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ON SC-MODULES

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Let R be a ring. A right R-module M is called an SC-module if every M-singular right R-module is continuous. The purpose of this note is to give some characterisations of SC-modules.

1. INTRODUCTION

Rings for which every singular right module is injective (briefly, SI-rings) were introduced and studied by Goodearl [5]. Later, Dinh van Huynh and R. Wisbauer [3] studied the structure of SI-modules. A right R-module M is called an SI-module provided every M-singular right R-module is M-injective. A generalisation of SI-rings is SC-rings, that is, rings R for which every singular right R-module is continuous. SC-rings were introduced and studied by Rizvi and Yousif [9]. In this paper we introduce and investigate SC-modules. A right R-module M is called an SC- module provided every M-singular right R-module is continuous. By investigating a (finitely generated) self-projective SC-module we have more general statements which also include Propositions 3.4, 3.6 and 3.7 of [9].

2. DEFINITIONS AND PRELIMINARIES

Throughout the paper R is an associative ring with identity and Mod-R the category of unitary right R-modules. For $M \in$ Mod-R we denote by $\sigma[M]$ the full subcategory of Mod-R whose objects are submodules of M-generated modules (see [11]). M is called self-projective (respectively self-injective) if it is M-projective (respectively M-injective). $Soc(M)$ (respectively $Rad\,(M)$) denotes the socle (respectively radical) of the module M.

We consider the following conditions on a module M:

(C_1) Every submodule of M is essential in a direct summand of M;

(C_2) Every submodule isomorphic to a direct summand of M is itself a direct summand;

(C_3) If M_1 and M_2 are direct summands of M with $M_1 \cap M_2 = 0$ then $M_1 \oplus M_2$ is a direct summand of M.

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M is called *continuous* if it satisfies conditions (C_1) and (C_2), *quasi-continuous* if it satisfies (C_1) and (C_3) and a *CS-module* if M satisfies condition (C_1) only.

It is easy to see that $(C_2) \Rightarrow (C_3)$ and the hierarchy is as follows :

injective $\Rightarrow$ self-injective $\Rightarrow$ continuous $\Rightarrow$ quasi-continuous $\Rightarrow$ CS;

For more details we refer to [**6**].

In the following, we list a few known results which will be used often.

LEMMA 1. *Let M be a cyclic module such that K/L is a CS-module for every cyclic submodule K of M and a submodule L of K. Then M has finite uniform dimension.*

PROOF: See [**8**, Theorem 1] □

Let M and N be R-modules. N is called *singular in* $\sigma[M]$ or *M-singular* if there exists a module L in $\sigma[M]$ containing an essential submodule K such that $N \simeq L/K$ (see [**10**]).

By definition, every M-singular module belongs to $\sigma[M]$. For $M = R$ the notion R-singular is identical to the usual definition of singular R-module (see [**5**]).

The class of all M-singular modules is closed under submodules, homomorphic images and direct sums (for example, [**11**, 17.3 and 17.4]). Hence every module $N \in \sigma[M]$ contains a largest M-singular submodule which we denote by $Z_M(M)$. The following properties of M-singular modules are shown in [**10**, 1.1] and [**12**, 2.4].

LEMMA 2. *Let M be an R-module.*

(1) *A simple R-module E is M-singular or M-projective.*
(2) *If $Soc(M) = 0$, then every simple module in $\sigma[M]$ is M-singular.*
(3) *If M is self-projective and $Z_M(M) = 0$, then the M-singular modules form a hereditary torsion class in $\sigma[M]$.*

We extend the definition of right SC-rings (see [**9**]) to modules.

DEFINITION: An R-module M is called an SC-module if every M-singular module is continuous. The module M is defined to be an SI-module if every M-singular module is M-injective (see [**3**]).

3. RESULTS

Recall that an R-module M is called *V-module* if every simple module (in $\sigma[M]$) is M-injective. In [**10**] V-modules are also called co-semisimple modules. The following assertions also include Theorems 3.2 and 3.6 in [**9**]:

THEOREM 3. *Let M be a right R-module. Then the following conditions are equivalent :*

(1) *M is an SC-module;*

(2) *Every M-singular module is semisimple;*
(3) *Every (finitely generated) M-singular R-module is semisimple;*
(4) *Every (finitely generated) M-singular R-module is self-injective;*
(5) *Every finitely generated M-singular R-module is continuous;*
(6) *Every (finitely generated) M-singular R-module is quasi-continuous;*
(7) *M/K is semisimple for every essential submodule K of M;*
(8) *Every (cyclic) M-singular module is $(M/Soc(M))$-injective;*
(9) *$M/Soc(M)$ is a locally noetherian V-module and for every essential submodule K of M, $Soc(M/K) \neq 0$;*

If M is finitely generated, then (1)-(9) are also equivalent to:

(10) *$M/Soc(M)$ is a V-module and for every essential submodule K of M, M/K is finitely cogenerated.*

PROOF: The equivalences (2) $\Leftrightarrow$ (7) $\Leftrightarrow$ (8) $\Leftrightarrow$ (9) $\Leftrightarrow$ (10) follow from [**10**, 3.7].
(1) $\Rightarrow$ (5) by definition.
(2) $\Rightarrow$ (1) since every semisimple module is continuous.
(2) $\Leftrightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) $\Rightarrow$ (6) is clear.
(6) $\Rightarrow$ (7) We use an argument similar to one given in [**9**]. Let K be an essential submodule of M. Put $E = M/K$. We have to show that every cyclic submodule L of E is semisimple. Obviously L is M-singular. Consider a non-zero $x \in L$. Then $L \oplus xR$ is finitely generated and M-singular, hence quasi-continuous by assumption. Therefore xR is L-injective by [**11**, 41.20] and consequently a direct summand of L. On the other hand, since L is cyclic and M-singular, every subquotient of L is M-singular, quasi-continuous and so is a CS-module. By Lemma 1, L has finite uniform dimension. It follows that L is semisimple. ∎

COROLLARY 4. *For an SC-module M, we have :*

(1) $Rad(M) \subseteq Soc(M)$;
(2) $Z_M(M) \subseteq Soc(M)$.

PROOF: (1) For every essential submodule K of M, M/K is M-singular, hence is semisimple by Theorem 3. This implies $Rad\,(M/K) = 0$ and therefore $Rad(M) \subseteq K$, that is, $Rad(M) \subseteq Soc(M)$.

(2) For every $x \in Z_M(M)$, xR is M-singular, hence is semisimple by Theorem 3. Therefore $xR \subseteq Soc(M)$. This implies $Z_M(M) \subseteq Soc(M)$. ∎

COROLLARY 5. *Let M be a finitely generated SC-module. If M is CS then M is noetherian.*

PROOF: By Theorem 3, $M/Soc(M)$ is noetherian. If M is moreover a CS-module, then by using the same argument as that of [**4**, Lemma 1] we see that $Soc(M)$ is finitely generated. Hence M is noetherian. ∎

A module M is called a *GCO-module* if every singular simple module is M-injective or M-projective (see [**10**]).

Proposition 6. *For a finitely generated self-projective right R-module M, the following conditions are equivalent :*

(1) *M is an SC-module with* $Z_M(M) = 0$;
(2) *M is an SI-module;*
(3) *Every cyclic M-singular module is M-injective;*
(4) *M/K is semisimple for every essential submodule K of M and* $Z_M(M) = 0$;
(5) *M is hereditary in $\sigma[M]$ and M-singular modules are semisimple;*
(6) *M is a GCO-module, $M/Soc(M)$ is noetherian and $Soc(M/K) \neq 0$ for every essential submodule K of M;*
(7) *Soc(M) is M-projective and M is an SC-module.*

Proof: The equivalences (2)$\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6) follow from the proposition 1.3 in [**3**] (for not necessarily finitely generated self-projective modules) and from Theorem 3.
(2) $\Rightarrow$ (1) is clear.
(1) $\Rightarrow$ (4) by Theorem 3.
(1) $\Rightarrow$ (7) Since $Z_M(M) = 0$, every simple submodule of M is M-projective, and so is Soc(M).
(7) $\Rightarrow$ (1) By Corollary 7, $Z_M(M) \subseteq Soc(M)$. Since $Soc(M)$ is M-projective, every simple submodule of M is M-projective, therefore $Z_M(M)$ must be zero. □

Corollary 7. *If M is an SC-module, then $\overline{M} = M/Soc(M)$ is an SI-module.*

Proof: Since every $\overline{M}$-singular module is M-singular, every $\overline{M}$-singular module is $\overline{M}$-injective by Theorem 3. Hence $\overline{M}$ is an SI-module. □

Corollary 8. *For a module M the following conditions are equivalent:*

(1) *M is an SC-module with essential Soc(M);*
(2) *$\overline{M} = M/Soc(M)$ is semisimple.*

Proof: (1) $\Rightarrow$ (2) is clear.
(2) $\Rightarrow$ (1): By condition (7) in Theorem 3 it is enough to show that $Soc(M)$ is essential in M. Set $S = Soc(M)$ and let A be a non-zero submodule of M such that $S \cap A = 0$. Then $A \simeq A/(A \cap S) \simeq (A+S)/S \subseteq M/S$, hence A is semisimple. Therefore $A \subseteq S$, a contradiction. □

References

[1] V. Camillo and M.F. Yousif, 'CS-modules with acc or dcc on essential submodules', *Comm. Algebra* **19** (1991), 655–662.

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[2] D. van Huynh, P.F. Smith and R. Wisbauer, 'A note on GV-modules with Krull dimension', *Glasgow Math. J.* **32** (1990), 389–390.

[3] D. van Huynh and R. Wisbauer, 'A structure theorem for SI-modules', *Glasgow Math. J.* **34** (1992), 83–89.

[4] D. van Huynh, N.V. Dung and R. Wisbauer, 'Quasi-injective modules with acc or dcc on essential submodules', *Arch. Math.* **53** (1989), 252–255.

[5] K.R. Goodearl, 'Singular torsion and the splitting properties', *Mem. Amer. Math. Soc.* **124** (1972).

[6] S.H. Mohamed and B.J. Müller, *Continuous and discrete modules*, London Math. Soc. Lecture Notes **147** (Cambridge Univ. Press, 1990).

[7] B.J. Müller and S.T. Rizvi, 'On injective and quasi-continuous modules', *J. Pure Appl. Algebra* **28** (1983), 197–262.

[8] B.L. Osofsky and P.F. Smith, 'Cyclic modules whose quotients have complement direct summands', *J. Algebra* **139** (1991), 342–354.

[9] S.T. Rizvi and M.F. Yousif, 'On continuous and singular modules', in *Non-commutative ring theory*, Lecture Notes in Mathematics **1448**, Proc. Conf., Athens/OH(USA) 1989 (Springer-Verlag, Berlin, Heidelberg, New York, 1990), pp. 116–124.

[10] R. Wisbauer, 'Generalized co-semisimple modules', *Comm. Algebra* **18** (1990), 4235–4253.

[11] R. Wisbauer, *Foundation of module and ring theory* (Gordon and Breach, London, Tokyo, 1991).

[12] R. Wisbauer, 'Localization of modules and the central closure of rings', *Comm. Algebra* **9** (1981), 1455–1493.

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