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WEIGHTED POLYNOMIAL APPROXIMATION OF ENTIRE FUNCTIONS ON UNBOUNDED SUBSETS OF THE COMPLEX PLANE

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ABSTRACT. We study the asymptotic behavior of the n-widths of a class of entire functions in weighted approximation on subsets of the complex plane.

1. **Introduction.** In [2], [3], a Bernstein-type characterization of entire functions of finite exponential type and order was given in terms of the degree of weighted polynomial approximation of the functions on the whole real line. When the weight function is the Hermite weight, the theorems in [2] go beyond the classical Bernstein theorem in that the type of entire functions of order 1 is exactly given in terms of the degree of approximation. Subsequently, in [6], Micchelli and the author studied an n-width problem for weighted polynomial approximation of certain entire functions on Borel subsets of $\mathbf{C}$. In the case of approximation on $\mathbf{C}$, we obtained precise n-th root asymptotics for the n-widths.

In this paper, we continue the investigations in [2], [3], [6]. We consider the weight functions $\exp(-|z|^\alpha)$ on an unbounded Borel subset Σ of $\mathbf{C}$ which satisfies certain technical conditions to be described in Section 2. We give exact expressions for the type of entire functions of order less than α in terms of the degree of weighted polynomial approximation of such functions on Σ. We also obtain precise asymptotic formulas for certain nonlinear n-widths.

In Section 2, we develop the necessary notations and definitions as well as review some of the known theorems. The new theorems of this paper are discussed and proved in Section 3.

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2. **Notations and definitions.** Throughout this paper, Σ will denote a fixed Borel subset of $\mathbf{C}$ having positive inner logarthmic capacity (*cf.* [12], p. 55). In addition, we assume that Σ satisfies the following cone condition: For every $z \in \Sigma$, and $r > 0$, we have $rz \in \Sigma$. We fix $\alpha > 0$ and consider weight functions of the form $w(z) := \exp(-|z|^\alpha)$, $z \in \Sigma$. For integer $n \geq 0$, the symbol Π_n will denote the class of all algebraic polynomials (with complex coefficients) of degree at most n.

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Let $\mathcal{X}$ be any normed linear space defined for functions on Σ, $\|\cdot\|_{\mathcal{X}}$ be its norm, and $n \geq 0$ be an integer. If $wf \in \mathcal{X}$, we define

$$\epsilon_n(w, \mathcal{X}; f) := \inf_{P \in \Pi_n} \|w(f - P)\|_{\mathcal{X}}. \tag{2.1}$$

If $K \subset \mathcal{X}$, the weighted *Kolmogorov n-width* of K is defined by the formula

$$d_n(w, \mathcal{X}; K) := \inf_{Y \in \mathcal{F}_n} \sup_{f \in K} \inf_{g \in Y} \|w(f - g)\|_{\mathcal{X}} \tag{2.2}$$

where $\mathcal{F}_n$ denotes the collection of all linear subspaces of $\mathcal{X}$ with dimension at most n. The (weighted) *Bernstein n-width* of K is defined by

$$b_n(w, \mathcal{X}; K) := \sup_{Y \in \mathcal{F}_{n+1}} \sup\{\rho : g \in Y, \|wg\|_{\mathcal{X}} \leq 1 \Rightarrow \rho g \in K\}. \tag{2.3}$$

A good source of information on these n-widths is [10]. In particular, we need (*cf.* [10], Proposition 1.6)

$$d_n(w, \mathcal{X}; K) \geq b_n(w, \mathcal{X}; K). \tag{2.4}$$

Following [1], we define the (weighted continuous) *nonlinear n-width* of K by the formula

$$D_n(w, \mathcal{X}, K) := \inf_{a,M} \sup_{f \in k} \left\| w\Big(f - M\big(a(f)\big)\Big) \right\|_{\mathcal{X}}, \tag{2.5}$$

where the infimum is taken over all functions $M\colon \mathbf{R}^n \longrightarrow \mathcal{X}$ and continuous functions $a\colon K \longrightarrow \mathbf{R}^n$. Similarly to (2.4), we have ([1], Theorem 3.1)

$$D_n(w, \mathcal{X}; K) \geq b_n(w, \mathcal{X}; K), \tag{2.6}$$

In this paper, we restrict our attention to the case when $\mathcal{X}$ is either the class of bounded continuous functions with the usual supremum norm or some L^p space, $1 \leq p < \infty$. If μ is a positive Borel measure on Σ, $0 < p < \infty$ and $f\colon \Sigma \longrightarrow \mathbf{C}$ is a Borel measurable function, we set

$$\|f\|_{p,\mu,\Sigma} := \left\{ \int_\Sigma |f(t)|^p \, d\mu(t) \right\}^{1/p}. \tag{2.7}$$

The space $L^p_{\mu,\Sigma}$ then consists of all Borel measurable functions $f\colon \Sigma \longrightarrow \mathbf{C}$ such that $\|f\|_{p,\mu,\Sigma} < \infty$, where, as usual, two functions are identified if they are equal μ-almost everywhere. When $p = \infty$, instead of considering the usual L^∞ space, we shall consider the space $C_0(\Sigma)$ of all continuous functions $f\colon \Sigma \longrightarrow \mathbf{C}$ which vanish at infinity. This space is normed by

$$\|f\|_\Sigma := \|f\|_{\infty,\mu,\Sigma} := \sup_{z \in \Sigma} |f(z)|. \tag{2.8}$$

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In general, if $A \subseteq \mathbf{C}$ and $f: A \longrightarrow \mathbf{C}$, then we write

$$\|f\|_A := \sup_{z\in A} |f(z)|.$$

In the case when the underlying set Σ is $\mathbf{R}$, it is natural to consider the L^p spaces with respect to the Lebesgue measure. For arbitrary subsets of $\mathbf{C}$, there is no such natural choice. In particular, when Σ is a domain, then it is natural to consider the two dimensional Lebesgue measure, while if Σ is a curve, then it is more natural to consider the arclength measure. In [4], it was demonstrated that the following class of measures are suitable for certain potential theoretic applications to weighted approximation. We say that a positive Borel measure σ supported on Σ is *natural* if it satisfies each of the following conditions.

(M1) σ is a regular measure in the sense that for any Borel set $A \subseteq \Sigma$.

$$\begin{aligned}\sigma(A) &= \inf\{\sigma(O) : A \subseteq O \subseteq \Sigma, O\,\Sigma\text{-open}\}\\ &= \sup\{\sigma(K) : K \subseteq A,\ K \text{ compact}\}.\end{aligned}$$

(M2) For any compact set $K \subseteq \Sigma$, $\sigma(K) < \infty$ and the restriction of σ to K has finite logarithmic energy.

(M3) There exists an integer $N \geq 0$ such that

$$\int_{|z|\geq 1} |z|^{-N}\, d\sigma(z) < \infty.$$

(M4) For $\delta > 0$ and $z \in \mathbf{C}$, let

$$\lambda_n(\sigma,\delta,z) := \min_{P\in\Pi_n} |p(z)|^{-2} \int_{\Sigma_\delta(z)} |P(t)|^2\, d\sigma(t).$$

where

$$\Sigma_\delta(z) := \{t \in \Sigma : |t - z| \leq \delta\}.$$

Then, for every $\delta > 0$ and every compact set $K \subseteq \Sigma$,

$$\limsup_{n\to\infty} \|\lambda_n(\sigma,\delta,z)\|_K^{1/n} \leq 1. \tag{2.9}$$

In [4], various conditions were given which imply the condition (M4) above, but are easier to check. In particular, it was proved that the area measures on domains as well as arclength measures on C^{1+}-curves are 'natural'. It was also proved that if the restriction of σ to every compact subset of Σ is 'completely regular' in the sense of [11], then σ also satisfies the condition (M4). The significance of the definition will be clearer in Proposition 2.1(d) below.

If $E \subseteq \mathbf{C}$ and $r \geq 0$, we write

$$rE := \{rz : z \in E\}.$$

If σ is a measure supported on Σ, we say that σ is *homogeneous* if there are constants L, c_2 such that for any Borel subset $E \subseteq \Sigma$,

$$\sigma(\{rz : z \in E\}) \leq c_2 r^L \sigma(E), \quad r > 0. \tag{2.10}$$

In the sequel, a property is said to hold *quasi-everywhere* (q. e.) if the set where it does not hold has inner logarithmic capacity equal to zero. The following Proposition 2.1 summarizes certain potential theoretic facts which will be needed later.

PROPOSITION 2.1. *(a) There exists a unique Borel measure* $\mu^* := \mu^*(\alpha, \Sigma)$ *with the following properties. The measure* μ^* *has finite logarithmic energy and a compact support* $\mathcal{S}$. *There exists a constant* $F := F_{\alpha,\Sigma}$ *such that*

$$U(z) := U(\mu^*, z) := \int_\Sigma \log \frac{1}{|z-t|}\, d\mu^*(t) \le F - |z|^\alpha, \quad z \in \mathcal{S}, \tag{2.11}$$

and

$$U(z) \ge F - |z|^\alpha, \quad \text{q. e. } on\ \Sigma. \tag{2.12}$$

(b) Let $\mathcal{S}^*$ *denote the set* $\{z \in \Sigma : U(z) \le F - |z|^\alpha\}$. *If* $n \ge 0$ *is an integer,* $P \in \Pi_n$, *and*

$$|w^n(z)\, P(z)| \le 1, \quad z \in \mathcal{S}^*,$$

then

$$|P(z)| \le \exp\big(nF - nU(z)\big), \quad z \in \mathbf{C},$$

and, in particular,

$$|w^n(z)\, P(z)| \le 1, \quad z \in \Sigma.$$

(c) Let $\mathcal{S}^{**}$ *denote the (compact) convex hull of the set* $\mathcal{S}^*$. *There exists a sequence of polynomials* $P_n(z) = \prod_{k=1}^n (z - z_{k,n})$ *where* $z_{k,n} \in \mathcal{S}^{**}$, $k = 1, \ldots, n$, $n = 1, 2, \ldots,$ *with the following properties*

$$\lim_{n\to\infty} \|w^n(z)\, P_n(z)\|_\Sigma^{1/n} = \exp(-F), \tag{2.13}$$

$$\lim_{n\to\infty} |P_n(z)|^{1/n} = \exp\big(-U(z)\big), \quad z \in \mathbf{C} \setminus \Sigma, \tag{2.14}$$

uniformly on compact subsets of $\mathbf{C} \setminus \Sigma$.

(d) Let σ *be a natural measure on* Σ. *There exists a sequence of positive numbers* $\{N_n := N_n(\alpha, \sigma, \Sigma)\}$ *such that*

$$\lim_{n\to\infty} N_n^{1/n} = 1 \tag{2.15}$$

and, for $0 < p,\ r \le \infty$, *any integer* $n \ge 0$ *and* $P \in \Pi_n$,

$$\|w^n P\|_{r,\sigma,\Sigma} \le N_n^{|1/p - 1/r|} \|w^n P\|_{p,\sigma,\Sigma}. \tag{2.16}$$

Proposition 2.1 is developed in various papers. Parts (a) and (b) are proved in [5], [7], part (d) is proved in [4]. In part (c), we may choose P_n to be the extremal polynomials studied in [4], [5], [7], [8]. The fact that the zeros of P_n are in $\mathcal{S}^{**}$ follows easily from the extremal properties of P_n. We observe that if σ is a homogeneous, natural measure, then (2.16) can be reformulated as

$$\|wP\|_{r,\sigma,\Sigma} \le N_n'^{|1/p - 1/r|} \|wP\|_{p,\sigma,\Sigma} \tag{2.17}$$

for a suitable choice of constants N_n' which also satisfy

$$\lim_{n\to\infty} N_n'^{1/n} = 1.$$

Finally, we introduce a class of entire functions. If $f\colon \mathbf{C} \longrightarrow \mathbf{C}$ is entire and $r > 0$, we set

$$M(r,f) := \sup_{|z|\le r} |f(z)|. \tag{2.18}$$

We recall that the function f is said to be of type τ and order λ if

$$\limsup_{r\to\infty} \frac{\log M(r,f)}{r^\lambda} = \tau. \tag{2.19}$$

Let $0 < \tau < \infty$, $0 < \lambda < \alpha$, $q \ge 1$. We denote by $A_{q,\tau,\lambda}$ the class of all entire functions $f\colon \mathbf{C} \longrightarrow \mathbf{C}$ such that

$$|||f|||_{q,\tau,\lambda} := \left\{ \int_{\mathbf{C}} [\exp(-\tau|z|^\lambda)\,|f(z)|]^q \, dm_2(z) \right\}^{1/q} \le 1 \tag{2.20}$$

where m_2 denotes the two dimensional Lebesgue measure. Similarly, $A_{\infty,\tau,\lambda}$ denotes the class of all entire functions $f\colon \mathbf{C} \longrightarrow \mathbf{C}$ for which

$$|||f|||_{\infty,\tau,\lambda} := \sup_{r>0} \exp(-\tau r^\lambda)\, M(r,f) \le 1. \tag{2.21}$$

3. **Main theorems.** Our first theorem generalizes Theorem 2 in [3].

THEOREM 3.1. *Let $\alpha > 0$, $1 \le p \le \infty$, $0 < \lambda < \alpha$ and σ be a homogeneous, natural measure on Σ. Let $\mathcal{X}$ denote $L^p_{\sigma,\Sigma}$ (respectively $C_0(\Sigma)$ if $p = \infty$), $wf \in \mathcal{X}$ and*

$$\rho(f) := \rho(w, \mathcal{X}; f) := \limsup_{n\to\infty} \{ n^{1/\lambda - 1/\alpha} \epsilon_n^{1/n}(w, \mathcal{X}; f) \} < \infty. \tag{3.1}$$

Then f has an extension to the complex plane as an entire function of order λ and finite type τ given by

$$\rho(f) = (\tau\lambda)^{1/\lambda} \exp(1/\lambda - F). \tag{3.2}$$

Conversely, if f is the restriction to Σ of an entire function of order λ and finite type τ, then $wf \in \mathcal{X}$, and the quantity $\rho(f)$ defined in (3.1) satisfies (3.2).

When $\Sigma = \mathbf{R}$, the quantity F is given by (*cf.* [9])

$$F = F_\alpha = \frac{\log \lambda_\alpha}{\alpha} + \log 2 + \frac{1}{\alpha}. \tag{3.3}$$

where

$$\lambda_\alpha := \frac{\Gamma(\alpha)}{2^{\alpha-2}\{\Gamma(\alpha/2)\}^2}. \tag{3.4}$$

In particular, when $\alpha = 2$ and $\Sigma = \mathbf{R}$, $F = \log(2\sqrt{e})$ and Theorem 2 in [2] can be seen to be a special case of Theorem 3.1. We caution the reader that the notation in [2] is different.

Our next theorem examines the asymptotic behavior of the n-widths of the classes of entire functions defined near the end of Section 2.

THEOREM 3.2. *Let $\alpha > 0$, $1 \leq p, q \leq \infty$, $0 < \tau < \infty$, $0 < \lambda < \alpha$ and σ be a homogeneous, natural measure on Σ. Let $\mathcal{X}$ denote $L^p_{\sigma,\Sigma}$ (respectively $C_0(\Sigma)$ if $p = \infty$) and $A := A_{q,\tau,\lambda}$. We have*

(3.5)
$$\begin{aligned}\lim_{n\to\infty} n^{1/\lambda-1/\alpha} D_n(w,\mathcal{X};A)^{1/n} &= \lim_{n\to\infty} n^{1/\lambda-1/\alpha} d_n(w,\mathcal{X};A)^{1/n} \\ &= \lim_{n\to\infty} n^{1/\lambda-1/\alpha} b_n(w,\mathcal{X};A)^{1/n} = (\tau\lambda)^{1/\lambda}\exp\Big(\frac{1}{\lambda}-F\Big).\end{aligned}$$

The proof of the above theorems require a few lemmas. In the sequel, $\alpha, p, q, \sigma, \Sigma$ will be fixed as in Theorems 3.1 and 3.2. We assume that p, q are finite; the other cases are only simpler. The letters $c, c_1, \ldots$ will denote positive constants generally depending upon these and other parameters such as τ and λ, but independent of the integer n and the functions being approximated. Their values may be different at different occurrences, even within the same formula. We will also need the convention that M_n will denote positive numbers, not necessarily independent of the functions involved, but with the property that

$$\lim_{n\to\infty} M_n^{1/n} = 1. \tag{3.6}$$

The value of this symbol may be different at different occurrences, even within the same formula.

LEMMA 3.3. *Let $0 < \tau < \infty$, $0 < \lambda < \alpha$. If $n > 1$ is an integer, $P \in \Pi_n$ and*

$$\|wP\|_{p,\sigma,\Sigma} \leq 1 \tag{3.7}$$

then

$$|||P|||_{q,\tau,\lambda} \leq M_n \frac{n^{n/\lambda-n/\alpha}\exp(nF-n/\lambda)}{(\tau\lambda)^{n/\lambda}}. \tag{3.8}$$

PROOF. Using (2.17), (3.7) implies

$$|w(z)P(z)| \leq M_n, \quad z \in \Sigma,$$

and hence

$$|w^n(z)P(n^{1/\alpha}z)| \leq M_n, \quad z \in \Sigma. \tag{3.9}$$

In view of Proposition 2.1(b), we get

$$|P(n^{1/\alpha}z)| \leq M_n \exp\big(nF - nU(z)\big), \quad z \in \mathbf{C}. \tag{3.10}$$

Let, in this proof only, $R := \sup\{|z| : z \in \mathcal{S}^*\}$, and

$$\Lambda_n := \max_{r\geq 0}(r+R)^n \exp(-\tau n^{\lambda/\alpha} r^\lambda) =: (\rho_n + R)^n \exp(-\tau n^{\lambda/\alpha}\rho_n^\lambda). \tag{3.11}$$

Proposition 2.1(a) implies that the support of the measure μ^* is a subset of $\mathcal{S}^*$. Therefore,

$$U(z) \geq \log \frac{1}{|z| + R}, \quad z \in \mathbf{C},$$

and hence (3.10) implies that

$$|P(n^{1/\alpha} z)| \leq M_n(|z| + R)^n \exp(nF), \quad z \in \mathbf{C}. \tag{3.12}$$

It follows that

$$|||P|||_{\infty,\tau,\lambda} \leq M_n \exp(nF)\Lambda_n. \tag{3.13}$$

Since the two dimensional Lebesgue measure m_2 is a homogeneous, natural measure, we may apply (2.17) with λ instead of α and m_2 instead of σ to deduce that

$$|||P|||_{q,\tau,\lambda} \leq M_n \exp(nF)\Lambda_n. \tag{3.14}$$

It is elementary calculus to check that

$$\rho_n^{\lambda-1}(\rho_n + R) = n^{1-\lambda/\alpha}/\tau\lambda.$$

Hence, $\rho_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and, in fact,

$$\lim_{n\to\infty} \rho_n / n^{1/\lambda - 1/\alpha} = (\tau\lambda)^{1\lambda}$$

It follows that

$$\begin{aligned} \Lambda_n &= M_n \rho_n^n \exp(-\tau n^{\lambda/\alpha} \rho_n^\lambda) \\ &= M_n \frac{n^{n/\lambda - n/\alpha}}{(\tau\lambda)^{n/\lambda}} \exp(-n/\lambda). \end{aligned} \tag{3.15}$$

The estimate (3.8) follows from (3.14) and (3.15). ∎

LEMMA 3.4. *Let $f: \mathbf{C} \longrightarrow \mathbf{C}$ be an entire function, $\{L_n \in \Pi_{n-1}\}$ be a sequence of polynomials, N be a positive integer and c, C, μ be positive constants such that*

$$w(z)|f(z) - L_n(z)| \leq c\Big(\frac{C}{n^\mu}\Big), \quad z \in n^{1/\alpha}\mathcal{S}^*, \quad n = N, N+1, \ldots. \tag{3.16}$$

Then, for every p', $1 \leq p' < \infty$,

$$\|w(f - L_n)\|_{p',\sigma,\Sigma} \leq c\Big(\frac{C}{n^\mu}\Big)^n, \quad n = N, N+1, \ldots, \tag{3.17}$$

where C and μ are the same constants as in (3.16).

PROOF. In view of (3.16), we have

$$w(z)\,|L_{n+1}(z) - L_n(z)| \leq M_n\Big(\frac{C}{n^\mu}\Big)^n, \quad z \in n^{1/\alpha}\mathcal{S}^*, \quad n \geq N.$$

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Using a change of variables and Proposition 2.1(b), and taking into account that $L_{n+1} - L_n \in \Pi_n$, we get that

$$w(z)\,|L_{n+1}(z) - L_n(z)| \le M_n\Big(\frac{C}{n^\mu}\Big)^n, \quad z \in \Sigma, \quad n \ge N.$$

In view of (2.17), we get

$$\|w(L_{n+1} - L_n)\|_{p,\sigma,\Sigma} \le M_n\Big(\frac{C}{n^\mu}\Big)^n, \quad n \ge N. \tag{3.18}$$

Using Lemma 3.3 with a value of λ sufficiently close to α and $q = \infty$, it is easy to conclude that the sequence $\{L_n\}$ converges uniformly on compact subsets of $\mathbf{C}$. We observe that (3.16) implies that the sequence L_n converges to f uniformly on compact subsets of Σ. Therefore, $\{L_n\}$ converges to f uniformly on compact subsets of $\mathbf{C}$. Using (3.18), it is easy to verify that for $n \ge N$.

$$\sum_{k=n}^{\infty} \|w(L_{k+1} - L_k)\|_{p,\sigma,\Sigma} \le M_n\Big(\frac{C}{n^\mu}\Big)^n. \tag{3.19}$$

Hence, (3.19) shows that

$$f = L_n + \sum_{k=n}^{\infty}(L_{k+1} - L_k)$$

in the sense of weighted $L^p_{\sigma,\Sigma}$ convergence. We now get (3.17) by applying (3.19) again. ■

Our next lemma estimates the rate of convergence of the Lagrange interpolation polynomials at the points $\zeta_{k,n} := n^{1/\alpha} z_{k,n}$, where $z_{k,n}$ are introduced in Proposition 2.1(c). If $g\colon \mathbf{C} \longrightarrow \mathbf{C}$, $n \ge 1$ is an integer, we denote by $L_n(g,\cdot)$ the unique polynomial in Π_{n-1} that satisfies

$$L_n(g, \zeta_{k,n}) = g(\zeta_{k,n}), \quad k = 1, \ldots, n. \tag{3.20}$$

LEMMA 3.5. *Let* $0 < \tau < \infty$, $0 < \lambda < \alpha$, $q > 0$, *and* $f \in A_{q,\tau,\lambda}$. *Then for each* $p > 0$,

$$\limsup_{n\to\infty} n^{1/\lambda - 1/\alpha} \big\|w\big(f - L_n(f,\cdot)\big)\big\|_{p,\sigma,\Sigma}^{1/n} \le (\tau\lambda)^{1/\lambda} \exp(1/\lambda - F). \tag{3.21}$$

PROOF. For the purpose of this proof only, we let $Q_n(z) := P_n(z/n^\alpha)$, $r_n := (n/\tau\lambda)^{1/\lambda}$, $R := \max\{|z| : z \in \mathcal{S}^{**}\}$. Let $\epsilon > 0$ be arbitrarily fixed. Then, (2.13) implies that

$$\lim_{n\to\infty} \|wQ_n\|_\Sigma^{1/n} = \exp(-F). \tag{3.22}$$

Therefore, for sufficiently large integer n,

$$Rn^{1/\alpha}/r_n < \epsilon, \quad n^{1/\alpha}\int^{**} \subseteq \{|z| \le r_n/2\} \tag{3.23}$$

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and

$$\|wQ_n\|_\Sigma \le (1+\epsilon)^n \exp(-nF). \tag{3.24}$$

Moreover, for $|z| \ge R(1-\epsilon)^{-1} n^{1/\alpha}$, we have

$$|Q_n(z)| \ge \prod_{k=1}^{n} \left(\frac{|z|}{n^{1/\alpha}} - R \right) \ge (1-\epsilon)^n \left(\frac{|z|}{n^{1/\alpha}} \right)^n. \tag{3.25}$$

Next, we recall the Hermite formula for interpolation ([13], p. 50), to observe that for any $z \in \Sigma$, $|z| < r_n/2$ and any $r \ge r_n(1-1/n)$,

$$f(z) - L_n(f,z) = \frac{Q_n(z)}{2\pi i} \int_{|\zeta|=r} \frac{f(\zeta)\, d\zeta}{Q_n(\zeta)(\zeta - z)}. \tag{3.26}$$

In view of (3.26) and (3.25), for $z \in \Sigma$, $|z| \le r_n/2$ and $r \ge r_n(1-1/n)$,

$$\begin{aligned} |f(z) - L_n(f,z)| &\le \frac{|Q_n(z)|}{2\pi} \int_{|\zeta|=r} \frac{|f(\zeta)|\,|d\zeta|}{|Q_n(\zeta)|\,|\zeta - z|} \\ &\le c|Q_n(z)| \left\{ \frac{r(1-\epsilon)}{n^{1/\alpha}} \right\}^{-n} \int_0^{2\pi} |f(re^{i\theta})|\, d\theta. \end{aligned} \tag{3.27}$$

We integrate both sides of (3.27) with respect to rdr from $r = r_n(1-1/n)$ to $r = r_n$ and get

$$\frac{r_n^2}{n} |f(z) - L_n(f,z)| \le c|Q_n(z)| \left\{ \frac{r_n(1-\epsilon)}{n^{1/\alpha}} \right\}^{-n} \int_{r_n(1-1/n) \le \zeta \le r_n} |f(\zeta)|\, dm_2(\zeta).$$

Next, we use Hölder's inequality and take into account the fact that $f \in A_{q,\tau,\lambda}$ and simplify to get for $z \in \Sigma$, $|z| \le r_n/2$,

$$|f(z) - L_n(f,z)| \le c|Q_n(z)| \left\{ \frac{r_n(1-\epsilon)}{n^{1/\alpha}} \right\}^{-n} \left(\frac{r_n^2}{n} \right)^{-1/q} \exp(n/\lambda). \tag{3.28}$$

In view of (3.24), (3.28) implies, for $z \in \Sigma$, $|z| \le r_n/2$, that

$$w(z)|f(z) - L_n(f,z)| \le c \left(\frac{r_n^2}{n} \right)^{-1/q} \left(\frac{1+\epsilon}{1-\epsilon} \right)^n \exp\left(\frac{n}{\lambda} - nF \right) \left(\frac{r_n}{n^{1/\alpha}} \right)^{-n}.$$

In view of (3.23) and the fact that $(r_n^2/n)^{1/n} \longrightarrow 1$ as $n \longrightarrow \infty$, we get, for sufficiently large values of n and for $z \in n^{1/\alpha} S^*$,

$$w(z)\,|f(z) - L_n(f,z)| \le c \left(\frac{C}{n^\mu} \right)^n \tag{3.29}$$

where $\mu := 1/\lambda - 1/\alpha$ and

$$C := \left(\frac{1+\epsilon}{1-\epsilon} \right)^2 (\tau\lambda)^{1/\lambda} \exp(1/\lambda - F).$$

The estimate (3.21) now follows using Lemma 3.4. ∎

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PROOF OF THEOREM 3.1. Let $\rho(f) < r < \infty$ and t be any number which satisfies

$$(t\lambda)^{1/\lambda} > r\exp(F - 1/\lambda). \tag{3.30}$$

Then there exists a sequence of polynomials $P_n \in \Pi_n$ such that

$$f = P_0 + \sum_{n=1}^{\infty} P_n \tag{3.31}$$

in the sense of convergence in $\mathcal{X}$ and

$$\|wP_n\|_{\mathcal{X}} \leq c\epsilon_n(w, \mathcal{X}; f) \leq c\Big(\frac{r}{n^{1/\lambda - 1/\alpha}}\Big)^n, \quad n \geq 1. \tag{3.32}$$

In view of Lemma 3.3 applied with t in place of τ, we get

$$|||P_n|||_{\infty,t,\lambda} \leq c\left(\frac{r\exp(F - 1/\lambda)}{(t\lambda)^{1/\lambda}}\right)^n, \quad n \geq 1. \tag{3.33}$$

In particular, (3.30) implies that the series $P_0 + \sum P_n$ converges uniformly on compact subsets of $\mathbf{C}$. In view of (3.31), the function f may be modified σ-almost everywhere so that this sum is an extension to $\mathbf{C}$ of the function so modified. Thus, we may denote this extension by f. The extended function f is then an entire function and (3.33) implies that

$$M(R,f) \leq c\exp(-tR^{\lambda}), \quad R > 0.$$

Since t was an arbitrary number which satisfies (3.30), it follows that f is of order λ and type τ given by (3.2). The converse statement is an immediate consequence of Lemma 3.5. ∎

PROOF OF THEOREM 3.2. Lemma 3.3 shows that

$$\liminf_{n\to\infty} n^{1/\lambda - 1/\alpha} b_n(w, \mathcal{X}; A)^{1/n} \geq (\tau\lambda)^{1/\lambda}\exp(1/\lambda - F). \tag{3.34}$$

Lemma 3.5 gives that

$$\limsup_{n\to\infty} d_n(w, \mathcal{X}; A)^{1/n} \leq (\tau\lambda)^{1/\lambda}\exp(1/\lambda - F) \tag{3.35}$$

and

$$\limsup_{n\to\infty} D_n(w, \mathcal{X}; A)^{1/n} \leq (\tau\lambda)^{1/\lambda}\exp(1/\lambda - F) \tag{3.36}$$

The theorem follows from (3.34), (3.35), (3.36), (2.4) and (2.6). ∎

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