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QUADRATIC FUNCTIONALS OF nth ORDER

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1. **Introduction.** It is well known that disconjugacy theorems for self-adjoint differential equations are very closely related to the study of the positivity of certain associated quadratic functionals. In this paper, this relationship is closely examined for quadratic functionals of nth order associated with self-adjoint differential equations of order $2n$. The motivation of this work comes from a recent paper of W. Leighton [8] on quadratic functionals of second order. It is shown that the method used by W. Leighton extends in a straightforward manner to the study of quadratic functionals of nth order, provided one makes use of an identity due to G. Cimmino [1]. Several authors have considered related problems. In particular we wish to mention the recent papers of G. Ladas [6], W. Simons [13], D. B. Hinton [5]. The book by C. A. Swanson [14] also contains many other related results. As a consequence of our results we can obtain several known and some new criteria for oscillation of $2n$th order self-adjoint differential equations.

2. **Basic assumptions.** The material given in this section and the following one is standard, but is included here for the sake of completeness. We will be mainly concerned with the problem of minimization of quadratic functionals of the form

$$J[y; a, b] = \int_a^b \left[\sum_{k=0}^{n} p_k(x)(y^{(k)})^2 \right] dx \tag{1}$$

where the functions $p_k(x)$ are of class $C^k[a, b]$ and $p_n(x) > 0$. $y^{(k)}$ denotes the kth derivative of the function y. The quadratic functional $J[y; a, b]$ will be recognized as the form of second variation of a functional

$$\int_a^b f(x, y, y', \ldots, y^{(n)})\, dx$$

and in particular is its own second variation if the variable y is replaced by a suitable variation η. A function y will be said to be admissible if it is of class $C^{n-1}[a, b]$ and if the interval can be divided into a finite number of subintervals on the closure of each of which y is of class C^n. Our problem, then, is to study the problem of minimizing the functional $J[y; a, b]$ among the admissible

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functions such that

$$y^{(k)}(a)=\alpha_k \qquad y^{(k)}(b)=\beta_k \qquad k=0,1,2,\ldots,n-1 \tag{2}$$

where α_k, β_k are arbitrary constants. It can be shown by using classical techniques of calculus of variations that the corresponding Euler equation is given by

$$Lu=\sum_{k=0}^{n}(-1)^k\frac{d^k}{dx^k}\left(p_k(x)\frac{d^ky}{dx^k}\right)=0. \tag{3}$$

3. **Definition of conjugate points.** The definitions given in this section can also be found in [6].

Let $\sigma_i(x; a)$, $i=1,2,\ldots,n$ denote n linearly independent solutions of

$$L\sigma_i=0 \qquad \sigma_i^{(k)}(a)=0 \qquad \begin{matrix} i=1,2,\ldots,n \\ k=0,1,\ldots,n-1. \end{matrix} \tag{4}$$

Let $W[\sigma_1,\sigma_2,\ldots,\sigma_n]$ denote the subwronskian whose jth row and ith column is $\sigma_i^{(j)}(t; a)$. If $y_i(t; a)$, $i=1,2,\ldots,n$ is any other system of n linearly independent solutions of (3), then

$$y_i=\sum_{j=1}^{n}\alpha_{ij}\sigma_j$$

where $\det(\alpha_{ij})\neq 0$, so that

$$\det(y_i^{(j)}(x; a))=\det(\alpha_i^{(j)}(x; a))\cdot\det\alpha_{ij}.$$

Therefore, we can associate with (3) the determinant $W[\sigma_1,\sigma_2,\ldots,\sigma_n]$ which is unique up to a nonzero constant multiple.

Lemma 6. *$W[\sigma_1,\sigma_2,\ldots,\sigma_n]$ vanishes at $x=x_1$, $x_1\neq a$ if and only if there exists a solution of* (3) *having n-fold zeros at both points, a and x_1.*

Definition. The conjugate point of $x=a$, if it exists, is the smallest zero $W[\sigma_1,\sigma_2,\ldots,\sigma_n]$ greater than a.

Next, using integration by parts, we note that

$$\begin{aligned} J[y; a, b]&=\left[\sum_{k=1}^{n}\left(\sum_{r=0}^{k-1}(-1)^r(p_ky^{(k)})^{(r)}y^{(k-r-1)}\right)\right]_a^b+\int_a^b yLy\,dx \\ &=B_y[a, b]+\int_a^b yLy\,dx. \end{aligned} \tag{5}$$

The second variation of J may easily be shown to be

$$J_2=\int_a^b\sum_{k=0}^{n}p_k(x)(\eta^{(k)})^2\,dx$$

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where η is admissible and $\eta^{(k)}(a)=\eta^{(k)}(b)=0$ for $k=0,1,2,\ldots,n-1$. From the calculus of variations, it is known that a necessary condition the $J[y;a,b]$ possess a minimum among admissible function under fixed end point conditions is that $J_2 \geq 0$ for all admissible variations $\eta(x)$. The above considerations and standard techniques of the calculus of variations give the following theorem.

THEOREM 1. *If $J[y;a,b]$ has a minimum among admissible functions y, it is necessary that the interval $[a,b)$ contain no point conjugate to $x=a$.*

4. **Cimmino-Picone identity and sufficient conditions.** The above mentioned identity, proved in [1] is useful in deriving sufficient conditions for J_2 to be positive. We begin with the identity.

LEMMA 1 [1]. *If $\sigma_1, \sigma_2, \ldots, \sigma_n$ is a system of n linearly independent solutions of* (3) *and $W[\sigma_1, \sigma_2, \ldots, \sigma_n] \neq 0$ on (a,b) then for any admissible function y we have the identity*

$$\sum_{k=0}^{n} p_k(x)(y^{(k)})^2 - p_n(x)\left\{\frac{W[y,\sigma_1,\sigma_2,\ldots,\sigma_n]}{W[\sigma_1,\sigma_2,\ldots,\sigma_n]}\right\}^2 = \frac{dA}{dx} \tag{6}$$

where $A(x)$ is a rational expression involving $\sigma_i(x)$, $y^{(i)}(x)$ and $p_i(x)$.

An explicit representation for A can be found in [1], where the identity is proved. The important fact, observed by G. Cimmino [1], is to note that $A(x_i)=0$ if $y^{(k)}(x_i)=0$, $k=0,1,2,\ldots,n-1$. The identity reduces to relation (2.6) in [8] and equation 2.9 in [6]. Also we note that $W[y,\sigma_1,\ldots,\sigma_n]$ is the $(n+1)\times(n+1)$ determinant

$$\begin{vmatrix} y & \sigma_1 & \sigma_2 & \cdots & \sigma_n \\ y' & \sigma_1' & \sigma_2' & \cdots & \sigma_n' \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ y^{(n-1)} & \sigma_1^{(n-1)} & \sigma_2^{(n-1)} & \cdots & \sigma_n^{(n-1)} \\ y^{(n)} & \sigma_1^{(n)} & \sigma_2^{(n)} & & \sigma_n^{(n)} \end{vmatrix}.$$

Hence, we have

$$J[y,a,b] = A(x)|_a^b + \int_a^b p_n(x)\left\{\frac{W[y,\sigma_1,\sigma_2,\ldots,\sigma_n]}{W[\sigma_1,\sigma_2,\ldots,\sigma_n]}\right\}^2. \tag{7}$$

THEOREM 2. *$J[y;x_1,x_2]>0$ for all admissible functions y such that*

$$y^{(k)}(x_1)=y^{(k)}(x_2)=0 \qquad \begin{matrix} k=0,1,2,\ldots,n-1 \\ a<x_1<x_2<b \end{matrix}$$

if there exists no point conjugate to $x=a$ in $[a,b]$, and $J=0$ if and only if $y\equiv 0$.

Proof. This is an easy consequence of identity (7) applied to the interval $[x_1, x_2]$, and $J[y; x_1, x_2]=0$ if and only if

$$W[y; \sigma_1, \sigma_2, \ldots, \sigma_n]=0$$

which implies $y\equiv 0$ since $W[\sigma_1, \sigma_2, \ldots, \sigma_n]\neq 0$ in $(a, b]$.

THEOREM 3. *If $x=b$ is not conjugate to $x=a$, there exists a unique solution of* (3) *determined by the conditions* (2).

Proof. A direct extension of theorem 2.2 in Leighton's paper [8] gives us the result.

THEOREM 4. *In the class of admissible functions satisfying*

$$y^{(k)}(a)=y^{(k)}(b)=0 \qquad k=0, 1, 2, \ldots, n-1 \tag{*}$$

$J[y; a, b]$ is positive unless $y\equiv 0$ provided there is no point conjugate to $x=a$ on the interval $[a, b]$; when $x=b$ is conjugate to $x=a$, $J[y; a, b]$ is positive except along those solutions of (3) *which have n-fold zeros at $x=a$, $x=b$.*

Proof. To prove the first part we define $p_k(x)$ in a slightly larger interval $[a-\varepsilon, b]$ in such a way as to preserve their continuity properties and note that there will be no conjugate point to $a-\varepsilon$ in $[a-\varepsilon, b]$ if $\varepsilon>0$ is sufficiently small. Then instead of $\sigma_i(x)$ appearing in Cimmino-Picone identity take the n linearly independent solutions with n-fold zeros at $a-\varepsilon$ and apply theorem 2. This proves the first part of the theorem.

To prove the second part of the theorem we shall first prove that $J[y, a, b]\geq 0$ for admissible functions satisfying (*). If this were not true there will exist an admissible function z satisfying (*) such that $J[z, a, b]<0$. Now we define a new admissible function y satisfying (*) as follows.

$$y=\begin{cases} u_1(x) & a\leq x\leq \tau_1 \\ z(x) & \tau_1\leq x\leq \tau_2 \\ v_1(x) & \tau_2\leq x\leq b \end{cases} \tag{8}$$

where u_1 and v_1 are solutions of (3) satisfying

$$\left.\begin{aligned} &u_1^{(k)}(a)=0, && u_1^{(k)}(\tau_1)=z^{(k)}(\tau_1) \\ &v_1^{(k)}(\tau_2)=z^{(k)}(\tau_2), && v_1^{(k)}(b)=0 \end{aligned}\right\} k=0, 1, 2, \ldots, n-1.$$

Such solutions exist since b is conjugate to a. Further we define new solutions y_1, y_2 of (3) as follows

$$\begin{aligned} &y_1^{(k)}(\tau_1)=0, && y_1^{(k)}(x_1)=u_1^{(k)}(x_1) \\ &y_2^{(k)}(\tau_2)=0, && y_2^{(k)}(x_2)=v_1^{(k)}(x_2) \end{aligned} \tag{9}$$

$$k=0, 1, \ldots, n-1, \qquad a<x_1<\tau_1, \qquad \tau_2<x_2<b.$$

where x_1, x_2 will be chosen suitably.

Let $(\bar{\sigma}_i)_{1\le i\le n}$ be n linearly independent solutions of (3) such that

$$\left.\begin{aligned}\bar{\sigma}_i^{(k)}(\tau_1)&=0\\ \bar{\sigma}_i^{(k+n)}(\tau_1)&=\delta_{i,k+1}\end{aligned}\right\}\begin{aligned}&k=0,1,2,\ldots,n-1\\ &i=1,2,\ldots,n.\end{aligned}\tag{10}$$

Then we see that

$$y_1(x)=\sum_{k=1}^{n}\alpha_i\bar{\sigma}_i(x)$$

observe that $W[\sigma_1,\sigma_2,\ldots,\sigma_n](x)\neq 0$ in $[a,\tau_1)$ due to our assumption concerning a and b. For if it were zero at, some point in $[a,\tau_1)$ then by theorem 2, $y_1\equiv 0$. Hence we can find α_i by Cramer's rule and we obtain

$$\alpha_i=\frac{W[\bar{\sigma}_1,\bar{\sigma}_2,\ldots,\bar{\sigma}_i,u,\bar{\sigma}_{i+1},\ldots,\bar{\sigma}_n]}{W[\bar{\sigma}_1,\bar{\sigma}_2,\ldots,\bar{\sigma}_n]}(x_1)\tag{11}$$

$1\le i\le n$. Thus α_i depends only on the value of $u_1^{(k)}$ at x_1, and can be made sufficiently small since $u_1^{(k)}(x_1)\to 0$ as $x_1\to a$ for $k=0,1,\ldots,n-1$. A similar argument can be made with respect to y_2 as $x_2\to b$.

Now define

$$\eta=\begin{cases}y(x)-y_1(x) & x_1\le x\le\tau_1\\ y(x) & \tau_1\le x\le\tau_2\\ y(x)-y_2(x) & \tau_2\le x\le x_2.\end{cases}\tag{12}$$

A simple calculation shows that

$$\begin{aligned}J[\eta;x_1,x_2]-J[y;a,b]=&\left[\sum_{k=1}^{n}\left(\sum_{r=0}^{k-1}(-1)^r(p_ky_2^{(k)})^{(r)}v_1^{(k-r-1)}\right](\tau_2)\\ &-\left[\sum_{k=1}^{n}\left(\sum_{r=0}^{k-1}(-1)^r(p_ky_1^{(k)})^{(r)}u_1^{(k-r-1)}\right](\tau_1).\end{aligned}\tag{13}$$

From the explicit representation of y_2, y_1, it can be easily seen that coefficients of $v^{(k)}(\tau_2)$, $u^{(k)}(\tau_1)$ in the above expression depend only on values of $u_1^{(k)}(x_1)$ and $v_1^{(k)}(x_2)$ for, $k=0,1,\ldots,n-1$ and can be made arbitrarily small.

So now if we choose x_1, and x_2 sufficiently close to a and b respectively, such that the right hand side of (13) is less than $-J[z,a,b]/2$, we get

$$J[\eta;x_1,x_2]\le J[y,a,b]-\frac{J[z,a,b]}{2}.$$

It can easily be proved that $J[y]\le J[z]$ [12, Theorem 2.2, p. 741] then

$$J[\eta;x_1,x_2]\le\frac{J[z,a,b]}{2}<0$$

a contradiction to theorem 2.

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Now if $J[y; a, b]=0$ then y must be a solution of (3) (being an extremal) and hence a linear combination of σ_i, $i=1,2,\ldots,n$ and hence has n-fold zeros at $x=a$, $x=b$ and further along such a solution $J=0$.

THEOREM 5. *If there is no point conjugate to $x=a$ on the interval (a, b), the unique solution of* (3) *satisfying* (2) *provides a minimum to J among admissible functions $y(x)$ that satisfy* (2).

Proof. We write $J[y]$ as follows

$$J[y]=\int_a^b \sum_{k=0}^n p_k(x)(y^{(k)})^2\,dx$$

where $y(x)$ is admissible and satisfies (2). If v is any solution of (3) we can easily show that

$$J[y]-J[v]=\int_a^b \sum_{k=0}^n p_k((y-v)^{(k)})^2\,dx+2Q, \tag{14}$$

where

$$Q=\left[\sum_{k=1}^n\left(\sum_{r=0}^{k-1}(-1)^{(r)}(p_k y^{(k)})^r(y-v)^{(k-r-1)}\right)\right]_a^b. \tag{15}$$

If y and v both satisfy conditions (2), then it follows that $Q=0$ and $J[y]-J[v]=J[\eta]$, where $\eta=y-v$ and, by theorem 2 $J[y]>J[v]$ unless $y\equiv v$.

The following comparison theorem is an easy consequence of the above theorems and we do not repeat the proofs.

THEOREM 6. *If, for some admissible function u, where u satisfies*

$$\sum_{k=0}^n(-1)^{n-k}\frac{d^k}{dx^k}\left(\bar{p}_k(x)\frac{d^k u}{dx^k}\right)=0 \tag{16}$$

$$u^{(k)}(a)=u^{(k)}(b)=0 \qquad k=0,1,2,\ldots,n-1$$

$$\int_a^b\sum_{k=0}^n(\bar{p}_k-p_k)(u^{(k)})^2\,dx>\int_a^b\sum_{k=0}^n\bar{p}_k(u^{(k)})^2\,dx$$

then there exists a conjugate point of $x=a$ with respect to

$$\sum_{k=0}^n(-1)^{n-k}\frac{d^k}{dx^k}\left(p_k(x)\frac{d^k y}{dx^k}\right)=0. \tag{17}$$

in the open interval (a, b).

COROLLARY. *If $x=b$ is the first conjugate point of $x=a$ with respect to* (10) *and if*

$$\int_a^b\sum_{k=0}^n(\bar{p}_k-p_k)(u^{(k)})^2\,dx>0 \tag{18}$$

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where u is a solution of (16) *defining the conjugate point, then the first conjugate point $x = c$ of $x = a$ with respect to* (17) *lies on the interval* (a, b).

5. **Oscillation criteria.** We say that the equation $Lu = 0$ is nonoscillatory if and only if there exists an a such that no nontrivial solution of this equation has more than one n-fold zero in $[a, \infty)$. It can be shown [6] that the above definition is equivalent to saying that there exists an a such that the subwronskian $W[\sigma_1, \sigma_2, \ldots, \sigma_n]$ has no zero in (b, ∞) for any $b \geq a$. The equation is oscillatory if it is not nonoscillatory. It is now quite easy to obtain oscillation criteria for the differential equation $Lu = 0$ by producing an admissible function and an interval on which the integral inequality is violated. For example, we have the following theorem.

THEOREM 7. *If for any $a_1 \in [a, \infty) \exists$ a number $b_1 > a_1$ and there exists a function $y(x)$ which satisfies*

(i) $$y^{(k)}(a_1) = y^{(k)}(b_1) = 0 \qquad k = 0, 1, 2, \ldots, n-1$$

(*ii*) $$\int_{a_1}^{b_1} \sum p_k(x)(y^{(k)})^2\, dx < 0,$$

then the equation $Lu = 0$ is oscillatory.

The proof of the theorem is immediate and is the same as those used by C. A. Swanson [14], E. S. Noussair [10], D. B. Hinton [5]. Consequently, we can obtain the following theorem of I. M. Glazman [9] as a corollary.

THEOREM 8 [9]. *The equation $(-1)^n(p_n y^n)^{(n)} - p_0 y = 0$ is oscillatory if*

$$b_1^{1-2n} \int_{b_1/2}^{b_1} p_n(x)\, dx$$

is bounded for sufficiently large b_1 and

$$\int^{\infty} p_0(x)\, dx = \infty.$$

The proof of the theorem is obtained by defining y as follows. Let $\omega(t) = K\int_0^t s^{n-t}(1-s)^{n-1}\, dx$, where K is chosen so that $\omega(1) = 1$. Let y be defined as follows

$$y(x) = \begin{cases} 0 & a \leq x \leq a_1 \\ \omega\left(\dfrac{x - a_1}{a_1}\right) & a_1 < x \leq 2a_1 \\ 1 & 2a_1 < x \leq \dfrac{b_1}{2} \\ \omega\left(\dfrac{2b_1 - 2x}{b_1}\right) & \dfrac{b_1}{2} < x \leq b_1 \\ 0 & x > b_1 \end{cases}$$

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where a_1 is arbitrary and b_1 will be chosen so that $b_1/2 > 2a_1$ and $J[y; a_1, b_1] < 0$. Now

$$\begin{aligned}\int_{a_1}^{b_1} p_n(y^{(n)})^2\,dx &= \int_{a_1}^{2a_1} \frac{1}{a_1^{2n}}\left[\omega^{(n)}\left(\frac{x-a_1}{a_1}\right)\right]^2 dx \\ &\quad + \int_{b_1/2}^{b_1} \frac{2^{2n}}{b_1^{2n}} p_n\left[\omega^{(n)}\left(\frac{2b_1-2x}{b_1}\right)^2\right] dx \\ &\leq k_1 + \frac{k_2}{b_1^{2n-1}} \int_{b_1/2}^{b_1} p_n(x)\,dx\end{aligned}$$

where k_1, k_2 are suitable constants since $\omega(t)$ is a polynomial of degree n. Moreover

$$\begin{aligned}\int_{a_1}^{b_1} p_0(x)y^2(x)\,dx &= \int_{a_1}^{2a_1} p_0(x)\omega^2\left(\frac{x-a}{a}\right) dx + \int_{2a_1}^{b_1/2} p_0(x)\,dx \\ &\quad + \int_{b_1/2}^{b_1} p_0(x)\omega^2\left(\frac{2b_1-2x}{b_1}\right) dx.\end{aligned}$$

Let

$$\int_{a_1}^{2a_1} p_0(x)\omega^2\left(\frac{x-a_1}{a_1}\right) dx = k_3.$$

where k_3 depends only on a_1. Now choose $b_0 > 4a_1$ large enough so that

$$k_1 + \frac{k_2}{b_1^{2n-1}} \int_{b_1/2}^{b_1} p_n(x)\,dx - k_3 - \int_{a_1}^{b_1/2} p_0(x)\,dx < 0$$

for all $b_1 \geq b_0$. This is possible by hypothesis of the theorem. Let

$$\bar{y}(x) = \int_{b_0/2}^{x} p_0(t)\,dt.$$

Then $\lim_{x\to\infty} \bar{y}(x) = \infty$. Choose $b_1/2$ in the definition of y to be the last zero of $\bar{y}(x)$. Then $b_1/2 \geq b_0/2$, $y(x) > 0$ for all $x > b_1/2$ and

$$\int_{b_1/2}^{b_1} \omega^2\left(\frac{2b_1-2x}{b_1}\right) p_0(x)\,dx > 0$$

as can be proved by integration by parts and the fact that $y(x) > 0$ for all $x > b_1/2$ and $\omega'((2b_1-2x)/b_1)$ is positive by definition of $\omega(t)$. From the above considerations we obtain $J[y; a_1, b_1] < 0$, and this completes the proof.

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