Canad. Math. Bull. Vol. **61** (3), 2018 pp. 509–517
http://dx.doi.org/10.4153/CMB-2017-053-3

Area Integral Means of Analytic Functions in the Unit Disk

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Abstract. For an analytic function f on the unit disk $\mathbb{D}$, we show that the L^2 integral mean of f on $c < |z| < r$ with respect to the weighted area measure $(1-|z|^2)^\alpha\, dA(z)$ is a logarithmically convex function of r on $(c,1)$, where $-3 \le \alpha \le 0$ and $c \in [0,1)$. Moreover, the range $[-3,0]$ for α is best possible. When $c = 0$, our arguments here also simplify the proof for several results we obtained in earlier papers.

1 Introduction

Let $H(\mathbb{D})$ denote the space of all analytic functions in the unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$. For any $f \in H(\mathbb{D})$ and $0 < p < \infty$, the classical integral means of f are defined by

$$M_p(f,r) = \frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p\, d\theta, \qquad 0 \le r < 1.$$

These integral means play a prominent role in classical analysis, especially in the theory of Hardy spaces. For example, the well-known Hardy convexity theorem asserts that $M_p(f,r)$, as a function of r on $(0,1)$, is logarithmically convex. Logarithmic convexity here means that the function $r \mapsto \log M_p(f,r)$ is convex in $\log r$. See [1] for an example.

In [9] Xiao and Zhu initiated the study of area integral means of f with respect to a family of weighted area measures on the unit disk. More specifically, for any real α we consider the measure

$$dA_\alpha(z) = (1-|z|^2)^\alpha\, dA(z),$$

where dA is the area measure on $\mathbb{D}$. Such measures are frequently used in the more recent theory of Bergman spaces; see [2,3]. For $f \in H(\mathbb{D})$ and $0 < p < \infty$, we consider the area integral means

$$M_{p,\alpha}(f,r) = \frac{\int_{|z|<r} |f(z)|^p\, dA_\alpha(z)}{\int_{|z|<r} dA_\alpha(z)}, \qquad 0 < r < 1.$$

Received by the editors March 30, 2017; revised July 17, 2017.
Published electronically January 17, 2018.
Research of K. Zhu is partially supported by the China National Natural Science Foundation (Grant Number 11720101003), the Project of International Science and Technology Cooperation Innovation Platform in Universities in Guangdong Province (Grant Number 2014KGJHZ007), and Shantou University (Grant Number NTF17009). Research of C. Wang is supported by the China National Natural Science Foundation (Grant Number 11671357).
AMS subject classification: **30H10**, 30H20.
Keywords: logarithmic convexity, area integral mean, Bergman space, Hardy space.

It was shown in [6, 7] that, just like the classical integral means, $M_{p,\alpha}(f,r)$ is also logarithmically convex on $(0,1)$ when $-2 \le \alpha \le 0$. Furthermore, if $p = 2$, then $M_{2,\alpha}(f,r)$ is logarithmically convex on $(0,1)$ when $-3 \le \alpha \le 0$, and this range for α is best possible. Despite the elegance of these results, the proofs in [6, 7] were very long and laborious. In a few instances we even had to use the computer algebra system Maple to help us with the computations.

In this paper we will present a new approach to the logarithmic convexity problem above, which not only yields a much-simplified proof for several results in [4–7] but also generalizes some of these results to the case where the area integral means are taken over the annuli $c < |z| < r$, where $c \in [0,1)$ is fixed. Our main result is the following theorem.

Main Theorem *Suppose $0 \le c < 1$, $-3 \le \alpha \le 0$, $p = 2$, and $f \in H(\mathbb{D})$. Then the function*

$$r \mapsto M_{p,\alpha,c}(f,r) =: \frac{\int_{c<|z|<r} |f(z)|^p (1-|z|^2)^\alpha \, dA(z)}{\int_{c<|z|<r} (1-|z|^2)^\alpha \, dA(z)}$$

is logarithmically convex for $r \in (c,1)$. Furthermore, the range $[-3,0]$ for the weight parameter α is best possible.

Here we assume that f is analytic in the entire unit disk $\mathbb{D}$, although the integral means are taken over the annuli $c < |z| < r$, $r \in (c,1)$. In fact, we will give an example to show that the result above is false if the function f is only analytic on the annulus $c < |z| < 1$. See [4, 5, 8] for recent work about integral means of analytic functions on the complex plane with respect to Gaussian measures.

Throughout the paper we use the symbol =: whenever a new notation is being introduced. We will use the notation $A \sim B$ to mean that A and B have the same sign, which is different from the meaning of this notation in most papers in the literature. Since our main concern will be the sign of various quantities, this new use of $A \sim B$ will significantly simplify our presentation.

2 Preliminaries

In this section we collect several preliminary results that will be needed for the proof of our main theorem.

Lemmas 2.1, 2.2, and 2.3 were stated and proved in [6, 7] for the interval $(0,1)$. But it is clear that the conclusions still hold if $(0,1)$ is replaced by any interval (a,b), where $0 \le a < b < \infty$.

Lemma 2.1 *Suppose that f is positive and twice differentiable on (a,b). Then*

(i) *$f(x)$ is convex in $\log x$ if and only if $f(x^2)$ is convex in $\log x$;*

(ii) *$\log f(x)$ is convex in $\log x$ if and only if*

$$D\big(f(x)\big) =: \frac{f'(x)}{f(x)} + x\frac{f''(x)}{f(x)} - x\Big(\frac{f'(x)}{f(x)}\Big)^2 \ge 0$$

for all $x \in (a,b)$.

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Lemma 2.2 *Suppose that $f = f_1/f_2$ is a quotient of two positive and twice differentiable functions on (a,b). Then*

$$D(f(x)) = D(f_1(x)) - D(f_2(x))$$

for $x \in (a,b)$. Consequently, $\log f(x)$ is convex in $\log x$ if and only if

$$D(f_1(x)) - D(f_2(x)) \geq 0$$

on (a,b).

Lemma 2.3 *Suppose $\{h_k(x)\}$ is a sequence of positive and twice differentiable functions on (a,b) such that the function $H(x) = \sum_{k=0}^{\infty} h_k(x)$ is also twice differentiable on (a,b). If for each k the function $\log h_k(x)$ is convex in $\log x$ for $x \in (a,b)$, then $\log H(x)$ is also convex in $\log x$ for $x \in (a,b)$.*

For the remainder of the paper we will fix a constant $c \in [0,1)$ and consider weighted area integral means $M_{p,\alpha,c}(f,r)$ of analytic functions for $r \in (c,1)$. Since $M_{p,\alpha,c}(f,r)$ is a quotient of two positive and twice-differentiable functions, Lemma 2.2 tells us that the logarithmic convexity of $M_{p,\alpha,c}(f,r)$ means the numerator of $M_{p,\alpha,c}(f,r)$ must "dominate" the denominator after we apply the second-order (non-linear) differential operator D defined in Lemma 2.1.

We begin with the denominator, which, by polar coordinates, equals

$$2\pi \int_c^r (1-t^2)^\alpha t\, dt = \pi \int_{c^2}^{r^2} (1-t)^\alpha\, dt.$$

To simplify notation, we let $x = r^2$, $x_0 = c^2$, and

$$\varphi = \varphi(x) = \int_{x_0}^{x} \varphi'(t)\, dt, \qquad \varphi' = \varphi'(x) = (1-x)^\alpha,$$

where $x_0 < x < 1$. By Lemma 2.1, we can work with the variable x on the interval $(x_0, 1)$ instead of r on the interval $(c,1)$. Let us also write

$$D = D(\varphi(x)) = \frac{\varphi'}{\varphi}\Big(1 + x\frac{\varphi''}{\varphi'} - x\frac{\varphi'}{\varphi}\Big),$$

and

$$D_1 = D(\varphi'(x)) = \frac{\varphi''}{\varphi'}\Big(1 + x\frac{\varphi'''}{\varphi''} - x\frac{\varphi''}{\varphi'}\Big) = \frac{-\alpha}{(1-x)^2}. \tag{2.1}$$

It is elementary to check that the derivative of D can be written as

$$\begin{aligned} D' &= \Big(\frac{\varphi''}{\varphi'} - \frac{\varphi'}{\varphi}\Big)\frac{\varphi'}{\varphi}\cdot\Big(1 + x\frac{\varphi''}{\varphi'} - x\frac{\varphi'}{\varphi}\Big) + \frac{\varphi'}{\varphi}(D_1 - D) \\ &= \frac{\varphi''}{\varphi'}D + \frac{\varphi'}{\varphi}(D_1 - 2D). \end{aligned} \tag{2.2}$$

A key step for us in the proof of the main theorem is the following estimates for D and D_1.

Lemma 2.4 *For $x \in (x_0, 1)$, we have $D_1 \geq 2D$ when $-2 \leq \alpha < 0$ and $D_1 \geq \alpha D/(\alpha+1)$ when $\alpha < -2$.*

Proof Let σ be a positive constant. Then $D_1 - \sigma D \geq 0$ is equivalent to $g(x) \geq 0$, where

$$g(x) = \frac{\varphi^2}{\varphi'}(D_1 - \sigma D) = \frac{\varphi^2}{\varphi'}D_1 - \sigma\varphi\Big(1 + x\frac{\varphi''}{\varphi'}\Big) + \sigma x\varphi'.$$

Since D_1 and φ are both positive for $\alpha < 0$, we have

$$g'(x) = D_1'\frac{\varphi^2}{\varphi'} + 2\varphi D_1 - D_1\frac{\varphi^2\varphi''}{(\varphi')^2} - \sigma\varphi D_1 \sim \frac{\varphi}{\varphi'}\Big(\frac{D_1'}{D_1} - \frac{\varphi''}{\varphi'}\Big) + 2 - \sigma.$$

From $D_1 = -\alpha/(1-x)^2$ and $\varphi' = (1-x)^\alpha$ we deduce that

$$\frac{D_1'}{D_1} - \frac{\varphi''}{\varphi'} = \frac{\alpha+2}{1-x}.$$

Thus,

$$\begin{aligned} g'(x) &\sim \frac{\alpha+2}{1-x}\frac{\varphi}{\varphi'} + 2 - \sigma \sim (\alpha+2)\varphi + (2-\sigma)(1-x)^{\alpha+1} \\ &= [(\alpha+2) - (2-\sigma)(\alpha+1)]\varphi + (2-\sigma)(1-x_0)^{\alpha+1} \end{aligned}$$

for $\alpha < 0$.

If $-2 \leq \alpha < 0$ and $\sigma = 2$, we clearly have $g'(x) \geq 0$. Similarly, if $\alpha < -2$ and $\sigma = \alpha/(\alpha+1)$, we also have $g'(x) \geq 0$. In both cases, we then have $g(x) \geq g(x_0) = \sigma x_0(1-x_0)^\alpha \geq 0$. ∎

3 Proof of the Main Theorem

By Lemma 2.3, we can reduce the proof of the main theorem to the case of (non-constant) monomials. Thus, we consider the function

$$M(x) = M_2(z^n, \sqrt{x}) = x^n,$$

where n is a positive integer. Mimicking what we did for the denominator of $M_{p,\alpha,c}(f,r)$ in the previous section, we consider the numerator of $M_{2,\alpha,c}(f,r)$ in the special case of a monomial and write it as

$$h = h(x) = \int_{x_0}^{x} M(t)\varphi'(t)dt, \qquad x \in (x_0, 1).$$

We easily verify that

$$h' = M\varphi', \quad h'' = M\varphi'' + M'\varphi',$$

and

$$\frac{h''}{h'} = \frac{M'}{M} + \frac{\varphi''}{\varphi'}. \tag{3.1}$$

Lemma 3.1 *Suppose $\alpha < 0$, $n > 0$, and $x \in (x_0, 1)$. Then*

$$x\frac{h'}{h} - x\frac{\varphi'}{\varphi} - \frac{1}{2}n \geq 0.$$

If we also have $-(n+2) \leq \alpha < -2$, then

$$x\frac{h'}{h} - x\frac{\varphi'}{\varphi} - \frac{\alpha+1}{\alpha+2}n \leq 0.$$

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Proof Let σ be a positive constant and consider the function

$$G(x) = \frac{xh'}{x\frac{\varphi'}{\varphi} + \sigma n} - h.$$

It follows from direct computations that

$$\begin{aligned} G'(x) &= \frac{h' + xh''}{x\frac{\varphi'}{\varphi} + \sigma n} - \frac{xh'D}{\left(x\frac{\varphi'}{\varphi} + \sigma n\right)^2} - h' \\ &\sim \left(1 + x\frac{h''}{h'}\right)\left(x\frac{\varphi'}{\varphi} + \sigma n\right) - xD - \left(x\frac{\varphi'}{\varphi} + \sigma n\right)^2 \\ &= \left(1 + x\frac{\varphi''}{\varphi'} - x\frac{\varphi'}{\varphi} + (1-\sigma)n\right)\left(x\frac{\varphi'}{\varphi} + \sigma n\right) - xD \\ &\sim 1 + (1-\sigma)n + x\frac{\varphi''}{\varphi'} + \left(\frac{1}{\sigma} - 2\right)x\frac{\varphi'}{\varphi} =: G_1(x). \end{aligned}$$

For $\alpha < 0$ and $\sigma = 1/2$, we clearly have

$$G'(x) \sim G_1(x) = 1 + \frac{n}{2} - \frac{\alpha x}{1-x} \geq 0,$$

which gives $G(x) \geq G(x_0) = 0$, or

$$x\frac{h'}{h} - x\frac{\varphi'}{\varphi} - \frac{1}{2}n \geq 0.$$

If $-(n+2) \leq \alpha < -2$ and $\sigma = (\alpha+1)/(\alpha+2)$, we have

$$\begin{aligned} G_1(x) &= \frac{n+\alpha+2}{\alpha+2} + x\frac{\varphi''}{\varphi'} - \frac{\alpha}{\alpha+1}x\frac{\varphi'}{\varphi} \\ &\leq x\frac{\varphi''}{\varphi'} - \frac{\alpha}{\alpha+1}x\frac{\varphi'}{\varphi} \sim -(\alpha+1)\varphi - (1-x)^{\alpha+1} \\ &= -(1-x_0)^{\alpha+1} \leq 0. \end{aligned}$$

Thus, $G'(x) \leq 0$ and $G(x) \leq G(x_0) = 0$, or

$$x\frac{h'}{h} - x\frac{\varphi'}{\varphi} - \frac{\alpha+1}{\alpha+2}n \leq 0.$$

This completes the proof of the lemma. ■

We can now prove the main result of the paper, which we restate as follows.

Theorem 3.2 *Suppose $0 \leq c < 1$, $-3 \leq \alpha \leq 0$, and $f \in H(\mathbb{D})$. Then the function $M_{2,\alpha,c}(f,r)$ is logarithmically convex for $r \in (c,1)$. Furthermore, the range $[-3,0]$ for the weight parameter α is best possible.*

Proof By Lemma 2.3, we just need to consider the case where $f(z) = z^n$ is a non-constant monomial. Also, by Lemma 2.1, we can use the variable $x = r^2$ instead of r. Thus, the proof will be completed if we can prove the logarithmic convexity of the function $h(x)/\varphi(x)$, where h and φ are the functions used in Lemmas 2.4 and 3.1.

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This will be accomplished, according to Lemma 2.2, if we can show that $\Delta(x) \geq 0$ for $x \in (x_0, 1)$, where

$$\begin{aligned}\Delta(x) &= D(h(x)) - D(\varphi(x))\\ &= \frac{h'}{h} + x\frac{h''}{h} - x\Big(\frac{h'}{h}\Big)^2 - \Big[\frac{\varphi'}{\varphi} + x\frac{\varphi''}{\varphi} - x\Big(\frac{\varphi'}{\varphi}\Big)^2\Big]\\ &\sim h\Big(1 + x\frac{h''}{h'}\Big) - xh' - \frac{h^2}{h'}D =: \delta(x).\end{aligned}$$

It follows from direct computations that

$$\begin{aligned}\delta'(x) &= h\Big(\frac{h''}{h'} + x\frac{h'''}{h'} - x\Big(\frac{h''}{h'}\Big)^2\Big) - 2hD - h^2\Big(\frac{D'}{h'} - \frac{h''D}{(h')^2}\Big)\\ &= hD(h') - 2hD - h^2\Big(\frac{D'}{h'} - \frac{h''D}{(h')^2}\Big).\end{aligned}$$

Since $h' = M\varphi'$, it follows from Lemma 2.2 that

$$D(h') = D(M) + D(\varphi').$$

It is easy to check that $D(M) = 0$, so $D(h') = D(\varphi') = D_1$. Thus,

$$\delta'(x) = h(D_1 - 2D) - \Big(\frac{D'}{h'} - \frac{h''D}{(h')^2}\Big)h^2 \sim x\frac{h'}{h}(D_1 - 2D) - x\Big(D' - \frac{h''}{h'}D\Big).$$

Combining this with (2.2) and (3.1), we obtain

$$\delta'(x) \sim \Big(x\frac{h'}{h} - x\frac{\varphi'}{\varphi}\Big)(D_1 - 2D) + x\frac{M'}{M}D. \tag{3.2}$$

Since $xM'/M = n$, we have

$$\begin{aligned}\delta'(x) &\sim \Big(x\frac{h'}{h} - x\frac{\varphi'}{\varphi}\Big)(D_1 - 2D) + nD\\ &= \Big(x\frac{h'}{h} - x\frac{\varphi'}{\varphi} - \frac{1}{2}n\Big)(D_1 - 2D) + \frac{1}{2}nD_1.\end{aligned}$$

By (2.1), $D_1 > 0$ for $\alpha < 0$. Therefore,

$$\delta'(x) \sim \Big(x\frac{h'}{h} - x\frac{\varphi'}{\varphi} - \frac{1}{2}n\Big)\Big(1 - 2\frac{D}{D_1}\Big) + \frac{1}{2}n =: \delta_1(x).$$

It is easy to check that

$$\lim_{x\to x_0}\frac{h}{\varphi} = M(x_0) = x_0^n.$$

By the definition of $\delta(x)$ and D, we have

$$\delta(x) = h\Big(1 + x\frac{h''}{h'}\Big) - xh' - \Big(\frac{h}{\varphi}\Big)^2\frac{\varphi'}{h'}\Big(\varphi + \frac{x\varphi\varphi''}{\varphi'} - x\varphi'\Big).$$

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Since

$$\lim_{x\to x_0} h\Big(1 + x\frac{h''}{h'}\Big) = 0,$$
$$\lim_{x\to x_0} \Big(\frac{h}{\varphi}\Big)^2 \frac{\varphi'}{h'}\Big(\varphi + \frac{x\varphi\varphi''}{\varphi'}\Big) = 0,$$

we obtain

$$\delta(x_0) = \lim_{x\to x_0}\Big(-xh' + \Big(\frac{h}{\varphi}\Big)^2 \frac{x(\varphi')^2}{h'}\Big)$$
$$= -x_0^{n+1}(1-x_0)^\alpha + x_0^{2n}\frac{x_0(1-x_0)^{2\alpha}}{x_0^n(1-x_0)^\alpha} = 0$$

whenever $x_0 = c^2 > 0$. It is easy to see that $\delta(x_0) = 0$ is valid for $x_0 = 0$ as well.

If $-2 \le \alpha < 0$, it follows from Lemmas 2.4 and 3.1 that $\delta_1(x) \ge 0$. So $\delta'(x) \ge 0$, $\delta(x) \ge \delta(x_0) = 0$, and $\Delta(x) \ge 0$.

If $-3 \le \alpha < -2$, it follows from Lemmas 2.4 and 3.1 that

$$\delta_1 \ge \Big(1 - 2\frac{\alpha+1}{\alpha}\Big)\Big(x\frac{h'}{h} - x\frac{\varphi'}{\varphi} - \frac{1}{2}n\Big) + \frac{1}{2}n$$
$$= -\frac{\alpha+2}{\alpha}\Big(x\frac{h'}{h} - x\frac{\varphi'}{\varphi} - \frac{\alpha+1}{\alpha+2}n\Big) \ge 0.$$

Hence $\delta'(x) \ge 0$, $\delta(x) \ge \delta(x_0) = 0$, and $\Delta(x) \ge 0$.

Finally, suppose that $\alpha \notin [-3, 0]$. It follows from the proof in [7] that $\Delta(x) < 0$ for certain monomial z^k and x sufficiently close to 1. This shows that the range $[-3, 0]$ for α is best possible. ∎

4 Further Remarks

Note that our main theorem is proved under the assumption that the function f is analytic on the whole unit disk $\mathbb{D}$. It is natural to ask if the result remains true when the function f is only analytic on the annulus $c < |z| < 1$. We show by an example that the answer is negative.

Consider the case where $\alpha = -1$, $f(z) = \frac{1}{z}$, and $c > 0$. It follows from a direct computation that

$$M_{2,-1,c}\Big(\frac{1}{z}, r\Big) = \frac{\log\frac{1-c^2}{1-r^2} + \log\frac{r^2}{c^2}}{\log\frac{1-c^2}{1-r^2}}.$$

As before, we let $x_0 = c^2$ and $x = r^2$. Then by Lemma 2.2, we just need to consider the logarithmical convexity of the function

$$H(x) = \frac{\log\frac{1-x_0}{1-x} + \log\frac{x}{x_0}}{\log\frac{1-x_0}{1-x}}.$$

By direct computations, $D(H(x))$ is the product of

$$\frac{1}{x(1-x)^2\Big(\log\frac{x}{x_0} + \log\frac{1-x_0}{1-x}\Big)^2} \quad\text{and}\quad \Big(x + \frac{x\log\frac{x}{x_0}}{\log\frac{1-x_0}{1-x}}\Big)\Big(x + \frac{x\log\frac{x}{x_0}}{\log\frac{1-x_0}{1-x}} - \log\frac{x}{x_0}\Big) - 1.$$

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We denote the last expression above by $g(x)$, which shares the same sign as $D(H(x))$, and observe that

$$\lim_{x\to 1} g(x) = -\log\frac{1}{x_0} < 0.$$

This implies that $\log H(x)$ is not convex in $\log x$ for $x \in (c,1)$.

Our analysis in the previous two sections is perfectly fine if $x_0 = c^2 = 0$. Thus, we obtain the main result in [7] as a special case.

Our approach here also yields a new and easier proof for the main result in [6], although it does not generalize the main theorem in [6] to the case of integral means over annuli $c < |z| < r$ when $c > 0$. In fact, if $c = 0$ and $-2 \le \alpha \le 0$, then $D_1 - 2D \ge 0$ and

$$D = D(\varphi(x)) = \frac{(\varphi - x)\varphi'}{(1-x)\varphi^2} \ge 0.$$

Furthermore, for the more general

$$M = M(x) = \frac{1}{2\pi}\int_0^{2\pi} |f(\sqrt{x}\,e^{i\theta})|^p\,d\theta$$

and

$$h = h(x) = \int_0^x M(t)\varphi'(t)\,dt,$$

we have

$$\frac{h'}{h} - \frac{\varphi'}{\varphi} = \frac{\varphi'}{h\varphi}(M\varphi - h) = \frac{\varphi'}{h\varphi}\Big(\int_0^x (M(x) - M(t))\varphi'(t)dt\Big).$$

Since $M(x)$ and $M'(x)$ are both positive (so $M(x)$ is increasing on $(0,1)$), it follows from these facts and (3.2) that $\delta'(x) > 0$ on $(0,1)$. It can be checked that $\delta(0) = 0$ in the general case as well. Thus, $\delta(x) > \delta(0) = 0$ for all $x \in (0,1)$. This implies that $\Delta(x) > 0$ on $(0,1)$, which proves that $M_{p,\alpha}(f,r) = M_{p,\alpha,0}(f,r)$ is logarithmically convex for all $0 < p < \infty$ and $-2 \le \alpha \le 0$.

Finally, we mention that the new method introduced in the paper can also be used to simplify some of the arguments in [4, 5]. In particular, the proof of [5, Theorem 1(ii)] can be simplified. Details are omitted here.

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