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Abstract

Let $J(m)$ be an $m \times m$ Jordan block with eigenvalue 1. For $\lambda \in \mathbb{C} \setminus \{0, 1\}$, we explicitly construct all rank 2 local systems of geometric origin on $\mathbb{P}^1 \setminus \{0, 1, \lambda, \infty\}$, with local monodromy conjugate to $J(2)$ at $0, 1, \lambda$ and conjugate to $-J(2)$ at ∞. The construction relies crucially on Katz's middle convolution operation. We use our construction to prove two conjectures of Sun, Yang and Zuo (one of which was proven earlier by Lin, Sheng and Wang; the other was proven independently of us by Yang and Zuo) coming from the theory of Higgs–de Rham flows, as well as a special case of the periodic Higgs conjecture of Krishnamoorthy and Sheng.

1. Introduction

It was known (in some sense) to Riemann [Kat96, Introduction], in his work on hypergeometric functions, that all rank 2 local systems on $\mathbb{P}^1 \setminus \{0, 1, \infty\}$ with trivial determinant and quasi-unipotent local monodromy are *of geometric origin*, that is, they arise in the cohomology of a family of varieties over $\mathbb{P}^1 \setminus \{0, 1, \infty\}$. The case of $\mathbb{P}^1$ minus four points is more complicated; Beauville [Bea82] famously classified the $D \subset \mathbb{P}^1, |D| = 4$ such that $\mathbb{P}^1 \setminus D$ carries a family of elliptic curves with stable reduction along D, or equivalently, carries a rank 2 $\mathbb{Z}$-local system of geometric origin with *unipotent* local monodromy. In general, rank 2 motivic local systems on $\mathbb{P}^1$ minus four points are poorly understood (and the situation when one increases the rank or the number of points deleted is completely mysterious).

The goal of this paper is to explicitly write down all local systems of geometric origin on $\mathbb{P}^1 \setminus \{0, 1, \infty, \lambda\}$, with local monodromies conjugate to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

at $0, 1, \lambda$ and with local monodromy conjugate to

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

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at ∞. In so doing we give short proofs of two conjectures of Sun, Yang and Zuos [SYZ21, Conjectures 4.8 and 4.10] on motivic local systems on $X = \mathbb{P}^1 \setminus \{0, 1, \infty, \lambda\}$. [SYZ21, Conjecture 4.8] was earlier proven in [LSW22] by different methods. Our proof of Conjecture 4.10 is conditional on an expected but technical statement about the periodicity of motivic parabolic bundles under the Higgs–de Rham flow, whose proof does not appear in the literature, but which we expect to follow without complication from the general approach of [LSZ19]; see, for example, [LYZ23, Theorem 2.34] for a precise statement (without proof).

Remark 1.1. While this paper was in preparation Yang and Zuo [YZ23] independently claimed a fascinating proof of [SYZ21, Conjecture 4.10], via completely different techniques from ours.[1] Their 128-page proof relies on p-adic Hodge theory and the Langlands correspondence, as well as crucially a previous conjecture of Sun, Yang and Zuo, proven by Lin, Sheng and Wang (Theorem 1.11); by contrast our proof is only a few pages and we *explicitly* write down the relevant local systems and the families in whose cohomology they appear, and we are also able to give a new proof of Theorem 1.11. We refer the reader to Remark 1.9 for more comparisons between the two approaches.

1.1 Main results

Let X be a smooth complex curve, and in the case where X is not proper, we pick a smooth compactification $\overline{X}$ of X. Recall that a $\mathbb{C}$-local system $\mathbb{V}$ on X is said to be of geometric origin if there exist a dense open $U \subset X$, and a family of smooth proper algebraic varieties $\pi : \mathcal{Y} \to U$ such that $\mathbb{V}$ appears as a subquotient of $R^i\pi_*\mathbb{C}$ for some $i \geq 0$. (The appearance of U in the definition is a slightly technical point which the reader may ignore at first pass, and indeed it is unnecessary for the local systems appearing in our main theorems.)

Motivated by the previous work of Faltings [Fal83], Deligne showed that, for a fixed X, there are only finitely many $\mathbb{Q}$-local systems of fixed rank n which are of geometric origin. One of our main motivations was whether a strengthened form of Deligne's theorem [Del87] could hold.

Question 1.2. Can there be infinitely many $\mathbb{C}$-local systems of rank n on X which are of geometric origin, and moreover whose local monodromies at the boundary $\overline{X} - X$ are fixed?

Remark 1.3. The condition on local monodromies is certainly necessary for the above question to be interesting. Indeed, the hypergeometric local systems of rank 2 on $X = \mathbb{P}^1 - \{0, 1, \infty\}$ are all of geometric origin with the local monodromies at $0, 1, \infty$ having eigenvalues mth roots of unity. The point is that, upon bounding m, there are only finitely many such hypergeometric local systems of fixed rank. We find the local monodromy condition natural since it implies that we are considering points of the *relative character varieties*, which are the natural replacement of character varieties in the case of non-proper X.

Note that [Lit21] shows that there are only finitely many K-local systems of geometric origin on X if K embeds in any finite extension of $\mathbb{Q}_p$, for any prime p.

We now specialize to our case of interest. For any $\lambda \neq 0, 1$, let X be the curve $\mathbb{P}^1 - \{0, 1, \lambda, \infty\}$. We consider rank 2 local systems on X satisfying the following condition ($\star$): namely with local monodromy conjugate to

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

[1] Our work began in December 2022, whereas the work in [YZ23] had been ongoing for several years, as we were informed by the authors.

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at $0, 1, \lambda$ and local monodromy conjugate to

$$\begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}$$

at ∞. We recall the following definition.

DEFINITION 1.4. A local system $\mathbb{V}$ on a manifold Y is said to be MCG-finite if the set of isomorphism classes of local systems $\{f^*\mathbb{V}\}$ obtained by acting on $\mathbb{V}$ by elements f of the mapping class group of Y is finite.

We may now state our first result.

THEOREM 1.5. *For any $\lambda \neq 0, 1$, there exist infinitely many rank 2 local systems of geometric origin on X satisfying condition $(\star)$. Moreover, a rank 2 local system on X satisfying $(\star)$ is of geometric origin if and only if it is MCG-finite.*

Remark 1.6. Theorem 1.5 answers Question 1.2 in the positive. It would be very interesting to further investigate Question 1.2 in the case of X proper and to understand whether there is a qualitative difference between the two cases.

This result was claimed independently, though inexplicitly, by Yang and Zuo [YZ23]; they rephrase MCG-finiteness in terms of algebraic solutions to the Painlevé VI equation. In that language, these solutions were, we believe, discovered originally by Hitchin [Hit95]; Lysovyy and Tykhyy [LT14], in their classification of algebraic solutions to Painlevé VI, refer to them as the 'Cayley solutions'. The reason for this name is that, under condition $(\star)$, the equation of the relative character variety is the *Cayley cubic*: see [LT14, equation (61)].

In fact, we classify all rank 2 local systems of geometric origin and satisfying $(\star)$. The following is the explicit form of our classification, and says that the local systems of geometric origin satisfying $(\star)$ arise in the cohomology of an explicit family of curves (and therefore abelian varieties).

THEOREM 1.7. *Let $f : E \to \mathbb{P}^1$ be the $2:1$ cover branched over $\{0, 1, \infty, \lambda\}$. Consider the fiber square*

$$\begin{tikzcd} Z'_s \arrow[r] \arrow[d] & W' \arrow[d] \\ E \times \mathbb{P}^1 \arrow[r, "{[s]\times \mathrm{id}}"] & E \times \mathbb{P}^1 \end{tikzcd}$$

where W' is the double cover of $E \times \mathbb{P}^1$ branched along the graph of f and $E \times \{\infty\}$, obtained by normalizing $E \times \mathbb{P}^1$ in the field

$$\mathbb{C}(E \times \mathbb{P}^1)(\sqrt{f(x) - y})$$

for x a coordinate on E and y a coordinate on $\mathbb{P}^1$. For any rank 2 local system $\mathbb{V}$ of geometric origin on X satisfying $(\star)$, $\mathbb{V}$ appears as a subquotient of the cohomology of the family

$$Z'_s \to E \times \mathbb{P}^1 \to \mathbb{P}^1$$

for some $s \in \mathbb{Z}_{>0}$.

Example 1.8. We write down explicitly all representations corresponding to rank 2 motivic local systems satisfying $(\star)$. Our formulas are obtained by plugging in the values of the Cayley solutions [LT14, p.145] into [Boa05, Equation (24)].[2] Since the fundamental group $\pi_1(\mathbb{P}^1 \setminus \{0, 1, \lambda, \infty\})$

[2]In the following, α, β correspond r_Y, r_Z, respectively, of [LT14].

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is the free group generated by loops around $0, 1, \lambda$, it suffices to write down three matrices M_0, M_1, M_λ, corresponding to the images of the three generators:

$$M_0 = \begin{pmatrix} 1 + x_2x_3/x_1 & -x_2^2/x_1 \\ x_3^2/x_1 & 1 - x_2x_3/x_1 \end{pmatrix}, \quad M_1 = \begin{pmatrix} 1 & -x_1 \\ 0 & 1 \end{pmatrix}, \quad M_\lambda = \begin{pmatrix} 1 & 0 \\ x_1 & 1 \end{pmatrix},$$

where

$$x_1 = 2\cos\left(\frac{\pi(\alpha+\beta)}{2}\right), \qquad x_2 = 2\sin\left(\frac{\pi\alpha}{2}\right), \quad x_3 = 2\sin\left(\frac{\pi\beta}{2}\right)$$

for $\alpha, \beta \in \mathbb{Q}$.

The idea of the proofs of Theorems 1.5 and 1.7 is to analyze local systems satisfying $(\star)$ via Katz's *middle convolution* operation [Kat96]. The key is to observe that local systems of geometric origin satisfying $(\star)$ may be obtained via middle convolution from certain finite monodromy local systems on X. Namely, we prove that if $f: E \to \mathbb{P}^1$ is the double cover of $\mathbb{P}^1$ branched at $\{0, 1, \infty, \lambda\}$, then any motivic local system satisfying $(\star)$ arises via middle convolution from $f_*\mathbb{L}|_X$, where $\mathbb{L}$ is a rank 1 local system on E with finite monodromy; moreover, this construction produces all local systems of geometric origin satisfying $(\star)$. Moreover, since $f_*\mathbb{L}|_X$ itself is MCG-finite, the same is true of its middle convolution.

Remark 1.9 (Comparison with the work of Yang and Zuo).

(i) As mentioned above, our main tool is Katz's middle convolution functor. Our approach gives new proofs of all the main results of [YZ23]. Indeed, [YZ23, Theorem 1.7] is a corollary of Theorem 1.7, and [YZ23, Theorem 1.5] follows immediately from Lemma 3.4.

(ii) An important invariant of a local system of geometric origin is the *trace field*, that is, the smallest field generated by the traces of elements of $\pi_1(X)$. From the description as middle convolution of rank 1 torsion local systems on E, it is straightforward to see that the trace fields of our local systems are $\mathbb{Q}(\zeta_m + \zeta_m^{-1})$ for ζ_m a primitive mth root of unity, with m being the order of the rank 1 local system. It does not seem possible to derive this information with the techniques of [YZ23]; Zuo has informed us that he conjectured these trace fields in 2018; see also [YZ23, Appendix A], where this conjecture is alluded to. This also explains a technical point of [YZ23]. Indeed, there, the authors assume that E has supersingular reduction at p, and crucially use the fact that the trace fields of all irreducible SL_2-local systems on X (satisfying $(\star)$) are unramified at p. The latter is certainly not true if E has ordinary reduction, in which case there will be rank 2 local systems with trace field $\mathbb{Q}(\zeta_{p^k} + \zeta_{p^k}^{-1})$ for any $k \geq 1$, which ramifies at p; correspondingly, there are pre-periodic Higgs bundles which are not periodic, in contrast to the case where E has supersingular reduction (cf. [YZ23, Corollary 3.4.1]).

1.2 Higgs bundles and Higgs–de Rham flows

As before, let $D = \{0, 1, \infty, \lambda\} \subset \mathbb{P}^1$, $X = \mathbb{P}^1 \setminus D$, and let $\mathbb{V}$ be a local system of geometric origin on X of rank 2, satisfying $(\star)$. By, for example, [LL24b, Proposition 4.1.4], $(\mathbb{V} \otimes \mathscr{O}_X, \mathrm{id} \otimes d)$ canonically extends to a filtered flat vector bundle on $\mathbb{P}^1$ with logarithmic flat connection,

$$(F^1, \mathscr{E}, \nabla : \mathscr{E} \to \mathscr{E} \otimes \Omega^1_{\mathbb{P}^1}(\log D)),$$

where $(\mathscr{E}, \nabla)$ is the Deligne canonical extension—that is, the residues of ∇ have eigenvalues with real parts lying in $[0, 1)$—and F^1 restricts to the Hodge filtration on X.

One can show (see Proposition 3.1) that if $\mathbb{V}$ satisfies $(\star)$, then $F^1 \simeq \mathscr{O}_{\mathbb{P}^1}$ and $\mathscr{E}/F^1 \simeq \mathscr{O}_{\mathbb{P}^1}(-1)$. The connection ∇ yields a natural map

$$\theta : F^1 \to \mathscr{E}/F^1 \otimes \Omega^1_{\mathbb{P}^1}(\log D), \tag{1.1}$$

which, being a non-zero map $\mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(1)$, vanishes at a unique point $w(\mathbb{V}) \in \mathbb{P}^1$. Moreover, $w(\mathbb{V})$ determines $\mathbb{V}$ up to isomorphism (Proposition 3.2).

The following is a simple-to-state variant of a conjecture of Sun, Yang and Zuo.

CONJECTURE 1.10 (Slight variant of [SYZ21, Conjecture 4.10]). Let $f : E \to \mathbb{P}^1$ be the double cover branched over D, viewed as an elliptic curve with identity the point over ∞. As $\{\mathbb{V}\}$ ranges over all local systems of geometric origin satisfying $(\star)$, the $w(\mathbb{V})$ are precisely the image under f of the torsion points of E.

We prove this conjecture in Proposition 3.6, and explain how to deduce [SYZ21, Conjecture 4.10] in § 6.2.

The proof has two parts. The first is to show that those $\mathbb{V}$ with $w(\mathbb{V}) = f(x)$, for $x \in E$ torsion, are of geometric origin; this will follows from a more precise form of Theorem 1.7.

The second is to show that these are the only points of geometric origin. We give two approaches, the first in Proposition 3.6, and the latter in § 5. The first approach is to again use the middle convolution, which preserves the property of geometric origin and sends $\mathbb{V}$ with $w(\mathbb{V})$ not the image of a torsion point to a local system manifestly not of geometric origin.

The second approach proceeds by proving [SYZ21, Conjecture 4.8], a conjecture in the theory of the Higgs–de Rham flows, as we now explain. The theory of Higgs–de Rham flows was introduced by Lan, Sheng and Zuo [LSZ19], using Ogus and Vologodsky's work [OV07]. The details of this theory are rather technical, but as a rough approximation, the reader may regard it as providing, for each variety X in positive characteristic, a functor from the category of Higgs bundles on X to itself generalizing the Frobenius pullback on vector bundles.

Krishnamoorthy and Sheng [KS20] showed that, for Higgs bundles associated to motivic local systems, almost all of their reductions mod p are f-*periodic* under the Higgs–de Rham flow for some fixed f, independent of p (following earlier work of [LS22]). They then conjectured the converse to hold; see, for example, [KS20] or [LSW22, Conjecture 1.7]. This is known as the *periodic Higgs conjecture*.

Returning to the situation in this paper, we consider semistable, nilpotent, graded parabolic Higgs bundles on $(\mathbb{P}^1, D)$ whose underlying graded vector bundle is $\mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$, as in (1.1). As discussed above, the moduli of such Higgs bundles is isomorphic to $\mathbb{P}^1$, by sending a Higgs bundle to the vanishing locus of its Higgs field. Sun, Yang and Zuo, using a 'twisted' version of the Higgs–de Rham flow, produce a self-map of this moduli space, $\psi_p : \mathbb{P}^1 \to \mathbb{P}^1$. In [KS20], this twisted flow was reinterpreted as a Higgs–de Rham flow for parabolic Higgs bundles.

The following theorem was conjectured in [SYZ21], and was proved very recently by Lin, Sheng and Wang; we will give a more precise statement in § 4.

THEOREM 1.11 ([SYZ21, Conjecture 4.8], [LSW22, Theorem 1.6]). *Viewing $\mathbb{P}^1$ as the moduli of graded nilpotent Higgs bundles as above, the map $\psi_p : \mathbb{P}^1 \to \mathbb{P}^1$ fits into the following commutative diagram.*

$$\begin{tikzcd} E \arrow[r, "{[p]}"] \arrow[d, "f"] & E \arrow[d, "f"] \\ \mathbb{P}^1 \arrow[r, "\psi_p"] & \mathbb{P}^1 \end{tikzcd}$$

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Combining with the periodicity of motivic Higgs bundles mentioned above, it immediately implies that motivic Higgs bundles are images of torsion points of E, and hence finishes the second proof of Conjecture 1.10. We will give a simple new proof of Theorem 1.11: see the end of § 5.

Finally, we give a sketch of our proof of Theorem 1.11. We view E as the moduli space of rank 1, degree 0 Higgs bundles with zero Higgs field on E. The middle convolution functor, on the level of moduli spaces, is then represented by the map $f: E \to \mathbb{P}^1$; where we view the source as a moduli of rank 1 Higgs bundles as above, and the target as a moduli of rank 2 Higgs bundles on $(\mathbb{P}^1, D)$. We then prove, in this case, that middle convolution commutes with the Higgs–de Rham flow. On the other hand, the Higgs–de Rham flow for Higgs bundles of rank 1 with zero Higgs field on E is simply represented by the map $[p]$ (as it is simply the Frobenius pullback on line bundles), and we therefore conclude. The detailed proof is given at the end of § 5, where we use a trick to show the commutation between middle convolution and Higgs–de Rham flow.

Remark 1.12. Our proof of Theorem 1.11 is also significantly different than that of [LSW22]: indeed, in [LSW22] the theorem is proved by using a relation to rank 2 Higgs bundles on E with non-trivial Higgs field, while we do so by exhibiting a relation, via middle convolution, to rank 1 Higgs bundles on E with trivial Higgs field.

2. Preliminaries on convolution in the Betti and ℓ-adic settings

2.1 Definitions

Let k be a field and let $D \subset \mathbb{P}^1_k$ be a reduced effective divisor containing ∞. Let $X = \mathbb{P}^1 \setminus D$. There are natural projection maps

$$\pi_i : X \times X \setminus \Delta \to X,$$

for $i = 1, 2$ as well as a map

$$m : X \times X \setminus \Delta \to \mathbb{G}_m$$

given by

$$m : (x, y) \mapsto x - y.$$

Let

$$j : X \times X \setminus \Delta \hookrightarrow \mathbb{P}^1_k \times X$$

be the evident inclusion.

Definition 2.1. Let χ be a rank 1 local system on $\mathbb{G}_m$, and let $\mathbb{V}$ be a local system on X. The *middle convolution* $MC_\chi(\mathbb{V})$ is defined to be

$$MC_\chi(\mathbb{V}) = R^1(\pi_2)_* j_*(\pi_1^* \mathbb{V} \otimes m^* \chi).$$

This definition is an unwinding of [Kat96, § 2.8], without the use of the language of perverse sheaves. For a proof that these two definitions agree, see, for example, [Kat96, § 5.1]. An alternative description well adapted to explicit computation is given by Dettweiler and Reiter; see [DR03, Theorem 1.1].

Remark 2.2. In the ℓ-adic or Betti setting, the middle convolution has a nice geometric interpretation. Indeed, if $\mathbb{V}$ is *pure of weight zero* and χ has finite order, then $MC_\chi(\mathbb{V})$ is precisely the weight 1 part of $R^1(\pi_2)_*(\pi_1^* \mathbb{V} \otimes m^* \chi)$.

We will also make use of the following variant of middle convolution. Let Y be a smooth proper curve, and let $f: Y \to \mathbb{P}^1$ be a finite, generically étale map, with branch divisor contained in D. Let $I = f^{-1}(\infty)$. Let Γ be the graph of f. Then again there are natural projections

$$\pi_1 : Y \times X \setminus (\Gamma \cup I \times X) \to Y, \pi_2 : Y \times X \setminus (\Gamma \cup I \times X) \to X$$

and a natural map

$$m : Y \times X \setminus (\Gamma \cup I \times X) \to \mathbb{G}_m$$

$$(y, x) \mapsto f(y) - x.$$

Let

$$i : Y \times X \setminus (\Gamma \cup I \times X) \hookrightarrow Y \times X$$

be the evident inclusion.

DEFINITION 2.3. Let χ be a rank 1 local system on $\mathbb{G}_m$, and let $\mathbb{V}$ be a local system on Y. The *middle convolution* $MC_\chi(f, \mathbb{V})$ is defined to be

$$MC_\chi(f, \mathbb{V}) = R^1(\pi_2)_* i_*(\pi_1^* \mathbb{V} \otimes m^* \chi).$$

The next proposition follows from, for example, proper base change.

PROPOSITION 2.4. *There is a natural isomorphism* $MC_\chi(f, \mathbb{V}) \simeq MC_\chi(f_* \mathbb{V}|_X)$.

The definitions as above are convenient for computing in the Betti and ℓ-adic settings; in the positive characteristic de Rham and Higgs settings we will use a slightly different formalism.

2.2 Betti and ℓ-adic computations

In this section we work over an algebraically closed field k of characteristic different from 2. Suppose D has even degree with $\deg(D) \geq 4$ (with the case $\deg(D) = 2$ being uninteresting) and let $f: Y \to \mathbb{P}^1$ be the double cover branched over D. Let $\mathbb{L}$ be a rank 1 local system on Y. Let χ be the unique non-trivial rank 1 local system on $\mathbb{G}_m$ with $\chi^2 = \text{triv}$. We write MC_{-1} instead of MC_χ to emphasize this particular choice of χ.

PROPOSITION 2.5. *The rank of* $MC_{-1}(f, \mathbb{L})$ *is* $\deg(D) - 2$. *For* $\mathbb{L}$ *non-trivial, the local monodromies of* $MC_{-1}(f, \mathbb{L})$ *are given by the following. Let* $J(\alpha, \ell)$ *denote the Jordan block given by a matrix of size* ℓ *and generalized eigenvalues* α.

- *At a point* $P \neq \infty$, *there is a Jordan block* $J(1, 2)$, *and all other Jordan blocks are* $J(1, 1)$.
- *At* $P = \infty$, *there is a Jordan block* $J(-1, 2)$, *and all other Jordan blocks are* $J(-1, 1)$.

Proof. Given $x \in X$, the fiber U_x of $\pi_2 : Y \times X \setminus (\Gamma \cup I \times X) \to X$ over x is isomorphic to $Y \setminus f^{-1}(\{x \cup \infty\})$, which has Euler characteristic $1 - \deg(D)$. The restriction $\pi_1^* \mathbb{L} \otimes m^* \chi|_{U_x}$ has non-trivial monodromy at the two points of $f^{-1}(x)$ and trivial monodromy at $f^{-1}(\infty)$. Hence, the Grothendieck–Ogg–Shafarevich formula yields

$$\chi(j_*(\pi_1^* \mathbb{L} \otimes m^* \chi|_{U_x})) = 2 - \deg(D).$$

As $\pi_1^* \mathbb{L} \otimes m^* \chi|_{U_x}$ is a non-trivial rank 1 local system, $H^0(\pi_1^* \mathbb{L} \otimes m^* \chi|_{U_x}) = H^2(\pi_1^* \mathbb{L} \otimes m^* \chi|_{U_x}) = 0$. Hence, $MC_{-1}(f, \mathbb{L})$ has rank equal to $\deg(D) - 2$.

For the computation of local monodromies, we refer the reader to [DR07, Lemma 5.1]. Indeed, $MC_{-1}(f, \mathbb{L})$ is equivalent to the standard middle convolution applied to (the restriction to X of) $f_* \mathbb{L}$, and the latter satisfies the conditions of [DR07, Lemma 5.1] by our assumptions that $\mathbb{L}$ is non-trivial. □

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The upshot of this proposition is that if $D = (0) + (1) + (\lambda) + (\infty)$, so that $Y = E$ is an elliptic curve, and $\mathbb{L}$ is a rank 1 local system on E, then $MC_{-1}(f, \mathbb{L})$ has rank 2 and satisfies $(\star)$. In the complex setting, if $\mathbb{L}$ is unitary (hence underlies a complex variation of Hodge structure, henceforth abbreviated as $\mathbb{C}$-VHS), then $MC_{-1}(f, \mathbb{L})$ underlies a $\mathbb{C}$-VHS as well.

3. Proofs of the main theorems in characteristic zero

We may now give a short proof of some of our main theorems. We assume standard material about complex variations of Hodge structure; for a brief primer see [LL24a, § 3] and [LL24b, § 4]. Before giving proofs we need some brief preliminaries.

3.1 Preliminaries

Proposition 3.1. *Let $X = \mathbb{P}^1 \setminus \{0, 1, \infty, \lambda\}$ and let $\mathbb{V}$ be an irreducible complex local system on X satisfying $(\star)$ and underlying a $\mathbb{C}$-VHS. Then the Deligne canonical extension $(\mathscr{E}, F^1, \nabla)$ of $(\mathbb{V} \otimes \mathscr{O}_X, \mathrm{id} \otimes d)$ satisfies $\mathscr{E} \simeq \mathscr{O}_{\mathbb{P}^1} \oplus \mathscr{O}_{\mathbb{P}^1}(-1)$, with $F^1 = \mathscr{O}_{\mathbb{P}^1}$.*

Proof. From $(\star)$, the residue matrices of ∇ at $\{0, 1, \lambda\}$ have trace 0, and the residue matrix at ∞ has trace 1. This means that $\mathscr{E}$ and hence $F^1 \oplus \mathscr{E}/F^1$ has degree -1, by [EV86, B.3], for example. Let

$$\theta : F^1 \to \mathscr{E}/F^1 \otimes \Omega^1_{\mathbb{P}^1}(\log D)$$

be the non-zero $\mathscr{O}$-linear map obtained from ∇. As in the introduction we consider the Higgs bundle

$$F^1 \oplus \mathscr{E}/F^1$$

with the Higgs field $\begin{pmatrix} 0 & \theta \\ 0 & 0 \end{pmatrix}$.

By Simpson's theory [Sim90, Theorem 8], this Higgs bundle is parabolically stable of parabolic degree 0. Parabolic stability implies the parabolic degree of F^1 is positive, and hence the honest degree satisfies $\deg F^1 > -1$. The fact that θ is non-zero implies that $\deg F^1 \leq \deg \mathscr{E}/F^1 + \deg \Omega^1_{\mathbb{P}^1}(\log D)$, and hence that $2 \deg F^1 \leq \deg \mathscr{E} + 2 = 1$. Hence, $\deg F^1 = 0$, and $\deg \mathscr{E}/F^1 = -1$. Finally, we must have that the extension

$$0 \to F^1 \to \mathscr{E} \to \mathscr{E}/F^1 \to 0$$

splits as $\mathrm{Ext}^1(\mathscr{O}_{\mathbb{P}^1}(-1), \mathscr{O}_{\mathbb{P}^1}) = 0$. □

Let $D = \{0, 1, \infty, \lambda\}$. Given $\mathbb{V}$ an irreducible local system satisfying $(\star)$ and underlying a (necessarily unique, by irreducibility) $\mathbb{C}$-VHS, the above proposition shows that we may canonically associate to $\mathbb{V}$ a non-zero map $\theta : \mathscr{O}_{\mathbb{P}^1} \to \mathscr{O}_{\mathbb{P}^1}(-1) \otimes \Omega^1_{\mathbb{P}^1}(\log D) = \mathscr{O}_{\mathbb{P}^1}(1)$, which vanishes at a unique point $w(\mathbb{V})$ of $\mathbb{P}^1$, as in the introduction.

Proposition 3.2. *With assumptions as in Proposition 3.1, the point $w(\mathbb{V})$ determines $\mathbb{V}$ up to isomorphism.*

Proof. The point $w(\mathbb{V})$ determines θ up to multiplication by a non-zero scalar, which by Proposition 3.1 above determines the stable parabolic Higgs bundle associated to $\mathbb{V}$ up to isomorphism. Now Simpson's theory [Sim90, Theorem 8] tells us that this parabolic Higgs bundle determines $\mathbb{V}$. □

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3.2 De Rham and Higgs computations

As before, let $f: E \to \mathbb{P}^1$ denote the double cover ramified $D = 0 + 1 + \lambda + \infty$, viewed as an elliptic curve with origin $e = f^{-1}(\infty)$. Let $(\mathscr{L}, \nabla)$ be a locally free sheaf on E of rank 1 and degree 0 with integrable unitary connection. If $k = \mathbb{C}$, the Riemann–Hilbert correspondence gives $\mathbb{L} := \ker(\nabla)$ a rank 1 unitary local system on E. We may write $\mathscr{L} = \mathcal{O}_E(e - q)$ for some $q \in E$. In this section we compute the parabolic Higgs bundle on $(\mathbb{P}^1, D)$ obtained from $(\mathscr{L}, \nabla)$ via middle convolution. That is, $MC_{-1}(f, \mathbb{L})$ (as defined in § 2.2) underlies a complex variation of Hodge structure $(\mathscr{E}, F, \nabla)$ of rank 2, where $(\mathscr{E}, \nabla)$ is a logarithmic flat vector bundle on $(\mathbb{P}^1, D)$ obtained as the Deligne canonical extension of $MC_{-1}(f, \mathbb{L})$, and F is the non-trivial piece of the Hodge filtration; we will compute the natural map

$$\theta : F \to \mathscr{E}/F \otimes \Omega^1_{\mathbb{P}^1}(\log D)$$

obtained from ∇. More precisely, we will compute the fiber θ_x of θ, at each point $x \in \mathbb{P}^1 \setminus D$.

Let $x_1, x_2 \in E$ be the preimages of x. Consider the rank 1 locally free sheaf with flat connection on $(E - \{x_1\} - \{x_2\}) \times \{x\}$ corresponding to the character $m^*\chi$ under the Riemann–Hilbert correspondence (with notation as in § 2), and let $(\mathscr{M}_0, \nabla)$ denote its Deligne canonical extension to E. The bundle $\mathscr{M}_0$ naturally obtains the structure of a parabolic bundle $\mathscr{M}_\star$ as in [LL24b, Definition 3.3.1]. (See [LL24b, § 2] or § 4 in this paper for a brief recollection on parabolic bundles.)

We claim that $\mathscr{M}_0 \simeq \mathscr{O}_E(-e)$. Indeed, we have the double cover (by abusing notation) $m : E \to \mathbb{P}^1$ ramified at $-x, 1-x, \lambda - x, \infty$, and a parabolic line bundle on $\mathbb{P}^1$ with parabolic weights $1/2$ at $0, \infty$. [AB23, Equation (3.4)] shows that the underlying bundle for the parabolic pullback is $\mathscr{O}_E(-e)$, as claimed.

We recall the following proposition, obtained by specializing [LL24a, Theorem 5.1.6] to our setting:[3]

Proposition 3.3. *The fiber θ_x of the Higgs field associated to the complex variation of Hodge structure $MC_{-1}(f, \mathbb{L})$ at x is computed via the map*

$$H^0(E, \mathscr{L} \otimes \mathscr{M}_0 \otimes \omega_E(x_1 + x_2)) \to H^1(E, \mathscr{L} \otimes \mathscr{M}_0) \otimes T^*_x X$$

given as adjoint (via Serre duality) to the multiplication map

$$H^0(E, \mathscr{L} \otimes \mathscr{M}_0 \otimes \omega_E(x_1 + x_2)) \otimes H^0(E, \mathscr{L}^\vee \otimes \mathscr{M}_0^\vee \otimes \omega_E) \to H^0(\omega_E^{\otimes 2}(x_1 + x_2)) \to T^*_x X,$$

where the map $H^0(\omega_E^{\otimes 2}(x_1 + x_2)) \to T^*_x X$ is the one given by pullback on cotangent spaces along the map $E \to \mathcal{M}_{1,2}$ given by

$$x_1 \mapsto (E, x_1, -x_1).$$

The following is the key computation required for our proofs.

Lemma 3.4. *The Higgs bundle corresponding to $MC_{-1}(f, \mathbb{L})$ has Higgs field vanishing at $f(q)$. That is, $\theta_{f(q)} = 0$.*

Proof. As above, we let $\{x_1, x_2\} = f^{-1}(x)$, and assume $x_1 = q$. By Proposition 3.3, the Higgs field factors through

$$H^0(\omega_E(-q + x_1 + x_2)) \otimes H^0(\omega_E(q)) \to H^0(\omega_E^{\otimes 2}(x_1 + x_2)).$$

[3] We match up the notation with [LL24a, Theorem 5.1.6] for the reader's convenience. Let E° denote $E - E[2]$. We set $\mathscr{B} \subset E^\circ$ to be a contractible neighborhood of x_1, $\mathscr{C}$ the constant family $E \times \mathscr{B}$, $s_1 : \mathscr{B} = E - E[2] \to \mathscr{C}$ the diagonal map and $s_2 = -s_1$, and $\mathbb{V}$ the unitary local system on $\mathscr{C}$ whose restriction to each fiber $\mathscr{C}_y$ is $\mathbb{L} \otimes (m^*\chi)|_{\mathscr{C}_y}$

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Since $x_1 = q$, there is a natural inclusion $\omega_E \hookrightarrow \omega_E(x_2) = \omega_E(-q + x_1 + x_2)$, so we have the following commutative diagram.

$$\begin{tikzcd} H^0(\omega_E) \otimes H^0(\omega_E) \arrow[r] \arrow[d, "\simeq"] & H^0(\omega_E^{\otimes 2}) \arrow[d] \\ H^0(\omega_E(-q+x_1+x_2)) \otimes H^0(\omega_E(q)) \arrow[r] & H^0(\omega_E^{\otimes 2}(x_1+x_2)) \end{tikzcd} \tag{3.1}$$

Indeed, the natural map $H^0(\omega_E) \to H^0(\omega(-q + x_1 + x_2))$ is an isomorphism, and the same is true of the map $H^0(\omega_E) \to H^0(\omega_E(q))$. Finally, the composition $H^0(\omega_E^{\otimes 2}) \to H^0(\omega^{\otimes 2}(x_1 + x_2)) \to T_x^* X$ is zero, since this is the map induced by pullback of differentials along the composite map

$$E \xrightarrow{x \mapsto (E, x, -x)} \mathcal{M}_{1,2} \to \mathcal{M}_{1,1},$$

which is constant; here, the map $\mathcal{M}_{1,2} \to \mathcal{M}_{1,1}$ is given by forgetting the second marked point. This implies $\theta_{f(q)} = 0$, as required. $\square$

3.3 Proofs

We now begin with the proofs of our main theorems in characteristic 0.

Lemma 3.5. *Let X be a smooth variety. A rank 1 local system $\mathbb{L}$ on X is of geometric origin if and only if it is torsion.*

Proof. Torsion evidently implies geometric origin, as all torsion rank 1 local systems arise as summands of $\pi_*\mathbb{C}$ for $\pi : Y \to X$ some cyclic étale cover, so we prove the converse.

Suppose $\mathbb{L}$ is rank 1 and of geometric origin. Then the monodromy representation of $\mathbb{L}$ is defined over the ring of integers $\mathscr{O}_K$ of some number field K. Moreover, for each embedding $\mathscr{O}_K \hookrightarrow \mathbb{C}$, $\mathbb{L} \otimes_{\mathscr{O}_K} \mathbb{C}$ is unitary, as it underlies a polarizable $\mathbb{C}$-VHS of rank 1. Hence, if $\gamma \in \pi_1(X)$ is a loop, the scalar in $\mathscr{O}_K^\times$ given by the monodromy of γ is an algebraic integer all of whose Galois conjugates have absolute value 1, hence a root of unity. $\square$

Proposition 3.6 (Conjecture 1.10). *Suppose $\mathbb{V}$ is an irreducible rank 2 local system on X satisfying $(\star)$. $\mathbb{V}$ is of geometric origin if and only if $\mathbb{V}$ underlies a $\mathbb{C}$-VHS with $w(\mathbb{V}) = f(x)$, for x a torsion point of E.*

Proof. Let q be a point of E. Then as the Narasimhan–Seshadri correspondence [NS65] is a tensor functor, the unique unitary connection ∇ on $\mathscr{L} = \mathscr{O}(q - e)$ has torsion monodromy if and only if q itself is torsion; in particular, the sheaf $\mathbb{L}$ of flat sections of $(\mathscr{L}, \nabla)$ is of geometric origin if and only if q is torsion, by Lemma 3.5.

Suppose q, and hence $\mathbb{L}$, is torsion. Now $\mathbb{V} = MC_{-1}(f, \mathbb{L}) = MC_{-1}(f_*\mathbb{L}|_X)$ is evidently of geometric origin (as middle convolution preserves the property of being of geometric origin), is irreducible [Kat96, Theorem 2.9.8(2)], and $w(\mathbb{V}) = q$ by Lemma 3.4. As $w(\mathbb{V})$ uniquely determines $\mathbb{V}$ by Proposition 3.2, we have shown that if $w(\mathbb{V})$ is torsion, then $\mathbb{V}$ is of geometric origin.

Conversely, suppose $\mathbb{V}$ is of geometric origin and $w(\mathbb{V}) = q$ is not torsion; we have that $\mathbb{V} = MC_{-1}(f, \mathbb{L}) = MC_{-1}(f_*\mathbb{L}|_X)$ for $\mathbb{L}$ the sheaf of flat sections to the unitary flat line bundle $(\mathscr{O}(q - e), \nabla)$ as above, by Lemma 3.4 and Proposition 3.2. Hence, as [Kat96, Theorem 2.9.8(1)] shows that MC_{-1} is involutive, we have $f_*\mathbb{L}|_X = MC_{-1}(\mathbb{V})$. So if $\mathbb{V}$ was of geometric origin, the same would be true for $f_*\mathbb{L}|_X$, and hence for $f^* f_*\mathbb{L}|_X = (\mathbb{L} \oplus \mathbb{L}^\vee)|_{f^{-1}(X)}$. Hence, $\mathbb{L}$ itself would be of geometric origin, and so have finite monodromy, by Lemma 3.5. But this contradicts our assumption that q is not torsion. $\square$

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Proof of Theorems 1.5 and 1.7. Theorem 1.5 is immediate from what we have proven above; we may simply consider $MC_{-1}(f,\mathbb{L})$ as $\mathbb{L}$ varies over all rank 1 local systems on E. These are of geometric origin if and only if $\mathbb{L}$ has finite monodromy, and there are infinitely many such, in which case MCG-finiteness follows from the fact that the middle convolution is equivariant for the action of the mapping class group [DR00, Theorem 5.1], and local systems with finite monodromy are evidently MCG-finite. On the other hand, again by the equivariance of middle convolution for the action of the mapping class group, $MC_{-1}(f,\mathbb{L})$ is MCG-finite if and only if the same is true for $\mathbb{L}$, which happens if and only if $\mathbb{L}$ has finite image, by [BGMW22, Lemma 3.2], for example.

To prove Theorem 1.7, note that the proof above tells us that each $\mathbb{V}$ of geometric origin satisfying ($\star$) is of the form $R^1(\pi_2)_*i_*(\pi_1^*\mathbb{L}\otimes m^*\chi)$ for χ of order 2 and $\mathbb{L}$ a finite order rank 1 local system on E, say of order s. But the local system $\pi_1^*\mathbb{L}\otimes m^*\chi$ on $U=E\times\mathbb{P}^1\setminus(\Gamma_f\cup E\times\{\infty\})$ is trivialized on the preimage of U in the variety Z'_s defined in the statement of Theorem 1.7, by construction. Hence, its cohomology appears in the cohomology of $Z'_s\to\mathbb{P}^1$, as desired. □

4. Recollections on parabolic bundles and Higgs–de Rham flows

For the rest of the paper we will not in general be working over $\mathbb{C}$. We prepare to state and prove a precise form of [SYZ21, Conjecture 4.8], stated earlier as Theorem 1.11.

4.1 Parabolic de Rham and Higgs bundles

Let $\mathcal{C}/k$ be a smooth proper curve over a field k, and $\mathcal{D}=\sum\mathcal{D}_i\subset\mathcal{C}$ a reduced effective divisor of degree l; we denote by $(\mathcal{C},\mathcal{D})$ the logarithmic curve with log structure specified by $\mathcal{D}$. We refer the reader to [KS20, Definition 2.3] for the definition of parabolic (respectively de Rham, respectively Higgs) bundles on $(\mathcal{C},\mathcal{D})$; roughly this consists of bundles $(V_\alpha)_\alpha$ indexed by $(\alpha_1,\dots,\alpha_l)$ with $\alpha_i\in\mathbb{R}$, such that:

- each V_α is itself a (respectively de Rham, respectively Higgs) bundle; and
- they are equipped with maps of (respectively de Rham, respectively Higgs) bundles $V_\alpha\hookrightarrow V_\beta$ for all $\alpha\geq\beta$ (where we write $\alpha\geq\beta$ whenever $\alpha_i\geq\beta_i$ for all i).

The bundles $(V_\alpha)_\alpha$ are required to satisfy several more conditions, and the parabolic weights of $(V_\alpha)_\alpha$ at $\mathcal{D}_i$ are, roughly, the values α_i such that the bundle changes at α_i: see [KS20, Definition 2.3] for details.

Following [KS20, Example 2.5], we have the following construction of parabolic bundles.

Definition 4.1. Let V be an arbitrary vector bundle on $\mathcal{C}$. For $(\alpha_1,\dots,\alpha_l)\in\mathbb{Q}^l$, we define the parabolic bundle $V(-\sum_{i=1}^l\alpha_i\mathcal{D}_i)$ by

$$V\bigg(-\sum_{i=1}^{l}\alpha_i\mathcal{D}_i\bigg)_\beta=V\bigg(\sum_{i=1}^{l}-\lceil\alpha_i+\beta_i\rceil\mathcal{D}_i\bigg),$$

for any $\beta=(\beta_1,\dots,\beta_l)\in\mathbb{R}^l$.

Remark 4.2. When $\alpha_i=0$, the above is referred to as the *trivial* parabolic structure along $\mathcal{D}_i$.

The following relates parabolic objects on a curve with those on a cover; see [Bis97] for the analogous result in characteristic 0.

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Proposition 4.3 [KS20, Proposition 2.14]. *Suppose $\mathcal{C}'$ and $\mathcal{C}$ are smooth curves over $\mathbb{F}_q$ and $\pi:\mathcal{C}'\to\mathcal{C}$ is a $\mathbb{Z}/N$-covering, branched at the divisor $\mathcal{D}\subset\mathcal{C}$; denote by $\mathcal{D}'$ the reduced divisor of $\pi^*(\mathcal{D})$. Then there is an equivalence of categories between $\mathbb{Z}/N$-equivariant de Rham (respectively, Higgs) bundles on $(\mathcal{C}',\mathcal{D}')$ (i.e., de Rham (respectively, Higgs) bundles with trivial parabolic structure along $\mathcal{D}'$) and the category of parabolic de Rham (respectively, Higgs) bundles on $(\mathcal{C},\mathcal{D})$ whose parabolic structure is supported on $\mathcal{D}$ with weights in $\frac{1}{N}\mathbb{Z}$.*

4.2 Conjecture of Sun, Yang and Zuo

We now recall the statement of [SYZ21, Conjecture 4.8]. We consider logarithmic graded semistable Higgs bundles on $(\mathbb{P}^1, D=0+1+\lambda+\infty)$ such that the underlying graded vector bundle is isomorphic to $\mathscr{O}\oplus\mathscr{O}(-1)$. By [SYZ21], the moduli space $\mathcal{M}_{\mathrm{HIG}}$ of such is isomorphic to $\mathbb{P}^1$, by taking the unique zero of the Higgs field; they then consider the *twisted Higgs–de Rham flow* on $\mathcal{M}_{\mathrm{HIG}}$, which in general depends on a lift of $(\mathbb{P}^1,D)$ to W_2, inducing a self-map $\psi_p:\mathbb{P}^1\to\mathbb{P}^1$.

Conjecture 4.4 (Sun, Yang and Zuo). For $p\neq 2$ and any lifting of $(\mathbb{P}^1,D)$ to W_2, the following diagram commutes

$$\begin{tikzcd} E \arrow[r, "{[p]}"] \arrow[d, "f"] & E \arrow[d, "f"] \\ \mathbb{P}^1 \arrow[r, "\psi_p"] & \mathbb{P}^1 \end{tikzcd} \tag{4.1}$$

where $f:E\to\mathbb{P}^1$ is the elliptic curve double cover branched over D, with $f^{-1}(\infty)$ being the identity.

4.3 Translation to parabolic Higgs bundles

Following [KS20, LSW22], we translate Conjecture 4.4 into the language of parabolic Higgs–de Rham flows, which is more natural for our purposes. Let k denote an algebraic closure of $\mathbb{F}_p$, with p different from 2, and let $F_k:\mathrm{Spec}(k)\to\mathrm{Spec}(k)$ denote absolute Frobenius.

Definition 4.5.

(i) Let $HIG^{\mathrm{par}}_{p-1,N}(\mathcal{C},\mathcal{D})$ denote the category of parabolic logarithmic Higgs bundles on $(\mathcal{C},\mathcal{D})$ which are nilpotent of exponent $\le p-1$, whose parabolic structures are supported on $\mathcal{D}$, with weights lying in $\frac{1}{N}\mathbb{Z}$.

(ii) Let $MIC^{\mathrm{par}}_{p-1,N}(\mathcal{C},\mathcal{D})$ be the category of adjusted (see [KS20, Definition 2.9]) logarithmic parabolic flat bundles on $(\mathcal{C},\mathcal{D})$, whose p-curvatures and nilpotent part of the residues are nilpotent of exponent $\le p-1$, and whose parabolic structures are supported on $\mathcal{D}$ with weights lying in $\frac{1}{N}\mathbb{Z}$.

From [KS20, Theorem 2.10], we have the parabolic version of the inverse Cartier transform:

$$C^{-1}_{\mathrm{par}}:HIG^{\mathrm{par}}_{p-1,N}(\mathcal{C},\mathcal{D})\to MIC^{\mathrm{par}}_{p-1,N}(\mathcal{C},\mathcal{D}).$$

This in general depends on a chosen lift of $(\mathcal{C},\mathcal{D})$ to W_2. Note that if $(\mathscr{E}_\star,\nabla)$ is an object of $MIC^{\mathrm{par}}_{p-1,N}(\mathcal{C},\mathcal{D})$ of rank 2, the bundle $\mathrm{Gr}(\mathscr{E})$ obtained by taking the associated graded with respect to the Harder–Narasimhan filtration on $\mathscr{E}_0$ has a nilpotent Higgs field θ induced by ∇ and hence gives rise to an object $\mathrm{Gr}(\mathscr{E},\nabla)\in HIG^{\mathrm{par}}_{p-1,N}(\mathcal{C},\mathcal{D})$.

Definition 4.6. Let $\mathcal{M}_{\frac{1}{2}\infty}$ denote the moduli of graded semistable parabolic Higgs bundles on $(\mathbb{P}^1,D)$, with underlying graded parabolic bundle $(\mathscr{O}\oplus\mathscr{O}(-1))(-\frac{1}{2}\infty)$ in the notation of Definition 4.1.

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Proposition 4.7 [LSW22, Proposition 2.7]. *The operation* $\mathrm{Gr} \circ C_{\mathrm{par}}^{-1}$ *induces a self-map on* $\mathcal{M}_{\frac{1}{2}\infty}$. *Moreover, there is a natural isomorphism* $\mathcal{M}_{\mathrm{HIG}} \simeq \mathcal{M}_{\frac{1}{2}\infty}$, *identifying* ψ_p *with* $\mathrm{Gr} \circ C_{\mathrm{par}}^{-1}$.

5. Proof of theorems in positive characteristic

We now let $\lambda \in \mathbb{F}_q$ be distinct from 0 and 1, and denote by X the curve $\mathbb{P}^1 \setminus \{0, 1, \lambda, \infty\}$. As before, let $f : E \to \mathbb{P}^1$ be the elliptic curve double cover branched at $0, 1, \lambda, \infty$, and let E° denote the curve $E - E[2]$.

5.1 Middle convolution, again

We now reinterpret the constructions of § 2 to define de Rham and Higgs analogues of the local systems $MC_{-1}(f_*\mathbb{L})$ constructed earlier, in positive characteristic.

We fix for the rest of this section a cyclic étale cover $\eta : \tilde{E} \to E$ of degree m, for some m odd. We denote by S the surface $E \times \mathbb{P}^1$, and by p_1, p_2 the projection from S to E and $\mathbb{P}^1$, respectively. Let D_S be the divisor $\Gamma + E \times \infty$ on S, where Γ denotes the graph of $f : E \to \mathbb{P}^1$.

Definition 5.1. Let $w : W' \to S$ be the double cover branched over D_S, as specified in Theorem 1.7 (replacing $\mathbb{C}$ by $\mathbb{F}_q$). Define the fiber product $Z' := (\tilde{E} \times \mathbb{P}^1) \times_{E \times \mathbb{P}^1} W'$.

We now give a more concrete description of the families of curves defined above. For $x \in X$, let Z_x, W_x denote the fibers of Z', W' at x; these fit into the Cartesian square.

$$\begin{tikzcd} Z_x \arrow[r] \arrow[d, "2:1"] & W_x \arrow[d, "2:1"] \\ \tilde{E} \arrow[r, "\eta"] & E \end{tikzcd} \tag{5.1}$$

Then the map $W_x \to E$ is branched over $f^{-1}(x) \subset E$.

Notation 5.2. Denote by Z_X, W_X the fiber products $Z' \otimes_{\mathbb{P}^1} X, W' \otimes_{\mathbb{P}^1} X$, respectively; these are families of curves over X, and we denote by $J_{Z,X}, J_{W,X}$ the relative Jacobians of Z_X, W_X over X. Also we write $z_X : Z_X \to X, w_X : W_X \to X$ for the natural maps.

Proposition 5.3. *The pullbacks* $W_{E^\circ} := W' \times_{\mathbb{P}^1} E^\circ$, $Z_{E^\circ} := Z' \times_{\mathbb{P}^1} E^\circ$, *equipped with maps* $w_{E^\circ} : W_{E^\circ} \to E^\circ$, $z_{E^\circ} : Z_{E^\circ} \to E^\circ$, *may be compactified to semistable families of curves* $w_E : W_E \to E$, $z_E : Z_E \to E$ *over* E. *Furthermore, we may choose* Z_E *and* W_E *such that:*

- *the natural* $(\mathbb{Z}/m \times \mathbb{Z}/2) \times \mathbb{Z}/2$*-action*[4] *on* Z_{E° *extends to* Z_E*;*
- *the natural* $\mathbb{Z}/2 \times \mathbb{Z}/2$*-action on* W_{E° *extends to* W_E*; and*
- *there is a commutative diagram*

$$\begin{tikzcd} Z_E \arrow[rr, "\theta"] \arrow[dr, "z_E"'] & & W_E \arrow[dl, "w_E"] \\ & E & \end{tikzcd} \tag{5.2}$$

extending the natural one with $Z_{E^\circ}, W_{E^\circ}, E^\circ$, *and such that* θ *is equivariant for the first* $\mathbb{Z}/2$*-action, and the entire diagram is equivariant for the second* $\mathbb{Z}/2$*-action.*

Proof. The desired compactifications are obtained by normalizing W', Z' in the function fields of W_{E°, Z_{E°. The three bullets are clear. One may prove semistability in a number of ways,

[4]The $\mathbb{Z}/m \times \mathbb{Z}/2$-action comes from the construction of Z' as a fiber product in Definition 5.1, whereas the $\mathbb{Z}/2$-action acting on $Z_{E^\circ} = Z \times_{\mathbb{P}^1} E^\circ$ is via the -1-map on the second factor.

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for example by computing with local models; we give a proof by analyzing the monodromies of the local systems $R^1w_{E^\circ,*}\overline{\mathbb{Q}}_\ell, R^1z_{E^\circ,*}\overline{\mathbb{Q}}_\ell$ about the points of $E[2]$. As a curve has semistable reduction if and only if its Jacobian does, it suffices to show, by the Néron–Ogg–Shafarevich criterion, that these local monodromies are unipotent.

Recall that we have the map $z_X : Z_X \to X$. It suffices to show (as $E \to \mathbb{P}^1$ is ramified to order 2 at $0, 1, \infty, \lambda$) that the local monodromies of $R^1z_{X,*}\overline{\mathbb{Q}}_\ell$ around all punctures have generalized eigenvalues 1 or -1. This is true for $R^1z_{X,*}\overline{\mathbb{Q}}_\ell/R^1w_{X,*}\overline{\mathbb{Q}}_\ell$, since this is a sum of middle convolutions as in Theorem 1.7, and the claim about local monodromies follows from Proposition 2.5. It remains to show the same for $R^1w_{X,*}\overline{\mathbb{Q}}_\ell$.

Denote the map $E^\circ \to X$ by π°, and let τ denote a rank 1 local system on X, given by the nontrivial summand of $\pi^\circ_*\overline{\mathbb{Q}}_\ell$. Then $R^1w_{X*}\overline{\mathbb{Q}}_\ell$ is a direct sum of the local system associated to the constant family $E \times X$, and $MC_{-1}(\tau)$. For $MC_{-1}(\tau)$, the local monodromies at the punctures satisfy ($\star$), again by [DR07, Lemma 5.1]. □

Definition 5.4.

(i) For each character $\chi : \mathbb{Z}/m \to (\overline{\mathbb{F}}_p)^\times$, we define $\widetilde{MC}^{\mathrm{dR}}_{-1}(\chi)$ to be the $\chi \otimes (-1)$-isotypic[5] part of the relative log-de Rham cohomology of $(Z_E, z_E^{-1}(E[2]))$ over $(E, E[2])$. This is locally free since $z_E : Z_E \to E$ is log smooth of relative dimension 1 by Proposition 5.3.[6]

(ii) Moreover, the additional $\mathbb{Z}/2$-action in Proposition 5.3 implies that each $\widetilde{MC}^{\mathrm{dR}}_{-1}(\chi)$ has a $\mathbb{Z}/2$-equivariant structure with respect to the covering $E \to \mathbb{P}^1$, and so descends to a parabolic logarithmic de Rham bundle on $(\mathbb{P}^1, D)$ by Proposition 4.3, which we denote by $MC^{\mathrm{dR}}_{-1}(\chi)$; moreover, the latter is equipped with a Hodge filtration descended from E. Finally, let $MC^{\mathrm{HIG}}_{-1}(\chi)$ denote the parabolic, graded, logarithmic, Higgs bundle on $(\mathbb{P}^1, D)$ given by the associated graded of $MC^{\mathrm{dR}}_{-1}(\chi)$ with respect to the Hodge filtration. This Higgs bundle is meant to be the Higgs avatar of the middle convolution applied to the de Rham local system associated to χ.

Remark 5.5. One could construct the functors MC^{dR}_{-1} and MC^{HIG}_{-1} by imitating Definition 2.1 or Definition 2.3 in the de Rham and Higgs settings, respectively. We have opted for a more geometric construction to reduce the heavy notation that would be required to execute this strategy.

Remark 5.6. As in the proofs of Theorems 1.5 and 1.7, we will prove our main results in characteristic p by reducing statements about Higgs bundles and flat bundles of rank 2 on the projective line to statements about rank 1 objects on an elliptic curve. The primary issue in proving, for example, Theorem 1.11 will be to show that the middle convolution functors we have just constructed commute with the Higgs–de Rham flow.

As before, for $x \in X$, let W_x be the fiber of W' above x; W_x is a smooth curve and is moreover equipped with a $\mathbb{Z}/2$-covering map $h_{W,x} : W_x \to E_x = E \times \{x\}$.

[5]Here $\chi \otimes (-1)$ denotes the character of $(\mathbb{Z}/m \times \mathbb{Z}/2)$, the first factor of the group $(\mathbb{Z}/m \times \mathbb{Z}/2) \times \mathbb{Z}/2$ from Proposition 5.3, given by tensoring χ with the -1-character of $\mathbb{Z}/2$.

[6]One many argue here for local freeness as follows. The formation of log-de Rham cohomology commutes with base change and has constant rank, as may be verified by backwards induction on the degree combined with standard computations for log-smooth curves. The referee suggests the following perhaps more direct argument: by [Ill90, Corollaire 2.4], it suffices to verify that, for an étale open U of $(E, E[2])$, the Frobenius twist of Z_E lifts to a W_2-lift of U; this follows from the smoothness of $\overline{\mathscr{M}_{g,n}}$.

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By Proposition 4.3, the $\mathbb{Z}/2$-equivariant bundle with connection $\mathscr{O}_{W_x}$ descends to a parabolic logarithmic connection J_x on E_x. Let $L_{\chi,x}$ denote the χ-isotypic component of $\eta_*\mathscr{O}_{\tilde{E}}$. Note that the isomorphism classes of E_x and $L_{\chi,x}$ are independent of the choice of x, and we simply denote them by E, L_χ, respectively. The line bundle L_χ has degree 0, and so is isomorphic to $\mathscr{O}_E(e-\tilde{y})$ for a unique $\tilde{y}\in E$; we say that $y := f(\tilde{y})$ is the point corresponding to L_χ.

Proposition 5.7. *The fiber of $MC_{-1}^{\mathrm{HIG}}(\chi)$ at x is given by*

$$H^1(L_\chi \otimes J_x) \oplus H^0(L_\chi \otimes J_x \otimes \Omega^1_E(f^{-1}(x))), \tag{5.3}$$

with the first (respectively, second) factor being the degree 1 (0) piece.

The Higgs field vanishes only at the point y corresponding to L_χ. Moreover, the graded vector bundle underlying $MC_{-1}^{\mathrm{HIG}}(\chi)$ is isomorphic to $\mathscr{O}\oplus\mathscr{O}(-1)$.

Proof. Since the maps $Z_X, W_X \to X$ are smooth, the restriction $MC_{-1}^{\mathrm{dR}}(\chi)|_X$ is given by the quotient of the relative de Rham cohomologies of Z_X and W_X over X. Therefore, the fiber $MC_{-1}^{\mathrm{dR}}(\chi)|_x$ is canonically identified with the $\chi\otimes(-1)$-isotypic component of $H^1_{\mathrm{dR}}(Z_x)$.

Let $h_{Z,x}: Z_x \to E$ be the natural map. To take the $\chi\otimes(-1)$ isotypic component, we can first take the $\chi\otimes(-1)$-component of $h^{\mathrm{dR}}_{Z,x,*}\mathscr{O}_{Z_x}$; the latter is a logarithmic connection on the line bundle $L_\chi\otimes J_x$, with poles along $f^{-1}(x)$. The fiber of $MC_{-1}^{\mathrm{dR}}(\chi)$ at x is then given by taking $\mathbb{H}^1$ of the associated de Rham complex

$$L_\chi\otimes J_x \to L_\chi \otimes J_x \otimes \Omega^1_E(f^{-1}(x)),$$

with the Hodge filtration given by the naive filtration. Therefore, the fiber of the Higgs cohomology is given by (5.3).

Denote the graded vector bundle underlying $MC_{-1}^{\mathrm{HIG}}(\chi)$ by $\mathcal{F}$; we now prove that $\mathcal{F}\simeq\mathscr{O}\oplus\mathscr{O}(-1)$ as a graded bundle.

Suppose $\lambda_W\in W(\mathbb{F}_q)$ is a lift of λ, so that the divisor D lifts canonically to a divisor $D_W\subset\mathbb{P}^1_W$, and we also have the pair $(E_W, E_W[2])$ lifting $(E, E[2])$, as well as the cyclic étale covering $\tilde{E}_W\to E_W$. Fixing a lift $\chi_W:\mathbb{Z}/m\to W(\overline{\mathbb{F}}_p)^\times$ of χ, we can then define a graded logarithmic Higgs bundle $MC^{\mathrm{HIG}}_{-1,\mathbb{P}^1_W}(\chi_W)$ on $(\mathbb{P}^1_W, D_W)$ exactly as in Definition 5.4, which lifts $MC_{-1}^{\mathrm{HIG}}(\chi)$.

Let us write the generic fiber $MC^{\mathrm{HIG}}_{-1,\mathbb{P}^1_W}(\chi_W)[1/p]$ as $M\oplus N$, with M (respectively, N) being the degree 1 (respectively, 0) piece for the Hodge grading; this is a stable Higgs bundle of degree 0 on $\mathbb{P}^1_{W[1/p]}$, with non-vanishing Higgs field. By our geometric construction, under the usual Simpson correspondence, this Higgs bundle corresponds to the local system $MC_{-1}(f,\mathbb{L})$ in the notation of Definition 2.3, with $\mathbb{L}$ being a rank 1 local system on E_W (which is in turn specified by the cover $\tilde{E}_W\to E_W$ and $\chi:\mathbb{Z}/m\to W(\overline{\mathbb{F}}_p)^\times$). Now $MC_{-1}(f,\mathbb{L})$ has monodromy $(\star)$ around by Proposition 2.5, and therefore the parabolic weights of $MC^{\mathrm{HIG}}_{-1,\mathbb{P}^1_W}(\chi_W)[1/p]$ are zero at $0,1,\lambda_W$ and $1/2$ at ∞. This forces $\deg(M)=0$, $\deg(N)=-1$ by Proposition 3.1; since Chern classes are invariant under specialization, the same is true of $\mathcal{F}$, as required.

Finally, let L_{χ_W} denote the χ_W-isotypic component of $\eta_*\mathscr{O}_{\tilde{E}_W}$, which is a line bundle lifting L_χ. By Lemma 3.4, θ vanishes at the point $\tilde{x}\in\mathbb{P}^1_{W[1/p]}$ corresponding to L_{χ_W}, and hence the Higgs field of $MC_{-1}^{\mathrm{HIG}}(\chi)$ vanishes at the point x corresponding to L_χ, as claimed. □

Corollary 5.8. *For all choices of $\eta:\tilde{E}\to E$ and χ, $MC_{-1}^{\mathrm{HIG}}(\chi)$ corresponds to a point in $\mathcal{M}_{\mathrm{HIG}}\simeq\mathcal{M}_{\frac{1}{2}\infty}$, where these spaces are defined in §§ 4.2 and 4.3.*

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5.2 Proof of Conjecture 4.4

Since $z_E: Z_E \to E$ and $w_E: W_E \to E$ are semistable families of curves, we may define their relative log-Higgs cohomologies (i.e., the associated graded of relative log-de Rham cohomology with respect to the Hodge filtration), and by Proposition 4.3 we may descend to obtain parabolic logarithmic Higgs bundles on $\mathbb{P}^1$, which we denote by $R^1z_*^{\mathrm{HIG}}\mathscr{O}_Z$ and $R^1w_*^{\mathrm{HIG}}\mathscr{O}_W$, respectively. Let $(\mathcal{E},\theta)$ denote the quotient Higgs bundle $R^1z_*^{\mathrm{HIG}}\mathscr{O}_Z/R^1w_*^{\mathrm{HIG}}\mathscr{O}_W$. Similarly, we have the logarithmic flat bundles $R^1z_*^{\mathrm{dR}}\mathscr{O}_Z, R^1w_*^{\mathrm{dR}}\mathscr{O}_W, (\mathcal{E}_{\mathrm{dR}},\nabla) := R^1z_*^{\mathrm{dR}}\mathscr{O}_Z/R^1w_*^{\mathrm{dR}}\mathscr{O}_W$.

Proposition 5.9. *$C^{-1}(\mathcal{E},\theta)|_X \simeq (\mathcal{E}_{\mathrm{dR}},\nabla)|_X$, and therefore $\mathrm{Gr}\circ C^{-1}(\mathcal{E},\theta)|_X \simeq (\mathcal{E},\theta)|_X$. Here C^{-1} denotes the usual inverse Cartier transform, and* Gr *the associated graded with respect to the Hodge filtration on de Rham cohomology.*

Proof. We have $(\mathcal{E},\theta)|_X \simeq R^1z_{X*}^{\mathrm{HIG}}\mathscr{O}_{Z_X}/R^1w_{X*}^{\mathrm{HIG}}\mathscr{O}_{W_X}$; here $R^1z_{X*}^{\mathrm{HIG}}$ denotes the pushforward in the category of Higgs bundles, along $z_X: Z_X \to X$, defined for example in [OV07, Definition 3.2, Remark 3.3], and similarly for $R^1w_{X*}^{\mathrm{HIG}}$. The first statement now follows from [OV07, Theorem 3.8] since for any lift $\tilde{X}$ of X to W_2, we have corresponding lifts of Z_X, W_X to W_2, by repeating Definition 5.1. The second statement is immediate. □

Proof of Conjecture 4.4. Let $g:\mathbb{P}^1 \to \mathbb{P}^1$ be the map making (4.1) commute. Then g has degree p^2, and the same is true of $\mathrm{Gr}\circ C_{\mathrm{par}}^{-1}$ by [SYZ21, § 4.3]; therefore it suffices to show that $\mathrm{Gr}\circ C_{\mathrm{par}}^{-1}$ and g agree on infinitely many closed points. For any $m \geq 1$ odd, let $\tilde{E} \to E$ be a non-trivial étale $\mathbb{Z}/m$-covering, and for any character $\chi: \mathbb{Z}/m \to \overline{\mathbb{F}}_p$ we have the Higgs bundles $MC_{-1}^{\mathrm{HIG}}(\chi)$ on $(\mathbb{P}^1, D)$, defined in Definition 5.4. By Lemma 5.8, these Higgs bundles correspond to points in $\mathcal{M}_{\mathrm{HIG}} \simeq \mathbb{P}^1$ and we will show that g and $\mathrm{Gr}\circ C_{\mathrm{par}}^{-1}$ agree on all of these points.

Recall that $F_k: k \to k$ denotes absolute Frobenius. The Higgs–de Rham functor $\mathrm{Gr}\circ C^{-1}$ is F_k-linear, and by Proposition 5.9[7] $\mathrm{Gr}\circ C^{-1}(\mathcal{E},\theta)|_X \simeq (\mathcal{E},\theta)|_X$. Now $MC_{-1}^{\mathrm{HIG}}(\chi)|_X$ is the χ-isotypic component for the $\mathbb{Z}/m$-action on $(\mathcal{E},\theta)|_X$, and therefore $\mathrm{Gr}\circ C^{-1}(MC_{-1}^{\mathrm{HIG}}(\chi))|_X$ is the χ^p-isotypic component. On the other hand, by definition of the $MC_{-1}^{\mathrm{HIG}}(\chi)$ the χ^p-isotypic component is given by $MC_{-1}^{\mathrm{HIG}}(\chi^p)|_X$. Therefore, $\mathrm{Gr}\circ C^{-1}(MC_{-1}^{\mathrm{HIG}}(\chi))|_X \simeq MC_{-1}^{\mathrm{HIG}}(\chi^p)|_X$, and hence $\mathrm{Gr}\circ C_{\mathrm{par}}^{-1}(MC_{-1}^{\mathrm{HIG}}(\chi)) \simeq MC_{-1}^{\mathrm{HIG}}(\chi^p)$ since they are both isomorphic to $\mathscr{O}\oplus\mathscr{O}(-1)$ as graded bundles and hence are determined by the vanishing locus of the Higgs field, which is a reduced point of X. By the explicit description of $MC_{-1}^{\mathrm{HIG}}(\chi)$ given in Proposition 5.7, in particular the claim about the vanishing locus of the Higgs field, we may conclude. □

6. Some loose ends

6.1 Motivicity

A well-known consequence of the Langlands correspondence is that all semisimple local systems (with finite order determinant, say) on curves over finite fields are of geometric origin. However, one expects there to be far fewer geometric local systems in characteristic 0, and that the local systems in characteristic p will not in general lift 'motivically' to characteristic 0. We show that the case studied in this paper is an exception to this expectation.

For each $\lambda \in \mathbb{F}_q$, suppose we have a rank 2 tame $\overline{\mathbb{Q}}_\ell$-local system $\mathbb{V}$ on $X = \mathbb{P}^1 - \{0,1,\lambda,\infty\}$ satisfying $(\star)$. Let $\lambda_W \in W(\mathbb{F}_q)$ be a lift of λ, and let $X_W = \mathbb{P}^1 - \{0,1,\lambda_W,\infty\}$ be the corresponding lift of X, and $\tilde{f}$ the lift of f. By definition $\mathbb{V}$ corresponds to a representation of the tame

[7] Note that Gr in the statement of Proposition 5.9, which is a priori the Hodge filtration restricted to $MC_{-1}^{\mathrm{HIG}}(\chi)$, is also the Harder–Narasimhan filtration, which is what Sun, Yang and Zuo use to formulate their conjecture; we refer to these unambiguously as Gr.

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fundamental group $\pi_1^t(X)$; the specialization map $sp: \pi_1^t(X_W) \to \pi_1^t(X)$ gives a local system on X_W, which we denote by $\mathbb{V}_W$.

Proposition 6.1. *The local system $\mathbb{V}_W$ is motivic, that is, there exists an abelian scheme $\pi: \mathcal{A} \to X_W$ such that $\mathbb{V}_W$ appears as a direct summand of $R^1\pi_*\overline{\mathbb{Q}}_\ell$.*

Proof. Note that, by using the Weil restriction of abelian schemes [BLR12, § 7.6], it suffices to show that the pullback of $\mathbb{V}_W$ to $X_W \otimes \mathrm{Spec}(W(\mathbb{F}_{q^2}))$ appears as a direct summand of an abelian scheme. This now follows by applying MC_{-1} to $\mathbb{V}$. Indeed, by Proposition 2.5 and invertibility of middle convolution, $MC_{-1}(\mathbb{V})$ has rank 2, and the local monodromies at $0, 1, \lambda, \infty$ are each semisimple with eigenvalues 1 and -1. As in the rest of the paper, let $f: E \to \mathbb{P}^1$ be the elliptic curve branched at $0, 1, \lambda, \infty$, and let $f^\circ: E^\circ = E - E[2] \to X$ denote the restriction to X. Therefore, $f^{\circ *}MC_{-1}(\mathbb{V})$ extends to a local system on E, and let us denote this by $\mathbb{W}$. Then either $\mathbb{W}$ is reducible, and $\mathbb{W} \simeq \mathbb{L} \oplus [-1]^*\mathbb{L}$ for a rank 1 local system $\mathbb{L}$ on E, or it is irreducible and of the form $h_*\mathbb{L}$ for a rank 1 local system $\mathbb{L}$ on $E_{\mathbb{F}_{q^2}}$, and h the natural map $E_{\mathbb{F}_{q^2}} \to E$. Now let $\tilde{f}: E_W \to \mathbb{P}^1$ denote the elliptic curve branched at $0, 1, \lambda_W, \infty$. In the first case above, $\mathbb{L}$ lifts canonically to a rank 1 local system $\widetilde{\mathbb{L}}$ on E_W, and we have $MC_{-1}(\tilde{f}, \widetilde{\mathbb{L}}) \simeq \mathbb{V}_W$, and therefore the latter appears in an abelian scheme. In the second case, the same argument applies after we pull $\mathbb{V}_W$ back to $X_W \otimes W(\mathbb{F}_{q^2})$. $\square$

Remark 6.2. We knew a priori that $\mathbb{V}$ was motivic, as it is an arithmetic local system on a curve over a finite field [Laf02, § 6]. The Higgs field of the associated Higgs bundle vanishes at a point of X, which is necessarily the image of a torsion point of E (as every closed point of an elliptic curve over a finite field is torsion). That the local system lifts to a motivic local system should perhaps be unsurprising, as we may lift the torsion point above to characteristic 0.

6.2 Lifting to motivic Higgs bundles over *W*

Fix $\lambda \in \mathbb{F}_q$, $\lambda_W \in W(\mathbb{F}_q)$ lifting λ, and set $X = \mathbb{P}^1 \setminus \{0, 1, \lambda, \infty\}$, $X_W = \mathbb{P}^1 \setminus \{0, 1, \lambda_W, \infty\}$.

We remark that each parabolic Higgs bundle $MC_{-1}^{\mathrm{HIG}}(\chi)$, as constructed in Definition 5.4, lifts to W motivically. In fact, this was used in the proof of Proposition 5.7. More precisely, there is a parabolic Higgs bundle $MC_{-1,\mathbb{P}^1_W}^{\mathrm{HIG}}(\chi_W)$ on $\mathbb{P}^1$, with:

- logarithmic poles along the divisor $(0) + (1) + (\lambda_W) + (\infty)$ whose reduction mod p is $MC_{-1}^{\mathrm{HIG}}(\chi)$;
- and which comes from an abelian scheme, in the sense that, after pulling back along the double cover $E_W \to \mathbb{P}^1_W$, this Higgs bundle occurs in the Higgs cohomology of a semistable family of abelian varieties (i.e., with semistable reduction at $0, 1, \lambda_W, \infty$).

Assuming that the parabolic Higgs bundles for a semistable family of abelian varieties are periodic for the parabolic Higgs–de Rham flow over W, and that the twisted Higgs–de Rham flow of [SYZ21] can be identified with the parabolic Higgs–de Rham flow, this proves [SYZ21, Conjecture 4.10]. Note that the mod p version of these statements has been worked out; see, for example, [KS20, Proposition 5.6] and [LSW22, Proposition 2.7]. The statements over W seem not to have been written down, but see [LYZ23, Lemma 2.35], which indicates that they follow by the methods developed in [LSZ19].

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Conflicts of interest

None.

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