A CHARACTERIZATION OF THE STRONGLY η-REPRESENTABLE MANY-ONE DEGREES

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Abstract. η-representations are a way of coding sets in computable linear orders that were first introduced by Fellner in his thesis. Limitwise monotonic functions have been used to characterize the sets with η-representations, and give characterizations for several variations of η-representations. The one exception is the class of sets with strong η-representations, the only class where the order type of the representation is unique.

We introduce the notion of a connected approximation of a set, a variation on Σ_2^0 approximations. We use connected approximations to give a characterization of the many-one degrees of sets with strong η-representations as well new characterizations of the variations of η-representations with known characterizations.

§1. Introduction. Coding one type of mathematical object into another is a common theme throughout mathematics. For instance, every Boolean algebra can be coded as a ring, and Fourier series are a way of coding periodic functions as vectors in ℓ_2. One of the simplest objects to code are sets of natural numbers. We can code subsets of ω as real numbers using binary expansions. Subsets of ω can be coded as graphs using daisy graph: we create an isolated vertex with n many loops for each $n \in A$. Using a similar idea we can code a set $A \subseteq \omega$ as a partial order: take a chain of length n for each $n \in A$. Encoding sets into fields is not as easy. One idea is to choose which polynomials are irreducible. This is complicated though as when adding a root to one polynomial other previously irreducible polynomials can become reducible, so a careful choice of polynomials that represent the natural numbers is needed. For linear orders, Fellner [4] came up with the following natural way of encoding sets.

DEFINITION 1.1. For a set A a linear order L is said to be an *η-representation* of A if there is a surjective function $F : \omega \to A$ such that L has order type

$$\sum_{n \in \omega} \eta + F(n),$$

where η is the order type of $\mathbb{Q}$. We say L is a *strong η-representation* if the function F is strictly increasing and an *increasing η-representation* if F is non-decreasing. If a set A has a computable (strong, increasing) η-representation then we say A is (*strongly,*

increasingly) *η-representable*. A degree is (strongly, increasingly) *η*-representable if it contains a set that is (strongly, increasingly) *η*-representable.

Note that an *η*-representation of A cannot tell us if 0 or 1 is in A so we will assume that $0, 1 \notin A$ when we are talking about representations of A.

Computability theory gives us a way of analyzing the effectiveness of such encodings and how easy it is to go from an encoding of a structure or set back to the original. For example, in the case of daisy graphs we can characterize the sets with computable daisy graph as the c.e. sets.

In this context, a natural question to ask is what sets, or degrees, have computable (strong, increasing) *η*-representations. In his thesis, Fellner [4] introduced the notion of a strong *η*-representation (predating the introduction of general *η*-representations) and proved that every set with a computable strong *η*-representation is Δ_3^0 and that every Σ_2^0 and every Π_2^0 set is strongly *η*-representable.

For the case of general *η*-representations we first look at the following definitions.

DEFINITION 1.2. For any linear order L the successor relation S_L on L is defined by $S_L(x, y) \iff |[x, y]| = 2$. The block relation B_L is given by $B_L(x, y) \iff [x, y]$ and $[y, x]$ are finite. A block of size n in L is a collection $x_0 <_L \cdots <_L x_{n-1}$ such that $B_L(x_0, y) \to \bigvee_{i<n} y = x_i$.

For any linear order L, one can see that S_L is Π_1^0 in L and B_L is Σ_2^0 in L. Feiner [3] proved the following:

THEOREM 1.3. *For a linear order L, the set $\{n : L$ has a block of size $n\}$ is Σ_3^0 in L.*

For an *η*-representation L of a set A, we have $A = \{n : L$ has a block of size $n\}$. This gives us the following.

COROLLARY 1.4. *If a set A has a computable η-representation then A is Σ_3^0.*

Coles et al. [1] show the reverse of theorem 1.3 is true for general linear orders.

THEOREM 1.5. *For any Σ_3^0 set A there is a computable linear order L, such that $A = \{n : L$ has a block of size $n\}$.*

Fellner [4] showed that every strongly *η*-representable set is Δ_3^0 and went on to conjecture that every Δ_3^0 set has a strong *η*-representation. However, Lerman [11] later showed that this is not the case.

THEOREM 1.6. (Lerman [11]) *There is a Δ_3^0 set with no computable η-representation.*

Lerman also characterized the *m*-degrees with computable *η*-representations showing that they are the Σ_3^0 degrees:

THEOREM 1.7. (Lerman [11]) *If A is Σ_3^0 then $A \oplus \omega$ has a computable η-representation.*

This leaves open the questions of what are the (strongly) *η*-representable sets and what are the strongly *η*-representable degrees. In the case of *η*-representations Harris [6] came up with a characterization involving limitwise monotonic functions. Limitwise monotonic functions were first introduced by Khoussainov et al. [10].

DEFINITION 1.8. A function $F : \omega \to \omega$ is *limitwise monotonic* if there is a computable function $f : \omega^2 \to \omega$ such that $F(n) = \lim_s f(n, s)$ and for all n, s, $f(n, s) \leq f(n, s + 1)$.

By the limit lemma, if F is limitwise monotonic then F is Δ_2^0, and hence if $A =$ range(F) then A is Σ_2^0.

Limitwise monotonic functions have been used to solve questions computable model theory [7, 9, 10]. In particular Coles et al. [1] proved that for any computable η-like linear order (a class that includes computable η-representations) that the set $\{n : L$ has a block of size $n\}$ is the range of a $\mathbf{0}'$-limitwise monotonic function. Harris [6] showed the reverse direction holds for computable η-representations.

THEOREM 1.9. *A set A is η-representable if and only if A is the range of a $\mathbf{0}'$-limitwise monotonic function.*

The construction of the η-representation L is performed uniformly, constructing linear orders $L_n \cong \eta + F(n)$ and taking $L = \sum_n L_n$. From this it can be seen that if A is the range of a strictly increasing $\mathbf{0}'$-limitwise monotonic function then A is strongly η-representable. However, Harris [6] showed that this is not a characterization of the strongly η-representable sets.

Harris also showed that the degrees with computable strong η-representations are not trivial.

THEOREM 1.10. (Harris [6]) *There is a Δ_3^0 degree that does not contain a set with a computable strong η-representation.*

Kach and Turetsky [8] modified the notion of limitwise monotonic to give the following:

DEFINITION 1.11. A function $F : \mathbb{Q} \to \omega$ is *support (strictly) increasing limitwise monotonic function on $\mathbb{Q}$* if there is computable $f : \mathbb{Q} \times \omega \to \omega$ such that

- $F(q) = \lim_s f(q, s)$.
- For all q, s $f(q, s) \leq f(q, s + 1)$.
- The set $S := \{q \in \mathbb{Q} : F(q) \neq 0\}$ has order type ω.
- $F \upharpoonright S$ is (strictly) increasing.

One can relativize this to a degree $\mathbf{d}$ by allowing f to be $\mathbf{d}$-computable. They define $\mathbf{SILM^d}(\mathbb{Q})$ to be the set of A such that A is the range of a $\mathbf{d}$-support increasing limitwise monotonic function on $\mathbb{Q}$ and $\mathbf{SSILM^d}(\mathbb{Q})$ to be the set of A such that A is the range of a $\mathbf{d}$-support strictly increasing limitwise monotonic function on $\mathbb{Q}$.

Kach and Turetsky were able to get the following result about increasing η-representations.

THEOREM 1.12. *A set A has a computable increasing η-representation if and only if $A \in \mathbf{SILM^{0'}}(\mathbb{Q})$.*

Similarly to the case of η-representable degrees, Kach and Turetsky proved that every Δ_3^0 degree has a computable increasing η-representation (proved earlier in [5]).

Like in the case of Theorem 1.9, the proof of Theorem 1.12 gives us that if $A \in \mathbf{SSILM^{0'}}(\mathbb{Q})$ then A is strongly η-representable. The converse, however, is not true in general.

THEOREM 1.13. (Turetsky [2])   *There is a set $A \notin \mathbf{SSILM}^{\mathbf{0}'}(\mathbb{Q})$ with a computable strong η-representation.*

This is close to a characterization of strongly η-representable sets. In Section 4 we are able to prove that for dense enough sets this is a characterization.

COROLLARY 1.14.   *Suppose $g : \omega \to \omega$ is a $\mathbf{0}'$-computable increasing function. If a set A has a strong η-representation and satisfies $|A \cap g(n)| \geq n$ for all n then $A \in$* $\mathbf{SSILM}^{\mathbf{0}'}(\mathbb{Q})$.

Using this we are then able to characterize the sets with computable strong η-representations up to many-one degree.

COROLLARY 1.15.   *The following coincide.*

- *The m-degrees of sets with computable strong η-representations.*
- *The m-degrees of sets in* $\mathbf{SSILM}^{\mathbf{0}'}(\mathbb{Q})$.
- *The m-degrees of sets with Δ_2^0 strong η-s-representations.*

§2. η-s-Representations. The existing characterizations of sets with computable η-representations and with computable increasing η-representations both involve relativizing some construction to $\mathbf{0}'$ and make use of the fact that $\mathbf{0}'$ can compute the successor relation on any computable linear order. For this reason we propose the following definition.

DEFINITION 2.1.   A (strong) η-s-representation of a set A is a computable (strong) η-representation L where the successor relation S_L is also computable.

The hope is that we can find a characterization of strongly η-s-representable sets and turn it into a characterization of the strongly η-representable sets. Towards this idea we have the following theorem.

THEOREM 2.2.   *If L is a $\mathbf{0}'$-computable η-representation of some set A and the block relation $B_L \leq_T \mathbf{0}'$ then there is a computable linear order D such that $D \cong L$ and $B_D \leq_T \mathbf{0}'$.*

PROOF.   Using that L is Δ_2^0 we can approximate L in stages. We keep track of the blocks that $\emptyset'_s$ thinks are in L_s and build corresponding blocks in D_s. When we see two blocks in L_s change order or merge, we keep the representative of the block with the smallest member (in the sense of $<_\mathbb{N}$) and remove the other one by densifying (i.e., adding points so that the block becomes part of a copy of $\mathbb{Q}$). Then we add a new block in the correct place if needed.

More formally, let $(L_s, <_s, B_s)_s$ be a sequence of linear orders with block relation that has limit $(L, <_L, B)$ where each $L_s \subseteq L_{s+1}$ is a subset of ω.

Define $D_0 = \emptyset$. We will keep a follower function f_s from the blocks of D_s to a corresponding element in L_s that represents the block. At stage s, for any $b_i, b_j \in \mathrm{dom}(f_s)$ if we have $f_s(b_i) <_\mathbb{N} f_s(b_j)$ and $f_s(b_i) <_s f_s(b_j)$ but $f_s(b_i) >_{s+1} f_s(b_j)$ then in D_{s+1} we will remove b_j from $\mathrm{dom}(f_{s+1})$. Similarly if $f_s(b_i) >_s f_s(b_j)$ but $f_s(b_i) <_{s+1} f(b_j)$ or $\neg B_s(f_s(b_i), f_s(b_j))$ but $B_{s+1}(f_s(b_i), f_s(b_j))$.

Next, for each block $b \in \mathrm{dom}(f_s)$ that has not been removed we make sure it has the correct size. Let $y = \min_\mathbb{N}\{x : B_s(x, f_s(b)) \wedge \neg B_{s+1}(x, f_s(b))\}$. If y exists,

remove points from the end of b so that it has size $|\{x <_{\mathbb{N}} y : B_s(x, f_s(b))\}|$. Now we add points to the end of the block so that the block of $f_s(b)$ in L_{s+1} will have the same size as b does in D_{s+1}. Then, in case small numbers have been added we set $f_{s+1}(b) = \min_{\mathbb{N}}\{x : B_{s+1}(x, f_s(b))\}$.

Then for each block c in L_{s+1} that does not have an element in range(f_s) we create a corresponding block b in D_{s+1} of the same size as c and set $f_{s+1}(b) = \min_{\mathbb{N}}(c)$. Finally we densify; for all adjacent x, y which are not part of the same block in dom(f_{s+1}), we add a new point between x and y. We now have D_{s+1}.

Now the verification. It is clear that D is a computable linear order. We need to make sure it has the right order type. At each stage we densify around the points that are not part of a block, so between adjacent blocks we must have order type η.

CLAIM 2.2.1. *For every block $c \in L$ there is a unique block $b \in D$ that has the same length as c.*

PROOF. Let $n = \max_{\mathbb{N}}(c) + 1$. There is a stage t such that for all $s \geq t$, $B_s \restriction n = B$ and $<_s \restriction n = <_L \restriction n$. At this stage t there will be a b such that $f_t(b) = \min_{\mathbb{N}}(c)$. By our choice of t we have $f_s(b) = \min_{\mathbb{N}}(c)$ and $|b| \geq |c|$ for all $s \geq t$ as there can be no reason to destroy b and we will never see any number smaller than n leave c.

Given any $s > t$ and $m = \min_{\mathbb{N}}\{x \geq_{\mathbb{N}} n : \exists r > s[B_r(f_r(b), x)]\}$ there is a stage $r > s$ such that $B_r(f_r(b), x) \wedge \neg B_{r+1}(f_{r+1}(b), x)$. So at stage $r+1$ we will have $|b| = |c|$ and as s is arbitrary, we have $|b| \leq |c|$ in D. ⊣

CLAIM 2.2.2. *For every block $b \in D$ there is a block $c \in L$ and t such that $|b| = |c|$ and for all $s \geq t$, $f_s(b) = \min(c)$. Furthermore, if $b_i <_D b_j$ then for the corresponding blocks $c_i, c_j \in L$ we have $c_i <_L c_j$.*

PROOF. Consider a block b. Suppose $f_s(b) = n$ and $f_t(b) = m$ for $s < t$. Then it must be that $f_s(b) \geq f_t(b)$. So $\lim_s f_s(b)$ exists. If $x = \lim_s f_s(b)$ and c is the block of x in L then by the same argument as above we have that $|b| = |c|$.

If $x = \lim_s f_s(b_i)$ and $y = \lim_s f_s(b_j)$ and $b_i <_D b_j$ then $x <_L y$ as otherwise we would have removed one of the blocks. ⊣

So we can see that there is an order preserving bijection F from the blocks of D to the blocks of L with $|b| = |F(b)|$. Hence the order type of D is the same as that of L.

From the construction, if a point is removed from a block then it is never put back in a block at a later stage. So $\mathbf{0}'$ can compute the set of points in D that are not in blocks. As D is computable, $\mathbf{0}'$ can also compute the successor relation on D. From both of these, $\mathbf{0}'$ can compute the block relation. ⊣

Theorem 2.2 is not quite what we would like as it requires the block relation to be $\mathbf{0}'$-computable. However, this is a property that occurs if the blocks are created in isolation and never merged. This is precisely what happens in the constructions given in the proofs of the characterizations of η-representable and increasingly η-representable sets.

THEOREM 2.3. *A set A is in $\mathbf{SSILM}(\mathbb{Q})$ if and only if there is a strong η-s-representation with computable block relation.*

Proof. For the left to right direction we observe that the usual construction (unrelativizing the one given in [8]) has computable block relation as the blocks that are created are never merged.

For the other direction, since we can compute if two blocks are actually the same block we can make sure we only assign one follower to each block. ⊣

By combining Theorems 2.3 and 2.2 we get a characterization of $\textbf{SSILM}^{\textbf{0}'}(\mathbb{Q})$ in terms of computable η-representations:

Corollary 2.4. *A set A is in $\textbf{SSILM}^{\textbf{0}'}(\mathbb{Q})$ if and only if there is a strong η-representation of A with $\textbf{0}'$-computable block relation.*

Theorem 1.13 states that there are strongly η-representable sets which are not in $\textbf{SSILM}^{\textbf{0}'}(\mathbb{Q})$, so as a result any characterization of the sets with strong η-representations must involve merging blocks as part of the construction.

§3. Connected approximations. The limit lemma says that we can approximate any Δ_2^0 set A with a computable sequence $(A_n)_n$ such that $A(x) = \lim_n A_n(x)$. Limitwise monotonic functions are one way of building on this idea. From what we have seen, the problem with trying to use these to characterize strongly η-representable sets is that each limit of a sequence $F(q) = \lim_s f(q,s)$ is taken in isolation, and there is no natural way of merging sequences. So we propose a different way of approximating sets that captures the idea of merging sequences.

Definition 3.1. A *connected approximation* to a set A is a sequence of finite functions $(c_n)_n$ with associated sequences of finite sets $(A_{n,m})_m$ that satisfy the following:

1. range$(c_n) \subseteq$ dom(c_{n+1}) for all n.
2. $A_{n,0} :=$ dom(c_n), $A_{n,m+1} := c_{n+m}(A_{n,m})$.
3. The limit $A_{n,\omega} := \lim_m A_{n,m}$ always exists.
4. $A = \cup_n A_{n,\omega}$.

We can assume each c_n is coded by a canonical index for the finite set of its graph $\{\langle x, c_n(x)\rangle : x \in$ dom$(c_n)\}$, so we can say a connected approximation $(c_n)_n$ is computable if the corresponding sequence of indices is computable.

We call a connected approximation $(c_n)_n$ *monotonic* if $c_n(x) \geq x$ for each n and $x \in$ dom(c_n), and *order preserving* if each c_n preserves $\leq$. We use the acronym MOP to denote monotonic and order preserving.

We give characterizations of all of the existing classes described so far using connected approximations.

Theorem 3.2. *For a set A we have the following characterizations.*

1. *A has a computable connected approximation if and only if A is Σ_2^0.*
2. *A has a computable monotonic connected approximation if and only if A is the range of a computable limitwise monotonic function.*
3. *A has a computable MOP connected approximation if and only if $A \in \textbf{SILM}(\mathbb{Q})$.*
4. *A has a computable MOP connected approximation where each c_n is injective if and only if $A \in \textbf{SSILM}(\mathbb{Q})$.*

PROOF OF (1) AND (2). We will handle the first two statements together. Given a Σ_2^0 set A we can assume $A = \text{range}(F)$ for $F(n) = \lim_s f(n,s)$ where f is computable. Then we can define a connected approximation of $A = \text{range}(F)$ as follows. Let $\text{dom}(c_n) = \{f(x,n) : x < n\}$ and define $c_n(y) = f(x, n+1)$ where x is least such that $f(x,n) = y$. Clearly $(c_n)_n$ is computable and $\text{range}(c_n) \subseteq \text{dom}(c_{n+1})$. For each n, m we have that $A_{n,m} = f(B_{n,m}, n+m)$ for some $B_{n,m} \subseteq n$. We take the $B_{n,m}$ which minimizes $\sum_{x \in B_{n,m}} x$. By construction $\sum_{x \in B_{n,m}} x \geq \sum_{x \in B_{n,m+1}} x$ and so the limit $B_{n,\omega} := \lim_m B_{n,m}$ exists. Hence $A_{n,\omega} = F(B_{n,\omega})$, so we have $(c_n)_n$ is a connected approximation of a subset of A. Consider an $n \in \omega$. Let $t > n$ be a stage after which $f(m,s) = F(m)$ for all $s \geq t, m \leq n$. Then $F[n] \subseteq A_{t,0}$, so $F[n] \subseteq A_{t,\omega}$. So $(c_n)_n$ is a connected approximation of A. Notice that if $f(n,s)$ is monotonic in s then $(c_n)_n$ is also monotonic.

Now consider a computable connected approximation $(c_n)_n$ of a set A. We define a computable function $f : \omega^2 \to \omega$ as follows. $f(n,0) = 0$, $t_0 = 0$. Define $f(n, s+1)$ as follows: $f(n, s+1) = c_s(f(n,s))$ if $n < t_s$. Let $m_0, \ldots, m_{k-1}$ list $\text{range}(c_s) \setminus \text{range}(c_s \circ c_{s-1})$ in order. Define $f(t_s + i, s+1) = m_i$ and $t_{s+1} = t_s + k$. For $n \geq t_{s+1}$ let $f(n, s+1) = 0$. We have that $A_{n,m} = \{f(x, n+m) : x \leq t_n\}$ and so $\text{range}(F \upharpoonright t_n) = A_{n,\omega}$ and hence $\text{range}(F) = A$. Notice that if $(c_n)_n$ is monotonic then F is limitwise monotonic. ⊣

A similar idea works for characterizations (3) and (4), but when going from a connected approximation we need to choose rationals so that the order is preserved.

Now we give a characterization of strongly η-s-representable sets using connected approximations. In a construction of a strong η-s-representation, blocks can do two things: they can grow and they can merge. Eventually they must stop doing either of these things, but we cannot put a computable bound of how late these actions take place. However, if two blocks are, in fact, different then we will see infinitely many points go in between them. Thus, if blocks merge at a late stage then the size of the resulting block should be very large. This is the main idea behind the formula in the following characterization and the proof.

THEOREM 3.3. *A set A has a strong η-s-representation if and only if it has a computable MOP connected approximation where each c_n satisfies*

$$\psi(n) = \forall x \in \text{range}(c_n) \left[\sum_{m \in c_n^{-1}(\{x\})} (m+n) \leq x + n \right].$$

PROOF. Suppose we have a strong η-s-representation L of A. We can assume that L has domain ω and let $L_s = L \upharpoonright s$. Let B_s be the blocks of L_s according to S_L. For blocks $b, c \in B_s$ and $t \geq s$ we use $|b|_t$ to denote the size the block has in L_t, and we use $b <_t c$ and $b =_t c$ to denote the order of the, possibly merged, blocks in L_t.

We start with $c_0 = \emptyset$, $t_0 = 0$. At stage s we assume we are given $\text{dom}(c_s)$, t_s and a block $b_s \in B_{t_s}$. We assume that for any $c, d \in B_{t_s}$ with $c <_{t_s} d \leq_{t_s} b$ we have $|c|_{t_s} < |d|_{t_s}$ and there are at least s many points between c and d in L_{t_s}. We also assume $\text{dom}(c_s) = \{|b|_{t_s} : b \in B_{t_s} \wedge b \leq_{t_s} b_s\}$.

We let $b_{s+1} = \max_{<_{t_s}} B_{t_s}$. Search for a $t > t_s$ such that for every $c, d \in B_t$ with $c <_t d \leq_t b_{s+1}$ we have $|c|_t < |d|_t$ and there are at least $s+1$ many points between c and d in L_{t_s}. The fact that L is a strong η-s-representation guarantees that we will

find such a t. We let $t_{s+1} = t$ and $\mathrm{dom}(c_{s+1}) = \{|b|_t : b \in B_t \land b \leq_t b_{s+1}\}$. We define c_s as follows. For $d \leq_{t_s} b_s$ we set $c_s(|d|_{t_s}) = |d|_t$. This clearly gives $\mathrm{range}(c_s) \subseteq \mathrm{dom}(c_{s+1})$. This completes the construction.

Now we need to check that c_s is MOP and meets the condition ψ. If $|d|_{t_s} < |c|_{t_s}$ for $d, c <_{t_s} b_s$ then we have that $d <_{t_s} c$, so $d \leq_t c$ and $|d|_t \leq |c|_t$. Thus c_s preserves $\leq$. Since L is a strong η-s-representation, we have that $|d|_n \leq |d|_m$ for $n \leq m$, and so c_s is monotonic. If we combine this with the fact that there are s many points between relevant blocks in B_{t_s} we have that if $d_1 <_{t_s} \cdots <_{t_s} d_n \leq_{t_s} b_s$ but $d_1 =_t \cdots =_t d_n$ then we have $|d_i|_t + s \geq \sum_{i=1}^{n}(|d_i|_{t_s} + s)$. So we can conclude that c_n meets the condition ψ.

All that is left to check is that the limits exist and that they give us A. We have that $A_{n,m} = \{|d|_{t_{n+m}} : d \in B_{t_n}, d \leq b_n\}$. Since B_{t_n} is a finite set and each block in B_{t_n} only changes size finitely often, we have that the limit $A_{n,\omega}$ exists and $A_{n,\omega} \subseteq A$. On the other hand, every $d \in B_L$ is in some B_n, so the there is a stage s such that $t_s > n$ and then we have $|d|_{t_{s+1}} \in \mathrm{dom}(c_{s+1})$. Thus $|d| \in A_{s+1,\omega}$. So we have that $A = \cup_n A_{n,\omega}$ and $(c_n)_n$ is a connected approximation of A as desired.

Now for the other direction. Suppose we have a connected approximation $(c_n)_n$ of A satisfying the conditions of the theorem. We construct an η-s-representation as follows. The main idea is that at stage s we will have a linear order L_s with successor relation and blocks B_s strictly ordered by size with s many points in between, and the sizes of blocks of B_s are the members of $\mathrm{dom}(c_s)$.

We define a computable function $H(L, c, m)$ that takes a finite linear order L with successor, a finite function c and a number m, and outputs a finite linear order D with successor extending L if it can. We assume that the blocks B_L are ordered by size the same way they are by $<_L$. We assume that $\{|b| : b \in B_L\} \subseteq \mathrm{dom}(c)$. We build D in steps as follows. If $c(|b|) = c(|d|)$ then we merge blocks b and d and all the points in between into one large block. This gives us a D_0 that differs from L only in the successor. We then go through each block b of D_0, and if d was a block of L and $d \subseteq b$ then we possibly add points to the end of b so that $|b| = c(|d|)$. If we have $|b| > c(|d|)$ already then H fails. If H does not fail then this gives us D_1. Now, for each $n \in \mathrm{range}(c) \setminus \{|b| : b \in B_{D_1}\}$, we add a new block of length n to D_1, keeping the ordering of blocks by size. This gives us a D_2. Finally, between each pair of adjacent blocks in D_2, we add points in a dense way so that there are exactly m many points between them. This is D. If one of the assumptions was wrong then H fails, otherwise it succeeds, and D is a linear order with blocks ordered by size the sizes of which are $\mathrm{range}(c)$, and there are exactly m many points between adjacent blocks.

We define our strong η-s-representation to be $L = \bigcup_s L_s$ where $L_0 = \emptyset$ and $L_{s+1} = H(L_s, c_s, s + 1)$. From the definition of H and the fact that each c_n preserves $\leq$ and satisfies ψ we can see, using an induction argument, that H will always succeed, so the L_s are all well-defined. From the definition of H we can see that if two points are never part of the same block for some L_s then there is a point in between them. So we have that the successor relation on L is $S_L = \bigcup_n S_{L_n}$. So L is a computable linear with c.e. successor relation. As the successor relation of a computable linear order is always co-c.e. we have that S_L is computable.

By construction we have that $A_{n,m} = \{|b|_{n+m} : b \text{ is a block in } L_{n+1}\}$, and for every block b in L_n, $c_m(|b|_m) = |b|_{m+1}$. So we have that $A_{n,\omega} = \{|b|_L : b \text{ is a block in } L_n\}$

and L is an η-s-representation of A. As the blocks of L_n are ordered by increasing size so too are the blocks of L, so L is a strong η-s-representation of A. ⊣

Note that if we replace $\psi(n)$ by the condition $\forall x \in \text{dom}(c_n)[(\sum_{m \in c_n^{-1}(\{x\})} m + f(n) \leq x + f(n)]$ for any computable non-decreasing f with $\lim_n f(n) = \omega$ then a slight modification of the arguments above should still work and we get another characterization. The relativized version of the proof with $\mathbf{0}'$-computable connected approximation, does not necessarily build us a computable strong η-representation, so we do not have a characterization of the strongly η-representable sets.

§4. The many-one degrees of η-representable sets. We know from Kach and Turetsky [8] that if $S \in \mathbf{SSILM}(\mathbb{Q})$ then S has a strong η-s-representation. The following is a condition on S under which the converse holds.

THEOREM 4.1. *Suppose $g : \omega \to \omega$ is a computable increasing function. If a set A has a strong η-s-representation and satisfies $|A \cap g(n)| \geq n$ for all n then $A \in \mathbf{SSILM}(\mathbb{Q})$.*

PROOF. The construction goes as follows. We use an enumeration of $L = \{x_i : i \in \omega\}$, and at stage s we look at the maximal blocks of L_s. We pick rationals to represent the blocks with the idea that $F(r)$ is the size of the block represented by r, but the block that r represents may change when blocks change. To keep track of what blocks rationals follow we will use a sequence of helper functions $h_s : \mathbb{Q} \to B_{t_s}$ with $\text{dom}(h_s) = \{r : f(r, s) > 0\}$. Once we see a block b appear in L_s it can only grow, so it will remain a block in L_t for $t > s$. Like we did in the proof of Theorem 3.3 we will use $|b|_t$, $b =_t c$ and $b <_t c$ to denote the size and order of the blocks from L_s according to L_t. We let B_t be the set of blocks from L_t.

At stage 0 we start with $f(r, 0) = 0$ for all $r \in \mathbb{Q}$ and $t_0 = 0$. At stage s let $b = \max(B_{t_s})$. Let $t_{s+1} = t$ be the least stage $t > t_s$ such that for each $c <_t d \leq_t b$ in B_t we have that $|c|_t < |d|_t$ and there are at least $g(|c|_t + |d|_t)$ many points between c and d in L_t, and furthermore for all n such that $g(n) \leq |b|_t + 1$ we have $|\{c \in B_t : c \leq_t b \wedge |c| < g(n)\}| \geq n$. As L is an η-s-representation of A there must be such a t.

Let $r_0 < \cdots < r_{n-1}$ be the domain of h_s; we begin defining h_{s+1} as follows. Let $h_{s+1}(r_0)$ be the smallest block $c_0 \in B_t$ such that $c_0 \leq_t h_s(r_0) \wedge |c_0|_t \geq f(r_0, s)$. Let $h_{s+1}(r_i)$ be the smallest block $c_i \in B_t$ such that $h_{s+1}(r_{i-1}) <_t c_i \leq_t h_s(r_i) \wedge |c_i|_t \geq f(r_i, s)$.

For each block $c \leq_t b$ that is not in $\text{range}(h_{s+1})$ we pick a rational r_c and set $h_{s+1}(r_c) = c$ so that h_{s+1} is order preserving and has image $\{c \in B : c \leq_t b\}$. We define

$$f(r, s+1) = \begin{cases} |h_{s+1}(r)|_t, & r \in \text{dom}(h_{s+1}), \\ 0, & \text{otherwise.} \end{cases}$$

Now for the verification: first we need to show that the recursive definition of $h_{s+1}(r_i)$ actually works. Suppose it does not. Then let i be least such that we cannot find a block for r_i. $|h_s(r_i)|_t \geq |h_s(r_i)|_{t_s} \geq f(r_i, s)$ so if $h_s(r_i)$ does not work then there must be some smaller r_j with $h_{s+1}(r_j) = h_s(r_i)$. So $i > 0$. We have $h_{s+1}(r_{i-1}) \leq_t h_s(r_{i-1}) <_{t_s} h_s(r_i)$, so it must be that $h_s(r_{i-1})$ and $h_s(r_i)$ have merged.

So $|h_s(r_i)|_t > g(|h_s(r_{i-1})|_{t_s} + |h_s(r_i)|_{t_s})$. So by our choice of t_s we have at least $|h_s(r_{i-1})|_{t_s} + |h_s(r_i)|_{t_s}$ many blocks before $h_s(r_i)$ and at least $|h_s(r_i)|_{t_s}$ have size at least $|h_s(r_{i-1})|_{t_s}$. But $i \leq |h_s(r_i)|_{t_s}$, so we would have chosen $h_{s+1}(r_{i-1})$ to be one of these, a contradiction.

From the definition of h_s we can see that $h_{s+1}(r) \leq_L h_s(r)$ for each r and s, so as the blocks of L are well ordered, $\lim_s h_s(r)$ exists. From the definition of f we have that it is limitwise monotonic and $F(r) = |\lim_s h_s(r)|_L$. So range$(F) \subseteq S$. If b is a block of L then after some stage t, all of b is in L_t as well as all smaller blocks. So at some stage s, $t_s > t$, so at stage $s+1$ we have an r such that $h_{s+1}(r) = b$ and for any $n > s$ we have $h_n(r) = b$ as the blocks in that part of the linear order no longer change.

So $S = $ range(F) as desired.   ⊣

Relativizing we get the following:

COROLLARY 1.14. *Suppose* $g : \omega \to \omega$ *is a* $\mathbf{0}'$*-computable increasing function. If a set* A *has a strong* η*-representation and satisfies* $|A \cap g(n)| \geq n$ *for all* n *then* $A \in$ **SSILM**$^{\mathbf{0}'}(\mathbb{Q})$.

This means that for dense enough sets, the notions of Δ_2^0 strong η-s-representation, strong η-representation and support strictly increasing limitwise monotonic on $\mathbb{Q}$ all coincide.

Note that we cannot use Theorem 4.1 to give a characterization of **SSILM**$(\mathbb{Q})$ as there are sparse sets in **SSILM**$(\mathbb{Q})$. For instance consider the function $F(n) = n + \sum_{e \in \emptyset' \cap n} h(e)$ where $h(e)$ is the least s such that $\varphi_{e,s}(e)\downarrow$. Then as F cannot be computably bounded, $S = $ range(F) would not meet the condition $|S \cap g(n)| \geq n$ for all n for any computable g, but by definition it is clearly limitwise monotonic and increasing, so $S \in$ **SSILM**$(\mathbb{Q})$.

We can, however, use Theorem 4.1 to characterize the degrees of sets with computable strong η-representations.

THEOREM 4.2. *If* **a** *is the* m*-degree of a set with a strong* η*-s-representation then there is* $S \in$ **a** *such that* $S \in$ **SSILM**$(\mathbb{Q})$.

To prove this we use the following lemma.

LEMMA 4.3. *If* A *is a set with a strong* η*-s-representation then* $A \oplus \omega$ *also has a strong* η*-s-representation*.

PROOF. Suppose that A is a set with a strong η-s-representation. Let $(c_n)_n$ be a computable MOP connected approximation of A satisfying condition ψ of Theorem 3.3. We build an connected approximation $(d_n)_n$ of $A \oplus \omega$ satisfying ψ as follows. The first idea is to use c_m with m much larger than n to build d_n. We want m to be large enough that when we see $c_m(x) = c_m(y)$ we can merge the corresponding numbers $2x, 2y \in $ dom(d_n) without violating ψ. The second idea is that when we see $c_m(x) > x$ without any merging, we shift the representative of x in dom(d_n) to a lager number so that we can handle the case where the gaps between numbers shrink, i.e., when $c_m(y) - c_m(x) < y - x$ for $y > x$.

To start let $d_0 = \emptyset$ and $m_0 = 0$. We will ensure that $m_n > \sum_{x \in \text{dom}(d_n)}(x + n)$, and if $2x \in $ range(d_n) then $x \in $ range(c_{m_n}). Given d_n and m_n, let $N > m_n$ be the least

number such that if $x = \max(A_{m_n, N+1})$ then $N > 2x(2x + n + 1)$. Let $m_{n+1} = N$. This will ensure that $m_{n+1} > \sum_{x \in \text{dom}(d_{n+1})}(x + n + 1)$. There must be such an N as $(c_n)_n$ is a valid connected approximation, so eventually x will stabilize. Let $c = c_N \circ \cdots \circ c_{m_n + 1}$.

Let $z \in \text{dom}(c)$ be the least such that there is $y > z$, $c(z) = c(y)$. For $y \geq 2z \in$ range(d_n) we define $d_{n+1}(y) = 2c(z)$. By assumption on m_n, this will not violate condition ψ. If there is no such z then set $z = \max(\text{dom}(c)) + 1$. Now we define d_{n+1} on values smaller than $2z$.

Let $a_0, \ldots, a_{s-1}$ list the elements of $(\text{dom}(c) \oplus \omega) \cap 2z - 1$. Let $b_0, \ldots, b_{s-1}$ list the first s elements of range$(c) \oplus \omega$. Note that $b_{s-1} < 2c(z)$ if $c(z)$ is defined. We define $d_{n+1}(a_i) = b_i$. This is definitely order preserving, and it is monotonic because c is injective and monotonic on $\text{dom}(c) \cap z$. This completes the construction of $(d_n)_n$.

Verification: By construction we can see that $(d_n)_n$ is MOP and satisfies ψ; all that is left to check is that it is a connected approximation of $A \oplus \omega$. Let $x \in \text{dom}(d_n)$ and let $2y$ be the least even number in $\text{dom}(d_n) \setminus x$. Following the construction we can see that $d_n(x) \leq d_n(2y) \leq 2(c_{m_n} \circ \cdots \circ c_{m_{n-1}+1})(y)$. So if we repeat this then we can see that the limit of $x, d_n(x), d_{n+1}(d_n(x)), \ldots$ if it exists is less than the limit of $2y, 2c_{m_n}(y), 2c_{m_n+1}(c_{m_n}(y)), \ldots$, so by monotonicity it exists. Thus $(d_n)_n$ is a valid connected approximation.

Fix x. Let s be a stage at which $c_n \restriction x = c_s \restriction x$ for all $n \geq s$. Then by the construction we have that $d_n \restriction 2x = d_s \restriction 2x$ for all $n \geq s$ and $\text{dom}(d_n) \cap 2x = (\text{dom}(c_n) \cap x) \oplus x$. So $(d_n)_n$ is a connected approximation of $A \oplus \omega$. ⊣

This allows us to characterize the m-degrees of sets with strong η-representations.

COROLLARY 1.15. *The following coincide.*

- *The m-degrees of sets with computable strong η-representations.*
- *The m-degrees of sets in* **SSILM$^{0'}$**$(\mathbb{Q})$.
- *The m-degrees of sets with Δ_2^0 strong η-s-representations.*

§5. Open questions.
We can characterize the m-degrees of sets with strong η-representations, but we leave open the following.

QUESTION 5.1. *Is there a characterization of the sets with strong η-representations that is not in terms of linear orders?*

A related question is in regards to the sets with Δ_2^0 strong η-s-representations.

QUESTION 5.2. *Is there a set with a Δ_2^0 strong η-s-representation, but no computable strong η-s-representation?*

A negative answer to Question 5.2 would give us an answer to Question 5.1, using the connected approximation characterization of sets with computable strong η-s-representations.

REFERENCES

[1] R. J. COLES, R. DOWNEY, and B. KHOUSSAINOV, *On initial segments of computable linear orders*. **Order**, vol. 14 (1997/98), no. 2, pp. 107–124.

[2] R. DOWNEY, A. M. KACH, and D. TURETSKY, *Limitwise monotonic functions and their applications*, **Proceedings of the 11th Asian Logic Conference**, World Scientific Publications, Hackensack, NJ, 2012, pp. 59–85.

[3] L. FEINER, *Hierarchies of Boolean algebras*, this JOURNAL, vol. 35 (1970), pp. 365–374.

[4] S. M. FELLNER, ***Recursiveness and Finite Axiomatizability of Linear Orderings***, ProQuest LLC, Ann Arbor, MI, 1976, Ph.D. thesis, Rutgers The State University of New Jersey, New Brunswick.

[5] A. N. FROLOV and M. V. ZUBKOV, *Increasing η-representable degrees*. **MLQ Mathematical Logic Quarterly**, vol. 55 (2009), no. 6, pp. 633–636.

[6] K. HARRIS, *η -representation of sets and degrees*, this JOURNAL, vol. 73 (2008), no. 4, pp. 1097–1121.

[7] D. R. HIRSCHFELDT, B. KHOUSSAINOV, and P. SEMUKHIN, *An uncountably categorical theory whose only computably presentable model is saturated*. **Notre Dame Journal of Formal Logic**, vol. 47 (2006), no. 1, pp. 63–71.

[8] A. M. KACH and D. TURETSKY, *Limitwise monotonic functions, sets, and degrees on computable domains*, this JOURNAL, vol. 75 (2010), no. 1, pp. 131–154.

[9] I. KALIMULLIN, B. KHOUSSAINOV, and A. MELNIKOV, *Limitwise monotonic sequences and degree spectra of structures*. **Proceedings of the American Mathematical Society**, vol. 141 (2013), no. 9, pp. 3275–3289.

[10] B. KHOUSSAINOV, A. NIES, and R. A. SHORE, *Computable models of theories with few models*. **Notre Dame Journal of Formal Logic**, vol. 38 (1997), no. 2, pp. 165–178.

[11] M. LERMAN, *On recursive linear orderings*, **Logic Year 1979–80 (Proc. Seminars and Conf. Math. Logic, Univ. Connecticut, Storrs, Conn., 1979/80)**, Lecture Notes in Mathematics, vol. 859, Springer, Berlin, 1981, pp. 132–142.

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