.$$

For n large enough the point ∞ is contained in U_n and from then on the sequence $(U_n)^c$ consists of nested disks, containing a, whose radii must necessarily converge to 0, proving that there is an N such that $(U_N)^c$ is contained in V. Note that N does not depend on the input parameters. Let D_0 be the interior of $(U_N)^c$, this is a rational disk whose size also does not depend on the input, and let $D_i = f_{\lambda_0}^i(D_0)$ for $i \in \{1, 2, 3\}$. Let $\delta > 0$ be a constant smaller than the minimum distance between points on the boundary of D_i and D_{i-1}. From now on we will assume that $\epsilon < \delta$. Finally let $h = f_{\lambda_0}^{-(N+3)}$.

If P lies outside D_2, then let $\widetilde{D}$ be the disk of radius ϵ represented by $P + \epsilon, P + i\epsilon$ and $P - \epsilon$. Note that $\widetilde{D}$ lies outside D_3 and thus $D = h(\widetilde{D}) \subset U$. Because the derivative of a Möbius transformation of the form $z \mapsto (az + b)/(cz + d)$ is $z \mapsto (ad - bc)/(cz + d)^2$ there is a constant c_1 such that the radius of D is at least $c_1 \cdot \min(\epsilon, \epsilon/|P|^2)$. In this case D and $K = N + 3$ are the output of the algorithm.

If P lies inside D_2 we determine N_0 such that for $P_{N_0} := f_\lambda^{-N_0}(P)$ we have $P_{N_0} \in D_1$ and $P_{N_0} \notin D_2$. Because $|P - a| \geq \epsilon/2$ and $D_1 \subset V$ we find that $N_0 = \mathcal{O}(\log(1/\epsilon))$. Let $\widetilde{D}$ be the disk of radius ϵ represented by $P_{N_0} + \epsilon, P_{N_0} + i\epsilon$ and $P_{N_0} - \epsilon$. Note that again $\widetilde{D}$ lies outside D_3 and thus $D = h(\widetilde{D}) \subseteq U$. Furthermore, because $\widetilde{D} \subset D_0 \subset V$ and f_{λ_0} is attracting on V it follows that $f_{\lambda_0}^{N_0}(\widetilde{D}) \subseteq B(P, \epsilon)$. Finally, if we let c_2 be the minimum of $|h'(z)|$ for $z \in D_0$, we find that the radius of D is at least $c_2 \cdot \epsilon$. So in this case the output is the disk D together with $K = N_0 + N + 3$.

We are now ready to complete the proof of Lemma 6·2.

Proof of Lemma 6·2. Recall we had defined a to be the attracting fixed point of f_{λ_0} and that we already described the algorithm in the case that $|P - a| < \epsilon/2$, therefore we assume that $|P - a| \geq \epsilon/2$.

It follows from Lemma 6·11 that we can generate a disk $D_1 \subseteq U$ of radius $r = \mathcal{O}(\min\{\epsilon, \epsilon/|P|^2\})$ and whose size is polynomial in $\mathrm{size}(P, \epsilon)$ together with a positive integer K_1 that is $\mathcal{O}(\log(1/\epsilon))$ such that $f_{\lambda_0}^{K_1}(D_1)$ is contained in $B(P, \epsilon)$. From Lemma 6·10 it follows that we can find an index $i \in \{1, \ldots, M\}$, a sequence of indices $j_1, \ldots, j_{K_2}$ and a disk D_2 such that $z_i \in D_2$, the radius of D_2 is at least r, its size is polynomial in $\mathrm{size}(r, D_1)$, which is again polynomial in $\mathrm{size}(\epsilon, P)$, and such that

$$(g_{j_1} \circ \cdots \circ g_{j_{K_2}})(D_2) \subseteq D_1.$$

Furthermore $K_2 = \mathcal{O}(\log(1/r)) = \mathcal{O}(\max(\log(1/\epsilon), \log(|P|/\epsilon)))$. Finally let Q be the center of D_2 and note that $\mathrm{size}(Q)$ is polynomial in $\mathrm{size}(D_2)$. Then, because $|Q - z_i| < r$, it

follows from Lemma 6·8 that we can generate a K_3 such that $g_i^{K_3}(0) \in D_2$, where $K_3 = \mathcal{O}(\log(1/r)) = \mathcal{O}(\max(\log(1/\epsilon), \log(|P|/\epsilon)))$. Concluding, we find that

$$(f_{\lambda_0}^{K_1} \circ g_{j_1} \circ \cdots \circ g_{j_{K_2}} \circ \circ g_i^{K_3})(0) \in B(P, \epsilon).$$

Furthermore, adding the running times of the individual algorithms, we find that the final algorithm runs in $poly(\text{size}(P, \epsilon))$ time.

7. *Activity and zeros for Cayley trees*

For fixed $\Delta \geq 2$ notions such as the activity-locus and the zero sets can be considered for subcollections of $\mathcal{G}_\Delta$. Particularly interesting subcollections from a physical viewpoint are given by subgraphs of regular lattices. However, it is notoriously difficult to rigorously deduce the properties for such collections.

A much simpler collection of rooted graphs in $\mathcal{G}_\Delta$ is given by finite Cayley trees, and we will describe the properties of those in this section. The trees are uniquely determined by the conditions that every leaf has fixed distance n to the root vertex v, and every non-leaf has down-degree $d = \Delta - 1$. The root vertex therefore has degree d, while every other non-leaf has degree Δ. We denote the Cayley tree of depth n by T_n, and its root by v_n.

As an immediate consequence of Lemma 2·4 we obtain

$$R_{T_n, v_n}(\lambda) = f_{\lambda, d}(R_{T_{n-1}, v_{n-1}}(\lambda)),$$

where $f_{\lambda, d}(z) = \lambda/(1+z)^d$. Since the ratio of a single point is given by $\lambda = f_{\lambda, d}(0)$, it follows inductively that

$$R_{T_n, v_n}(\lambda) = f_{\lambda, d}^{n+1}(0).$$

In fact, since

$$Z_{T_n, v_n}^{out}(\lambda) = \left(Z_{T_{n-1}, v_{n-1}}(\lambda)\right)^d$$

and

$$Z_{T_n, v_n}^{in}(\lambda) = \lambda \left(Z_{T_{n-1}, v_{n-1}}^{out}(\lambda)\right)^d$$

it follows by induction on n that for $\lambda \in \mathbb{C} \setminus \{0\}$ the polynomials $Z_{T_n, v_n}^{in}(\lambda)$ and $Z_{T_n, v_n}^{out}(\lambda)$ cannot vanish simultaneously. For $\lambda \in \mathbb{C} \setminus \{0\}$ it follows that $Z_{T_n}(\lambda) = 0$ if and only if $R_{T_n, v_n}(\lambda) = f_{\lambda, d}^n(0) = -1$.

In what follows we deduce properties of the zeros of $Z_{T_n}(\lambda)$ and the activity-locus of $f_{\lambda, d}^n(0)$ from well-known results in the field of holomorphic dynamical systems, occasionally adapting the proofs to our setting. We refer the reader to the standard references [CG93, Mil06].

Observe that $f_{\lambda, d}(-1) = \infty$ and $f_{\lambda, d}(\infty) = 0$, and $f'_{\lambda, d}(-1) = f'_{\lambda, d}(\infty) = 0$. Thus if $f_{\lambda_0, d}^n(0) = -1$ for some λ_0 and n, then 0 is an attracting periodic cycle of period $n + 2$. This cycle is stable under perturbations of λ_0, i.e. the attracting cycle persists and in fact varies holomorphically for nearby parameters $\lambda \sim \lambda_0$ by the implicit function theorem.

Recall that every attracting cycle attracts the orbit of a critical point. But $f_{\lambda, d}$ has only one critical orbit: the orbit of -1, ∞ and 0. Thus whenever $f_{\lambda, d}$ has an attracting cycle, the orbit $f_{\lambda, d}^n(0)$ converges to the attracting cycle. In fact, the convergence is uniform in a neighbourhood of the parameter λ_0, hence λ_0 cannot lie in the activity-locus. The situation

is therefore fundamentally different from the setting where the whole family of graphs $\mathcal{G}_\Delta$ is considered, as there λ_0 must lie in the activity-locus. The following however does hold:

PROPOSITION 7·1. *The activity-locus of the family* $\{T_n, v_n\}$ *equals the collection of accumulation points of the zeros of the collection* $\{Z_{T_n}\}$.

Proof. If there are no zeros in a neighbourhood of some λ_0, then the family $\{R_{T_n, v_n}\}$ avoids the values $0, -1$ and ∞, and is normal by Montel's Theorem.

Suppose on the other hand that λ_0 is an accumulation point of zeros $\lambda_1, \lambda_2, \ldots$. Let $n_1, n_2, \ldots$ be the minimal integers for which $f^{n_i}(\lambda_i) = -1$. Since for fixed n the zeros of Z_{T_n} are isolated, we may assume that $n_i \to \infty$ and (n_i) is strictly increasing.

When for a parameter λ the rational function f has an attracting periodic cycle, the unique critical orbit $\{f^n(0)\}_{n \geq 1}$ must converge to this periodic orbit. Since attracting periodic cycles are stable, i.e. they persist under small changes of the parameter λ, such parameters lie in a passivity component, i.e., a maximal connected open subset where the family $\{\lambda \mapsto f^n(0)\}$ is normal. The passivity component agrees exactly with the connected component where the attracting periodic cycle persists, since by [MnSS83] the parameter must become active when the periodic cycle becomes neutral.

Thus, λ_i lies in a connected component of the open set where the family $\{\lambda \mapsto f^n(0)\}$ is normal, and associated to this component is the unique period $n_i + 2$. Since the sequence $\{n_i\}_{i \geq 1}$ is strictly increasing, the parameters λ_i must all lie in distinct connected components. It follows that the limit parameter λ_0 cannot lie in an open component where the family is normal, and therefore λ_0 must be an active parameter.

The activity-locus for Cayley trees of down degree $d = 2, 3$ and 4 is illustrated in Figure 9. Each of these diagrams represents the spherical derivative of the function $\lambda \mapsto f_{\lambda,d}^{120}(0)$.

It follows from Proposition 7·1 above, plus the observation that zeros do not lie in the activity-locus for the Cayley tree setting, that the Cayley tree activity-locus never has interior. On the other hand, it follows from the universality of the Mandelbrot set, a result due to McMullen [McM00], that the activity-locus must contain a quasiconformal image of the Mandelbrot set of some degree. Therefore by Shishikura's result [Shi98] the Hausdorff dimension of the activity-locus is equal to 2 for any $d \geq 2$.

It follows from the proof of Proposition 7·1 that the complement of the activity-locus consists of infinitely many connected components. Each λ for which $f_{\lambda,d}$ has an attracting periodic cycle lies in such a passive component, a so-called *hyperbolic* component associated to the period k. Whether all connected components are hyperbolic is an open question, which is conjectured to hold for quadratic polynomials.

For any down-degree d there are two special connected components that can easily be identified. The unbounded component is always a hyperbolic component of period 2. For degree 2 this is the complement of the closed disk of radius 4. For down-degrees 3 and 4 the boundary has respectively 1 and 2 singular points.

For each down-degree $d = \Delta - 1$ there is a single hyperbolic component of period 1, which contains of course the parameter $\lambda = 0$ and equals the cardioid Λ_Δ.

Apart from these two special hyperbolic components, any hyperbolic component contains a unique zero of the partition function, i.e. a unique parameter λ for which $f_{\lambda,d}^n(\lambda) = -1$ for some $n \in \mathbb{N}$. Since $f_{\lambda,d}^2(-1) = 0$ and $f_{\lambda,d}^{-2}(0) = \{-1\}$, these are exactly the parameters
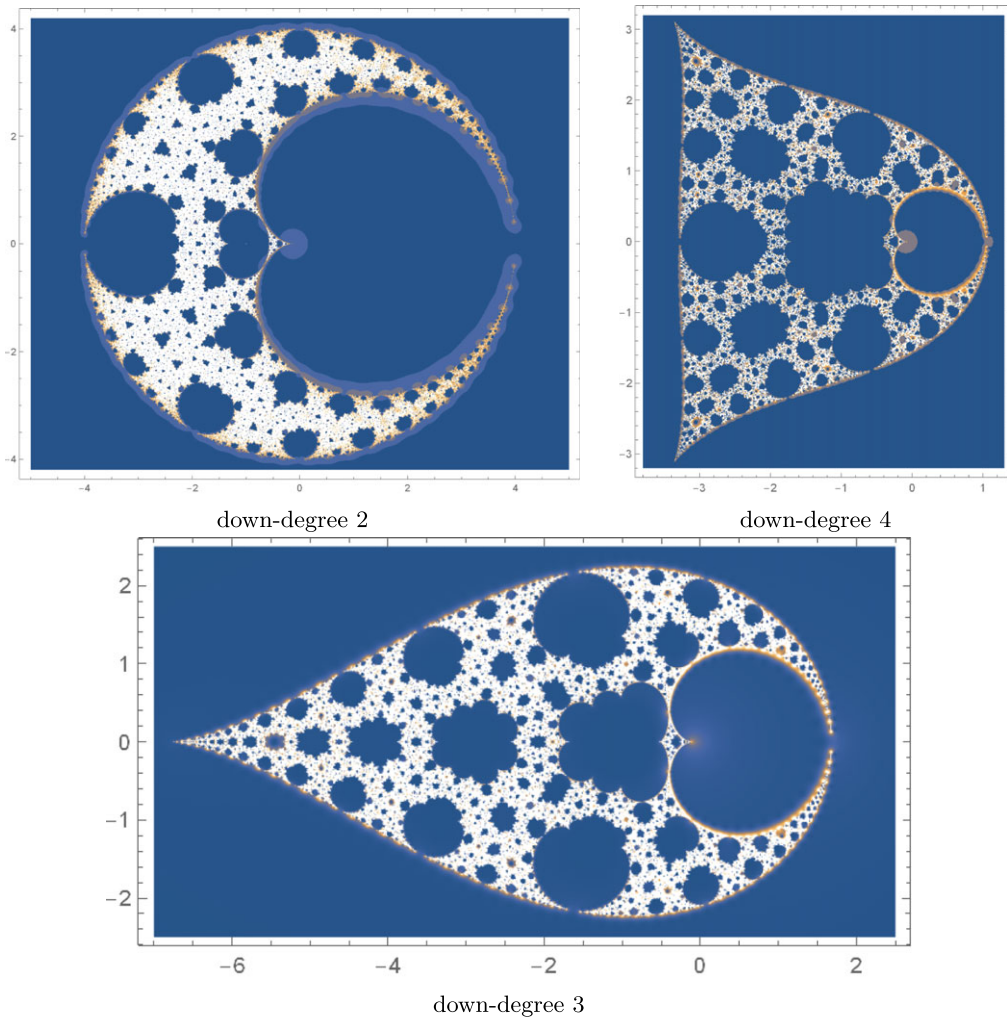
Fig 9. The activity-locus of Cayley trees for down-degrees 2, 3 and 4. For each pixel the spherical derivative of the occupation ratio is computed for the Cayley tree of depth 120. Pixels for which this derivative is sufficiently large are depicted in white, suggesting that the corresponding parameter λ lies approximately on the activity locus.

$\lambda \in \mathbb{C} \setminus \{0\}$ for which the unique critical orbit $\{f_{\lambda,d}^i(0)\}_{i\in\mathbb{N}}$ is periodic, i.e. for which $f_{\lambda,d}$ is super-attracting.

For the family $p_c(z) = z^2 + c$ the fact that every hyperbolic component of the Mandelbrot set contains a unique super-attracting parameter is a consequence of the Multiplier Theorem, due to Douady–Hubbard and Sullivan, see [**Dou83**].

Let us recall this fundamental result in the field. Let H be a hyperbolic component of the Mandelbrot set, say of period n. For every parameter $c \in H$ there exist an attracting periodic cycle $a_0, a_1, \ldots, a_n = a_0$. The multiplier $h(c) = (f_c^n)'(a_0)$ is independent from the choice of a_n, and gives a holomorphic map from H to the unit disk.

THEOREM 7·2 (Multiplier Theorem). *For every hyperbolic component H the map $c \mapsto h(c)$ gives a conformal bijection from H to the unit disk.*

The proof of the Multiplier Theorem can be found in [**CG93**, theorem 2.1, p. 133], and can be applied almost directly to our setting. We present a high-level discussion to outline how the proof adapts to our setting.

Let H be a hyperbolic component of period at least 3. One easily sees that $h(\lambda)$, the multiplier of the attracting periodic cycle of $f_{\lambda,d}$ is a holomorphic and surjective map from the hyperbolic component H to the unit disk $\mathbb{D}$, hence is a branched covering. Let Z be the set of super-attracting parameters in H, i.e. $Z = h^{-1}(0)$. If it can be shown that $h : H \setminus Z \to \mathbb{D} \setminus \{0\}$ is a covering map, it follows from the Riemann–Hurwitz Theorem that card$(Z) = 1$.

Thus, it needs to be shown that h is locally invertible near parameters $\lambda_0 \in H \setminus Z$. Write $\eta_0 = h(\lambda_0) \in \mathbb{D}$, and consider values of η near η_0. Following the proof of the Multiplier Theorem one applies quasiconformal surgery by modifying the ellipse field near the attracting periodic cycle in order to obtain attracting periodic cycles with multipliers η. Using the dynamics the ellipse field can be extended to the full basin of the attracting cycle, obtaining an invariant ellipse field that is invariant under the map $f_{\lambda_0,d}$. The ellipse field corresponds to a Beltrami coefficient, which can be extended to the entire Riemann sphere by setting it equal to 0 outside of the basin of attraction. The Measurable Riemann Mapping Theorem gives a holomorphic family of quasiconformal maps φ_η, with φ_{η_0} the identity. By composing with suitable Möbius transformations we can guarantee that the points $-1, \infty$ and 0 are fixed under all φ_η.

Since each ellipse field is invariant under $f_{\lambda_0,d}$, conjugating $f_{\lambda,d}$ by φ_η yields a holomorphic family of self-maps of the Riemann sphere $g_{\eta,d}$, which are necessarily rational functions of the same degree d. In fact, since each φ_η fixes the points $-1, \infty$ and 0, each rational function $f_{\eta,d}$ must send -1 to ∞ and ∞ to 0, each with local degree d. It follows that the rational function $g_{\eta,d}$ must be of the form

$$g_{\eta,d}(z) = \frac{\lambda(\eta)}{(1+z)^d}.$$

It follows that $\lambda(\eta)$ gives a local inverse of the multiplier function h, completing this step of the proof. This step guarantees that there exists a unique zero in each hyperbolic component of period at least 3, which equals the super-attracting *center* of the hyperbolic component. The proof of the Multiplier Theorem in our setting can be concluded by analyzing the local degree near the center. We have therefore obtained the following description of the zeros of the Cayley trees:

COROLLARY 7·3. *Every $\lambda \in \mathbb{C}$ for which $Z_{T_n}(\lambda) = 0$ for some $n \in \mathbb{N}$ is the center of a hyperbolic component of the complement of the activity locus. On the other hand: apart from the two special hyperbolic components, the unbounded component and the component containing 0, for each center λ of a hyperbolic component there exists an $n \in \mathbb{N}$ for which $Z_{T_n}(\lambda) = 0$. As a consequence zero-parameters are isolated.*

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